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On Convergence of Certain Nonlinear Bernstein Operators

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Abstract. In this article, we concern with the nonlinear Bernstein operators $NB_n f$ of the form

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \ , \ \ 0 \leq x \leq 1 \ \ , \quad n \in \mathbb{N},$$

acting on bounded functions on an interval $[0,1]$, where $P_{n,k}$ satisfy some suitable assumptions. As a continuation of the very recent paper of the authors [22], we estimate their pointwise convergence to a function f having derivatives of bounded (Jordan) variation on the interval $[0,1]$.

We note that our results are strict extensions of the classical ones, namely, the results dealing with the linear Bernstein polynomials.

1. Introduction

We consider the problem of approximating a given real-valued function f, defined on $[0,1]$, by means of a sequence of nonlinear Bernstein operators $NB_n f$. Positive linear operators, convolution, moment and sampling operators have an important role in several branches of Mathematics. For instance, in reconstruction of signals and images, in Fourier analysis, operator theory, probability theory and approximation theory (see e.g. [14, 28]).

In this paper, we will take into account the nonlinear Bernstein operators, generated by the classical Bernstein operators considered in [22].

Let f be a function defined on the interval $[0,1]$ and let $\mathbb{N} := \{1,2,...\}$. The classical Bernstein polynomial of $f : [0,1] \to \mathbb{R}$ of degree n is defined by

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \ , \ \ 0 \leq x \leq 1 \ \ , \quad n \in \mathbb{N}, \tag{1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis. These polynomials were introduced by Bernstein [10] in 1912 to give the first constructive proof of the Weierstrass approximation theorem. Some properties of the polynomials (1) can be found in Lorentz [23].

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We now state a brief and technical explanation of the relation between approximation by linear and nonlinear operators. Approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in [25] and widely developed in [7] (and the references contained therein). To the best of our knowledge, the approximation problem were limited to linear operators because the notion of singularity of an integral operator is closely connected with its linearity until the fundamental paper of Musielak [25]. In [25], the assumption of linearity of the singular integral operators was replaced by an assumption of a Lipschitz condition for the kernel function $K_\lambda(t,u)$ with respect to the second variable. Especially, nonlinear integral operators of the following type

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t-x, f(t))\, dt, \quad x \in (a,b),$$

and its special cases were studied by Bardaro-Karsli and Vinti [3, 4] and Karsli [17] in some Lebesgue spaces.

For further reading, we also refer the reader to [1, 2, 8, 9, 15, 21, 24, 29] and the very recent paper of the authors [22] as well as the monographs [7] and [13] where other kind of convergence results of linear and nonlinear operators in the Lebesgue spaces, Musielak-Orlicz spaces, BV-spaces and BV_φ-spaces have been considered.

Very recently, by using the techniques due to Musielak [25], Karsli-Tiryaki and Altin [22] considered the following type nonlinear counterpart of the well-known Bernstein operators;

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \ , \ 0 \le x \le 1 \ , \ n \in \mathbb{N}, \tag{2}$$

acting on bounded functions f on an interval $[0,1]$, where $P_{n,k}$ satisfy some suitable assumptions. They proved some existence and approximation theorems for the nonlinear Bernstein operators. In particular, they obtained some pointwise convergence for the nonlinear sequence of Bernstein operators (2) to some point x of f, as $n \to \infty$.

It should be note that the same definition of nonlinear Bernstein operators was given in the excellent papers due to Bardaro and Mantellini [5, 6], in which other kinds of convergence properties are studied.

In the present paper, the study of operators (2) will be continued.

As a continuation of [22], we estimate their pointwise convergence to functions f and $\psi \circ |f|$ having derivatives of bounded (Jordan) variation on the interval $[0,1]$.

The paper is organized as follows: The next section contains basic definitions and notations.

In Section 3, the main approximation results of this study are given. They are dealing with the rate of pointwise convergence of the nonlinear Bernstein operators $NB_n f$ to the limit f, where f and $\psi \circ |f|$ are functions whose derivatives are of bounded variation on the interval $[0,1]$. We shall prove that $(NB_n f)(x)$ converge to the limit $f(x)$ for $x \in (0,1)$. Let us note that the counterpart of such kind of results for positive linear operators are the rate of convergence for functions in $DBV(I)$, functions whose derivatives are bounded variation defined on a set $I \subset \mathbb{R}$, were first obtained by Bojanic and Cheng [11, 12]. Further papers on the subject are [16, 18, 19], and [26, 27].

In Section 4, we give some crucial results which are necessary to prove the main result.

The final section, that is Section 5, concerns with the proof of the main results presented in Section 3.

2. Preliminaries

In this section, we recall the following structural assumptions according to [22], which will be fundamental in proving our convergence theorems.

Let X be the set of all bounded Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the function ψ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$.
We now introduce a sequence of functions. Let $\{P_{n,k}\}_{n\in\mathbb{N}}$ be a sequence functions $P_{n,k} : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$P_{n,k}(t,u) = p_{n,k}(t) H_n(u) \tag{3}$$

for every $t \in [0,1], u \in \mathbb{R}$, where $H_n : \mathbb{R} \to \mathbb{R}$ is such that $H_n(0) = 0$ and $p_{n,k}(t)$ is the Bernstein basis.
Throughout the paper, we assume that $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing and continuous function such that $\lim\limits_{n\to\infty} \mu(n) = \infty$.
First of all, we assume that the following conditions hold:
a) $H_n : \mathbb{R} \to \mathbb{R}$ is a real-valued function such that

$$|H_n(u) - H_n(v)| \le \psi(|u - v|), \quad \psi \in \Psi,$$

holds for every $u, v \in \mathbb{R}$, and $n \in \mathbb{N}$. That is, H_n satisfies a $(L - \psi)$ Lipschitz condition (see [7]).

b) We now set

$$K_n(x,u) := \begin{cases} \sum\limits_{k\le nu} p_{n,k}(x) & , \quad 0 < u \le 1 \\ 0 & , \quad u = 0 \end{cases} \tag{4}$$

and

$$A_n(x) := \int\limits_{x - x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} d_t\left(K_n(x,t)\right) \quad \text{for any fixed } x \in (0,1)$$

where $\beta > 0$, $\gamma \ge 1$ and

$$\lambda_n(x,t) := \int\limits_0^t d_u\left(K_n(x,u)\right). \tag{5}$$

Similar approach and some particular examples can be found in [8, 20–22] and [27].

Suppose that there are $\beta > 0$, $\gamma \ge 1$ such that a finite function $C(x)$ exists with

$$n^{\gamma/\beta}\left(B_n\, |t - x|^\beta\right)(x) \le C(x), \; x \in (0,1) \tag{6}$$

uniformly with respect to $n \in \mathbb{N}$.

c) Denoting by $r_n(u) := H_n(u) - u$, $u \in \mathbb{R}$ and $n \in \mathbb{N}$. We suppose that for n sufficiently large

$$\sup_u |r_n(u)| = \sup_u |H_n(u) - u| \le \frac{1}{\mu(n)}$$

holds.
The symbol $[a]$ will denote the greatest integer not greater than a.

3. Convergence Results

We will consider the following type of nonlinear Bernstein operators,

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right)$$

defined for every $f \in X$ for which $NB_n f$ is well-defined, where

$$P_{n,k}(x,u) = p_{n,k}(x) H_n(u)$$

for every $x \in [0,1], u \in \mathbb{R}$.
For any function f for which the one-sided limits $f(x+), f(x-)$ exist at every point $x \in (0,1)$, we let

$$f_x(t) = \begin{cases} f(t) - f(x+) & , \ x < t \leq 1 \\ 0 & , \ t = x \\ f(t) - f(x-) & , \ 0 \leq t < x \end{cases} \tag{7}$$

and $\bigvee_0^1 \psi(|f_x|)$ is the total variation of $\psi(|f_x|)$ on $[0,1]$.
Let $DBV(I)$ denotes the class of differentiable functions defined on a set $I \subset \mathbb{R}$ whose derivatives are bounded variation on I and will be denoted as

$$DBV(I) = \{f : f' \in BV(I)\}.$$

We are now ready to establish the main results of this study:

Theorem 3.1. *Let $\psi \in \Psi$ and f be a function with derivatives of bounded variation on* $[0,1]$. *Then for every $x \in (0,1)$, we have for sufficiently large n,*

$$\begin{aligned} \left|(NB_n f)(x) - f(x)\right| \leq & \left|\frac{f'(x+) - f'(x-)}{2}\right| \sqrt{\frac{x(1-x)}{n}} \\ & + \frac{2}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x+(1-x)/k} \left(f'_x\right) + \frac{1}{\mu(n)} \end{aligned} \tag{8}$$

where $\bigvee_a^b \left(f'_x\right)$ is the total variation of f'_x on $[a,b]$.

Theorem 3.2. *Let $\psi \in \Psi$ and $f \in X$ be such that $\psi \circ |f| \in DBV([0,1])$. Then for every $x \in (0,1)$, we have for sufficiently large n,*

$$\begin{aligned} \left|(NB_n f)(x) - f(x)\right| \leq & \frac{\left|\left(\psi \circ |f|\right)'(x-) - \left(\psi \circ |f|\right)'(x+)\right|}{2} \sqrt{\frac{x(1-x)}{n}} \\ & + \frac{2}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x+(1-x)/k} \left(\psi \circ |f|\right)'_x + \frac{1}{\mu(n)} \end{aligned} \tag{9}$$

where $\bigvee_a^b \left(\psi \circ |f|\right)'_x$ is the total variation of $\left(\psi \circ |f|\right)'_x$ on $[a,b]$.

4. Auxiliary Result

In this section, we give crucial results which are necessary to prove our theorems.

Lemma 4.1. *For* $(B_n t^s)(x)$, $s = 0, 1, 2$, *one has*

$$(B_n 1)(x) = 1, (B_n t)(x) = x, (B_n t^2)(x) = x^2 + \frac{x(1-x)}{n}.$$

For the proof of this Lemma see [23].

By direct calculation, we find the following equalities:

$$(B_n (t-x)^2)(x) = \frac{x(1-x)}{n} \quad , \quad (B_n (t-x))(x) = 0.$$

Lemma 4.2. *For all* $x \in (0, 1)$ *and for each* $n \in \mathbb{N}$, *one has*

$$\lambda_n(x,t) =: \int_0^t d_u \left(K_n(x,u)\right) \leq \frac{C(x)}{(x-t)^\beta n^{\gamma/\beta}} \ , \quad 0 \leq t < x, \tag{10}$$

and

$$1 - \lambda_n(x,t) = \int_t^1 d_u \left(K_n(x,u)\right) \leq \frac{C(x)}{(t-x)^\beta n^{\gamma/\beta}} \ , \quad x < t < 1 \tag{11}$$

where $C(x)$ *is as given in (6).*

Proof. We have

$$\begin{aligned} \lambda_n(x,t) &=: \int_0^t d_u \left(K_n(x,u)\right) \leq \int_0^t \left(\frac{x-u}{x-t}\right)^\beta d_u \left(K_n(x,u)\right) \\ &\leq \frac{1}{(x-t)^\beta} \int_0^1 |u-x|^\beta \, d_u \left(K_n(x,u)\right). \end{aligned}$$

According to (6), we have

$$\lambda_n(x,t) \leq \frac{C(x)}{(x-t)^\beta n^{\gamma/\beta}}.$$

Proof of (11) is analogous. □

5. Proof of the Theorems

Proof of Theorem 3.1. In general, a singular integral may be written in the form

$$(T_n f)(x) = \int_a^b f(t) M_n(x,t) \, dt \tag{12}$$

where $M_n(x,t)$ is the kernel, defined for $a \leq x \leq b$, $a \leq t \leq b$, which has the property that for functions $f(x)$ of a certain class and in a certain sense, $(T_n f)(x)$ converges to $f(x)$ as $n \to \infty$.

The Bernstein polynomial (1) is a finite sum of a type corresponding to the integral (12). Both (1) and (12) are special cases of singular Stieltjes integrals. (1) may be written in the form of a Stieltjes integral in the variable t,

$$(B_n f)(x) = \int_0^1 f(t)\, d_t K_n(x,t)$$

with the kernel

$$\begin{aligned} K_n(x,t) &= \sum_{k \le nt} \binom{n}{k} x^k (1-x)^{n-k} \quad , \quad 0 < t \le 1 \\ K_n(x,0) &= 0 \end{aligned}$$

which is constant in any interval $k/n \le t < (k+1)/n$, $k = 0, 1, ..., n-1$.

We can write the difference between $(NB_n f)(x)$ and $f(x)$ as a singular Stieltjes integral as follows;

$$\begin{aligned} (NB_n f)(x) - f(x) &= \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f(x) \\ &= \sum_{k=0}^{n} p_{n,k}(x)\, H_n\left(f\left(\frac{k}{n}\right)\right) - f(x) \\ &= \int_0^1 H_n(f(t))\, d_t K_n(x,t) - f(x) \\ &= \int_0^1 [H_n(f(t)) - f(t)]\, d_t K_n(x,t) + \int_0^1 [f(t) - f(x)]\, d_t K_n(x,t) \\ &= I_{n,1}(x) + I_{n,2}(x)\,. \end{aligned}$$

Firstly, we consider

$$I_{n,2}(x) = \int_0^1 [f(t) - f(x)]\, d_t K_n(x,t)\,. \tag{13}$$

Since $f(t) \in DBV[0,1]$, we can rewrite (13) as follows

$$\begin{aligned} I_{n,2}(x) &= \int_0^x [f(t) - f(x)]\, d_t K_n(x,t) + \int_x^1 [f(t) - f(x)]\, d_t K_n(x,t) \\ &= -\int_0^x \left[\int_t^x f'(u)\, du\right] d_t K_n(x,t) + \int_x^1 \left[\int_x^t f'(u)\, du\right] d_t K_n(x,t) \\ &= -I_{n,2,1}(x) + I_{n,2,2}(x)\,, \end{aligned}$$

where

$$I_{n,2,1}(x) = \int_0^x \left[\int_t^x f'(u)\, du\right] d_t K_n(x,t) \tag{14}$$

and

$$I_{n,2,2}(x) = \int_x^1 \left[\int_x^t f'(u)\,du \right] d_t K_n(x,t). \tag{15}$$

For any $f(t) \in DBV[0,1]$, we decompose $f'(t)$ into four parts by using (7) as

$$\begin{aligned} f'(t) &= \frac{f'(x+) + f'(x-)}{2} + f'_x(t) + \frac{f'(x+) - f'(x-)}{2} sgn(t-x) \\ &\quad + \delta_x(t) \left[f'(x) - \frac{f'(x+) + f'(x-)}{2} \right], \end{aligned}$$

where

$$\delta_x(t) = \begin{cases} 1 & , \quad x = t \\ 0 & , \quad x \neq t \end{cases}.$$

If we use this equality in (14) and (15), we have the following expressions.

$$\begin{aligned} I_{n,2,1}(x) &= \int_0^x \left\{ \int_t^x \frac{1}{2}\left(f'(x+) + f'(x-)\right) + f'_x(u) + \frac{f'(x+) - f'(x-)}{2} sgn(u-x) \right. \\ &\quad \left. + \delta_x(u) \left[f'(x) - \frac{1}{2}\left(f'(x+) + f'(x-)\right) \right] du \right\} d_t K_n(x,t) \end{aligned}$$

and

$$\begin{aligned} I_{n,2,2}(x) &= \int_x^1 \left\{ \int_x^t \frac{1}{2}\left(f'(x+) + f'(x-)\right) + f'_x(u) + \frac{f'(x+) - f'(x-)}{2} sgn(u-x) \right. \\ &\quad \left. + \delta_x(u) \left[f'(x) - \frac{1}{2}\left(f'(x+) + f'(x-)\right) \right] du \right\} d_t K_n(x,t). \end{aligned}$$

Firstly, we evaluate $I_{n,2,1}(x)$.

$$\begin{aligned} I_{n,2,1}(x) &= \frac{f'(x+) + f'(x-)}{2} \int_0^x (x-t)\, d_t K_n(x,t) + \int_0^x \left[\int_t^x f'_x(u)\,du \right] d_t K_n(x,t) \\ &\quad - \frac{f'(x+) - f'(x-)}{2} \int_0^x (x-t)\, d_t K_n(x,t) \\ &\quad + \left[f'(x) - \frac{f'(x+) + f'(x-)}{2} \right] \int_0^x \left[\int_t^x \delta_x(u) du \right] d_t K_n(x,t). \end{aligned}$$

It is obvious that $\int_t^x \delta_x(u)du = 0$. From this fact that, we get

$$\begin{aligned} I_{n,2,1}(x) &= \frac{f'(x+) + f'(x-)}{2} \int_0^x (x-t)\, d_t K_n(x,t) + \int_0^x \left[\int_t^x f'_x(u)\,du \right] d_t K_n(x,t) \\ &\quad - \frac{f'(x+) - f'(x-)}{2} \int_0^x (x-t)\, d_t K_n(x,t). \end{aligned} \tag{16}$$

Using a similar method, for evaluating $I_{n,2,2}(x)$, we find that

$$I_{n,2,2}(x) = \frac{f'(x+)+f'(x-)}{2}\int_x^1 (t-x)\,d_tK_n(x,t) + \int_x^1\left[\int_x^t f'_x(u)\,du\right]d_tK_n(x,t) - \frac{f'(x+)-f'(x-)}{2}\int_x^1 (t-x)\,d_tK_n(x,t). \tag{17}$$

Combining (16) and (17), we get

$$-I_{n,2,1}(x)+I_{n,2,2}(x) = \frac{f'(x+)+f'(x-)}{2}\int_0^1 (t-x)\,d_tK_n(x,t) +$$

$$+\frac{f'(x+)-f'(x-)}{2}\int_0^1 |t-x|\,d_tK_n(x,t) - \int_0^x\left[\int_t^x f'_x(u)\,du\right]d_tK_n(x,t) + \int_x^1\left[\int_x^t f'_x(u)\,du\right]d_tK_n(x,t).$$

From the last expression, we can rewrite (13) as follows.

$$I_{n,2}(x) = \frac{f'(x+)+f'(x-)}{2}\int_0^1 (t-x)\,d_tK_n(x,t) + \frac{f'(x+)-f'(x-)}{2}\int_0^1 |t-x|\,d_tK_n(x,t) - \int_0^x\left[\int_t^x f'_x(u)\,du\right]d_tK_n(x,t) + \int_x^1\left[\int_x^t f'_x(u)\,du\right]d_tK_n(x,t). \tag{18}$$

On the other hand, since

$$\int_0^1 |t-x|\,d_tK_n(x,t) = B_n(|t-x|;x)$$

and

$$\int_0^1 (t-x)\,d_tK_n(x,t) = B_n((t-x);x),$$

by using these equalities in (18) and taking the absolute value, we can re-expressed (18) as follows;

$$\left|I_{n,2}(x)\right| \le \left|\frac{f'(x+)+f'(x-)}{2}\right| |B_n(t-x;x)| + \left|\frac{f'(x+)-f'(x-)}{2}\right| |B_n(|t-x|;x)| + \left|-\int_0^x\left[\int_t^x f'_x(u)\,du\right]d_tK_n(x,t)\right| + \left|\int_x^1\left[\int_x^t f'_x(u)\,du\right]d_tK_n(x,t)\right|. \tag{19}$$

Using Lebesgue-Stieltjes integration, and according to (5), we obtain

$$\int_0^x\left[\int_t^x f'_x(u)\,du\right]d_tK_n(x,t) = \int_0^x f'_x(t)\,\lambda_n(x,t)\,dt.$$

Thus

$$\left|-\int_0^x\left[\int_t^x f_x'(u)\,du\right]d_tK_n(x,t)\right|\leq\int_0^x\left|f_x'(t)\right|\lambda_n(x,t)\,dt$$

and

$$\left|-\int_0^x\left[\int_t^x f_x'(u)\,du\right]d_tK_n(x,t)\right|\leq\int_0^{x-\frac{x}{\sqrt{n}}}\left|f_x'(t)\right|\lambda_n(x,t)\,dt+\int_{x-\frac{x}{\sqrt{n}}}^x\left|f_x'(t)\right|\lambda_n(x,t)\,dt.$$

Since $f_x'(x)=0$ and $\lambda_n(x,t)\leq 1$, one has

$$\int_{x-\frac{x}{\sqrt{n}}}^x\left|f_x'(t)\right|\lambda_n(x,t)\,dt=\int_{x-\frac{x}{\sqrt{n}}}^x\left|f_x'(t)-f_x'(x)\right|\lambda_n(x,t)\,dt\leq\int_{x-\frac{x}{\sqrt{n}}}^x\bigvee_t^x(f_x')\,dt.$$

By the change of variables $t=x-\frac{x}{\sqrt{n}}$, we obtain

$$\int_{x-\frac{x}{\sqrt{n}}}^x\bigvee_t^x(f_x')\,dt\leq\bigvee_{x-\frac{x}{\sqrt{n}}}^x(f_x')\int_{x-\frac{x}{\sqrt{n}}}^x dt.$$

Besides from (10), we can write

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}\left|f_x'(t)\right|\lambda_n(x,t)\,dt &\leq \frac{x(1-x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\left|f_x'(t)-f_x'(x)\right|\frac{dt}{(x-t)^2}\\ &\leq \frac{x(1-x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x(f_x')\frac{dt}{(x-t)^2}.\end{aligned}$$

By the change of variables $t=x-\frac{x}{u}$ again, we have

$$\begin{aligned}\frac{x(1-x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x(f_x')\frac{dt}{(x-t)^2} &= \frac{x(1-x)}{n}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^x(f_x')\frac{\left(\frac{x}{u^2}\right)du}{\left(\frac{x}{u}\right)^2}\\ &\leq \frac{(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x(f_x')\end{aligned}$$

and hence we obtain

$$\left|-\int_0^x\left[\int_t^x f_x'(u)\,du\right]d_tK_n(x,t)\right|\leq\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(f_x')+\frac{(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x(f_x').$$

Since

$$\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(f_x')\leq\frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x(f_x'),$$

it follows that

$$\begin{aligned}\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f_x')+\frac{(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_x') &\leq \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_x')+\frac{2(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_x')\\ &\leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_x').\end{aligned}$$

Therefore

$$\left|-\int_0^x\left[\int_t^x f_x'(u)\,du\right]d_tK_n(x,t)\right|\leq\frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_x').$$

Using a similar method for estimating, then we have

$$\left|\int_x^1\left[\int_x^t f_x'(u)\,du\right]d_tK_n(x,t)\right|\leq\frac{1-x}{\sqrt{n}}\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}}(f_x')+\frac{x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x').$$

Furthermore, since

$$\frac{1-x}{\sqrt{n}}\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}}(f_x')\leq\frac{2(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x'),$$

we can write the following inequality

$$\begin{aligned}\frac{1-x}{\sqrt{n}}\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}}(f_x')+\frac{x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x') &\leq \frac{2(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x')+\frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x')\\ &\leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x').\end{aligned}$$

Thus we get

$$\left|\int_x^1\left[\int_x^t f_x'(u)\,du\right]d_tK_n(x,t)\right|\leq\frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x').$$

Collecting the estimates, we get (8), i.e.,

$$\left|I_{n,2}(x)\right|\leq\left|\frac{f'(x+)-f'(x-)}{2}\right|\sqrt{\frac{x(1-x)}{n}}+\frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}}(f_x'),$$

and

$$\left|I_{n,1}(x)\right|=\left|\int_0^1\left[H_n(f(t))-f(t)\right]d_tK_n(x,t)\right|\leq\int_0^1\left|H_n(f(t))-f(t)\right|d_tK_n(x,t)\leq\frac{1}{\mu(n)}$$

holds for sufficiently large n.

This completes the proof of the theorem.

Proof of Theorem 3.2. We can write the difference between $(NB_n f)(x)$ and $f(x)$ as a singular Stieltjes integral as follows;

$$\left|(NB_n f)(x) - f(x)\right| = \left|\int_0^1 H_n(f(t))\, d_t K_n(x,t) - f(x)\right|$$

$$\begin{aligned}
&= \left|\int_0^1 \left[H_n(f(t)) - H_n(f(x))\right] d_t K_n(x,t) + \int_0^1 \left[H_n(f(x)) - f(x)\right] d_t K_n(x,t)\right| \\
&\leq \int_0^1 \left|H_n(f(t)) - H_n(f(x))\right| d_t K_n(x,t) + \int_0^1 \left|H_n(f(x)) - f(x)\right| d_t K_n(x,t) \\
&\leq \int_0^1 \left|H_n(f(x)) - f(x)\right| d_t K_n(x,t) + \int_0^1 \psi\left(\left|f(t) - f(x)\right|\right) d_t K_n(x,t) \\
&= I_{n,1}(x) + I_{n,2}(x).
\end{aligned}$$

Note that for a concave function ψ

$$-\psi\left(\left|f(t) - f(x)\right|\right) \leq \psi\left(\left|f(t)\right|\right) - \psi\left(\left|f(x)\right|\right)$$

holds. Firstly, we consider

$$I_{n,2}(x) = \int_0^1 \psi\left(\left|f(t) - f(x)\right|\right) d_t K_n(x,t). \tag{20}$$

Since $\left(\psi \circ |f|\right)(t) \in DBV[0,1]$, we can rewrite (20) as follows:

$$\begin{aligned}
-I_{n,2}(x) &\leq \int_x^0 \left[\psi\left(\left|f(t)\right|\right) - \psi\left(\left|f(x)\right|\right)\right] d_t K_n(x,t) + \int_1^x \left[\psi\left(\left|f(t)\right|\right) - \psi\left(\left|f(x)\right|\right)\right] d_t K_n(x,t) \\
&= \int_0^x \left[\int_t^x \left(\psi \circ |f|\right)'(u)\, du\right] d_t K_n(x,t) + \int_x^1 \left[\int_t^x \left(\psi \circ |f|\right)'(u)\, du\right] d_t K_n(x,t) \\
&= I_{n,2,1}(x) - I_{n,2,2}(x)
\end{aligned}$$

where

$$I_{n,2,1}(x) = \int_0^x \left[\int_t^x \left(\psi \circ |f|\right)'(u)\, du\right] d_t K_n(x,t) \tag{21}$$

and

$$I_{n,2,2}(x) = \int_x^1 \left[\int_x^t \left(\psi \circ |f|\right)'(u)\, du\right] d_t K_n(x,t). \tag{22}$$

For any $\left(\psi \circ \left|f\right|\right)(t) \in DBV\,[0,1]$, we decompose $\left(\psi \circ \left|f\right|\right)(t)$ into four parts by using (7) as

$$\begin{aligned}
\left(\psi \circ \left|f\right|\right)'(t) &= \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2} + \left(\psi \circ \left|f\right|\right)'_x(t) \\
&+ \frac{\left(\psi \circ \left|f\right|\right)'(x+) - \left(\psi \circ \left|f\right|\right)'(x-)}{2} sgn\,(t-x) \\
&+ \delta_x(t)\left[\left(\psi \circ \left|f\right|\right)'(x) - \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2}\right],
\end{aligned}$$

where

$$\delta_x(t) = \begin{cases} 1 & , \quad x = t \\ 0 & , \quad x \neq t \end{cases}.$$

If we use this equality in (21) and (22), we have the following expressions.

$$\begin{aligned}
I_{n,2,1}(x) &= \int_0^x \left\{ \int_t^x \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2} + \left(\psi \circ \left|f\right|\right)'_x(u) \right. \\
&+ \frac{\left(\psi \circ \left|f\right|\right)'(x+) - \left(\psi \circ \left|f\right|\right)'(x-)}{2} sgn\,(u-x) \\
&\left. + \delta_x(u)\left[\left(\psi \circ \left|f\right|\right)'(x) - \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2}\right] du \right\} d_t K_n(x,t)
\end{aligned}$$

and

$$\begin{aligned}
I_{n,2,2}(x) &= \int_x^1 \left\{ \int_x^t \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2} + \left(\psi \circ \left|f\right|\right)'_x(u) \right. \\
&+ \frac{\left(\psi \circ \left|f\right|\right)'(x+) - \left(\psi \circ \left|f\right|\right)'(x-)}{2} sgn\,(u-x) \\
&\left. + \delta_x(u)\left[\left(\psi \circ \left|f\right|\right)'(x) - \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2}\right] du \right\} d_t K_n(x,t).
\end{aligned}$$

Firstly, we evaluate $I_{n,2,1}(x)$.

$$\begin{aligned}
I_{n,2,1}(x) &= \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2} \int_0^x (x-t)\, d_t K_n(x,t) \\
&+ \int_0^x \left[\int_t^x \left(\psi \circ \left|f\right|\right)'_x(u)\, du \right] d_t K_n(x,t) \\
&- \frac{\left(\psi \circ \left|f\right|\right)'(x+) - \left(\psi \circ \left|f\right|\right)'(x-)}{2} \int_0^x (x-t)\, d_t K_n(x,t) \\
&+ \left[\left(\psi \circ \left|f\right|\right)'(x) - \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2}\right] \int_0^x \left[\int_t^x \delta_x(u)\, du \right] d_t K_n(x,t).
\end{aligned}$$

It is obvious that $\int_t^x \delta_x(u)\,du = 0$. From this fact that, we get

$$\begin{aligned} I_{n,2,1}(x) &= \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2} \int_0^x (x-t)\,d_tK_n(x,t) \\ &+ \int_0^x \left[\int_t^x \left(\psi \circ \left|f\right|\right)_x'(u)\,du \right] d_tK_n(x,t) \\ &- \frac{\left(\psi \circ \left|f\right|\right)'(x+) - \left(\psi \circ \left|f\right|\right)'(x-)}{2} \int_0^x (x-t)\,d_tK_n(x,t). \end{aligned} \tag{23}$$

Using a similar method, for evaluating $I_{n,2,2}(x)$, we find that

$$\begin{aligned} I_{n,2,2}(x) &= \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2} \int_x^1 (t-x)\,d_tK_n(x,t) \\ &+ \int_x^1 \left[\int_x^t \left(\psi \circ \left|f\right|\right)_x'(u)\,du \right] d_tK_n(x,t) \\ &- \frac{\left(\psi \circ \left|f\right|\right)'(x+) - \left(\psi \circ \left|f\right|\right)'(x-)}{2} \int_x^1 (t-x)\,d_tK_n(x,t). \end{aligned} \tag{24}$$

Combining (23) and (24), we get

$$\begin{aligned} I_{n,2,1}(x) - I_{n,2,2}(x) &= \frac{\left(\psi \circ \left|f\right|\right)'(x+) + \left(\psi \circ \left|f\right|\right)'(x-)}{2} \int_0^1 (t-x)\,d_tK_n(x,t) \\ &- \frac{\left(\psi \circ \left|f\right|\right)'(x+) - \left(\psi \circ \left|f\right|\right)'(x-)}{2} \int_0^1 |t-x|\,d_tK_n(x,t) \\ &+ \int_0^x \left[\int_t^x \left(\psi \circ \left|f\right|\right)_x'(u)\,du \right] d_tK_n(x,t) - \int_x^1 \left[\int_x^t \left(\psi \circ \left|f\right|\right)_x'(u)\,du \right] d_tK_n(x,t). \end{aligned}$$

On the other hand, note that

$$\int_0^1 |t-x|\,d_tK_n(x,t) = B_n(|t-x|;x) \leq \sqrt{\left(B_n(t-x)^2\right)(x)} = \sqrt{\frac{x(1-x)}{n}}$$

and

$$\int_0^1 (t-x)\,d_tK_n(x,t) = B_n((t-x))(x) = 0.$$

Therefore we can estimate $I_{n,2,1}(x) - I_{n,2,2}(x)$ as follows;

$$\begin{aligned} I_{n,2,1}(x) - I_{n,2,2}(x) \quad &\leq \quad \frac{\left(\psi \circ |f|\right)'(x+) - \left(\psi \circ |f|\right)'(x-)}{2}\sqrt{\frac{x(1-x)}{n}} \\ &+ \int\limits_0^x \left[\int\limits_t^x \left(\psi \circ |f|\right)'_x(u)\,du\right] d_tK_n(x,t) - \int\limits_x^1 \left[\int\limits_x^t \left(\psi \circ |f|\right)'_x(u)\,du\right] d_tK_n(x,t). \end{aligned}$$

Using Lebesgue-Stieltjes integration, and according to (5), we obtain

$$\int\limits_0^x \left[\int\limits_t^x \left(\psi \circ |f|\right)'_x(u)\,du\right] d_tK_n(x,t) = \int\limits_0^{x-\frac{x}{\sqrt{n}}} \left(\psi \circ |f|\right)'_x(t)\,\lambda_n(x,t)\,dt + \int\limits_{x-\frac{x}{\sqrt{n}}}^x \left(\psi \circ |f|\right)'_x(t)\,\lambda_n(x,t)\,dt.$$

By using the method in the proof of the previous Theorem 1, one has

$$\int\limits_0^x \left[\int\limits_t^x \left(\psi \circ |f|\right)'_x(u)\,du\right] d_tK_n(x,t) \leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} \left(\psi \circ |f|\right)'_x,$$

and

$$-\int\limits_x^1 \left[\int\limits_x^t f'_x(u)\,du\right] d_tK_n(x,t) \leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}} \left(\psi \circ |f|\right)'_x.$$

In conclusion we obtain

$$I_{n,2}(x) \leq \frac{\left|\left(\psi \circ |f|\right)'(x+) - \left(\psi \circ |f|\right)'(x-)\right|}{2}\sqrt{\frac{x(1-x)}{n}} + \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{\sqrt{n}}} \left(\psi \circ |f|\right)'_x.$$

Since

$$I_{n,1}(x) = \int\limits_0^1 \left|H_n(f(t)) - f(t)\right| d_tK_n(x,t) \leq \frac{1}{\mu(n)}$$

holds for sufficiently large n, the proof of the Theorem 2 is now complete.

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