Filomat 29:8 (2015), 1839–1845
DOI 10.2298/FIL1508839S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Coefficient Bounds for a Certain Class of Analytic and Bi-Univalent Functions

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Abstract. In this paper, we introduce and investigate a subclass of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. By using the Faber polynomial expansions, we obtain upper bounds for the coefficients of functions belonging to this analytic and bi-univalent function class. ome interesting recent developments involving other subclasses of analytic and bi-univalent functions are also briefly mentioned.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are *analytic* in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized by the following Taylor-Maclaurin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ (see, for details, [8]).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right), \tag{1.2}$$

2010 *Mathematics Subject Classification*. Primary 30C45, 30C50; Secondary 30C80
Keywords. Analytic functions; Univalent functions; Faber polynomials; Bi-Univalent functions; Taylor-Maclaurin series expansion; Coefficient bounds and coefficient estimates; Taylor-Maclaurin coefficients.
Received: 06 November 2013; Accepted: 12 April 2014
Communicated by Dragan S. Djordjević
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according to the *Koebe One-Quarter Theorem* (see, for example, [8]). In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of analytic and bi-univalent functions in $\mathbb{U}$ given by the Taylor-Maclaurin series expansion (1.1). Some examples of functions in the class Σ are presented below:

$$\frac{z}{1-z}, \qquad -\log(1-z), \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of the class Σ. Other common examples of functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \qquad \text{and} \qquad \frac{z}{1-z^2}$$

are also not members of the class Σ.

For a brief history of functions in the class Σ, see [22] (see also [4], [14], [18] and [25]). In fact, judging by the remarkable flood of papers on the subject (see, for example, [5–7, 9–12, 15–17, 19–21, 23, 26, 27, 29, 30]), the recent pioneering work of Srivastava *et al.*[22] appears to have revived the study of analytic and bi-univalent functions in recent years (see also [3], [13] and [24]).

The object of the present paper is to introduce a new subclass of the function class Σ and use the Faber polynomial expansion techniques to derive bounds for the general Taylor-Maclaurin coefficients $|a_n|$ for the functions in this class. We also obtain estimates for the first two coefficients $|a_2|$ and $|a_3|$ of these functions.

2. Bounds Derivable by the Faber Polynomial Expansion Techniques

We begin by introducing the function class $\mathcal{N}_\Sigma^{(\alpha,\lambda)}$ by means of the following definition.

Definition. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{N}_\Sigma^{(\alpha,\lambda)}$ $(0 \leqq \alpha < 1;\ \lambda \geqq 0)$ if the following conditions are satisfied:

$$f \in \Sigma \qquad \text{and} \qquad \Re\left\{f'(z) + \lambda z f''(z)\right\} > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1;\ \lambda \geqq 0). \tag{2.1}$$

By using the Faber polynomial expansions of functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see [1] and [2]; see also [12]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}\left(a_2, a_3, \cdots, a_n\right) w^n. \tag{2.2}$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ &\quad + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &\quad + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] \\ &\quad + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6}\left[a_6 + (-2n+5)a_3 a_4\right] + \sum_{j \geqq 7} a_2^{n-j} V_j, \end{aligned}$$

where such expressions as (for example) $(-n)!$ are to be interpreted *symbolically* by

$$(-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2)\cdots \qquad \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \ \ (\mathbb{N} := \{1,2,3,\cdots\})\right) \tag{2.3}$$

and V_j $(7 \leqq j \leqq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$ (see, for details, [2]). In particular, the first three terms of K_{n-1}^{-n} are given below:

$$K_1^{-2} = -2a_2,$$

$$K_2^{-3} = 3\left(2a_2^2 - a_3\right)$$

and

$$K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

In general, an expansion of K_n^p is given by (see, for details, [1])

$$K_n^p = pa_n + \frac{p(p-1)}{2}D_n^2 + \frac{p!}{(p-3)!3!}D_n^3 + \cdots + \frac{p!}{(p-n)!n!}D_n^n \qquad (p \in \mathbb{Z})$$

where

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\} \qquad \text{and} \qquad D_n^p = D_n^p(a_2, a_3, \cdots)$$

and, alternatively, by (see, for details, [28])

$$D_n^m(a_1, a_2, \cdots, a_n) = \sum \left(\frac{m!}{\mu_1! \cdots \mu_n!}\right) a_1^{\mu_1} \cdots a_n^{\mu_n},$$

where $a_1 = 1$ and the sum is taken over all nonnegative integers $\mu_1, \cdots, \mu_n$ satisfying the following conditions:

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_n = m \\ \\ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \end{cases}$$

It is clear that

$$D_n^n(a_1, a_2, \cdots, a_n) = a_1^n.$$

Our first main result is given by Theorem 1 below.

Theorem 1. *Let f given by* (1.1) *be in the class* $\mathcal{N}_\Sigma^{\alpha,\lambda}$ $(0 \leqq \alpha < 1$ *and* $\lambda \geqq 0)$. *If* $a_k = 0$ *for* $2 \leqq k \leqq n-1$, *then*

$$|a_n| \leqq \frac{2(1-\alpha)}{n[1+\lambda(n-1)]} \qquad (n \in \mathbb{N} \setminus \{1,2\}). \tag{2.4}$$

Proof. For analytic functions f of the form (1.1), we have

$$f'(z) + \lambda z f''(z) = 1 + \sum_{n=2}^{\infty} [1 + \lambda(n-1)] n a_n z^{n-1} \tag{2.5}$$

and, for its inverse map $g = f^{-1}$, it is seen that

$$g'(w) + \lambda w g''(w) = 1 + \sum_{n=2}^{\infty} [1 + \lambda(n-1)] n b_n w^{n-1}$$
$$= 1 + \sum_{n=2}^{\infty} [1 + \lambda(n-1)] K_{n-1}^{-n}(a_2, a_3, \cdots, a_n)\, w^{n-1}. \tag{2.6}$$

On the other hand, since $f \in \mathcal{N}_{\Sigma}^{\alpha,\lambda}$ and $g = f^{-1} \in \mathcal{N}_{\Sigma}^{\alpha,\lambda}$, by definition, there exist two positive real-part functions

$$\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

and

$$\mathfrak{q}(w) = 1 + \sum_{n=1}^{\infty} d_n w^n,$$

where

$$\Re\Big(\mathfrak{p}(z)\Big) > 0 \qquad \text{and} \qquad \Re\Big(\mathfrak{q}(w)\Big) > 0 \qquad (z, w \in \mathbb{U}),$$

so that

$$f'(z) + \lambda z f''(z) = \alpha + (1-\alpha)\mathfrak{p}(z)$$
$$= 1 + (1-\alpha) \sum_{n=1}^{\infty} K_n^1(c_1, c_2, \cdots, c_n)\, z^n \tag{2.7}$$

and

$$g'(w) + \lambda w g''(w) = \alpha + (1-\alpha)\mathfrak{q}(w)$$
$$= 1 + (1-\alpha) \sum_{n=1}^{\infty} K_n^1(d_1, d_2, \cdots, d_n)\, w^n. \tag{2.8}$$

Thus, upon comparing the corresponding coefficients in (2.5) and (2.7), we get

$$[1 + \lambda(n-1)] n a_n = (1-\alpha) K_{n-1}^1(c_1, c_2, \cdots, c_{n-1}). \tag{2.9}$$

Similarly, by using (2.6) and (2.8), we find that

$$[1 + \lambda(n-1)] K_{n-1}^{-n}(a_1, a_2, \cdots, a_n) = (1-\alpha) K_{n-1}^1(d_1, d_2, \cdots, d_{n-1}). \tag{2.10}$$

We note that, for $a_k = 0 \ (2 \leqq k \leqq n-1)$, we have

$$b_n = -a_n$$

and so

$$[1 + \lambda(n-1)] n a_n = (1-\alpha) c_{n-1} \tag{2.11}$$

and

$$-[1 + \lambda(n-1)] n a_n = (1-\alpha) d_{n-1}. \tag{2.12}$$

Thus, according to the Carathéodory Lemma (see [8]), we also observe that

$$|c_n| \leqq 2 \qquad \text{and} \qquad |d_n| \leqq 2 \qquad (n \in \mathbb{N}).$$

Now, taking the moduli in (2.11) and (2.12) and applying the Carathéodory Lemma, we obtain

$$|a_n| \leqq \frac{(1-\alpha)\,|c_{n-1}|}{n[1+\lambda(n-1)]} = \frac{(1-\alpha)\,|d_{n-1}|}{n[1+\lambda(n-1)]} \leqq \frac{2(1-\alpha)}{n[1+\lambda(n-1)]}, \tag{2.13}$$

which evidently completes the proof of Theorem 1. □

3. Estimates for the Initial Coefficients a_2 and a_3

In this section, we choose to relax the coefficient restrictions imposed in Theorem 1 and derive the resulting estimates for the initial coefficients a_2 and a_3 of functions $f \in \mathcal{N}_\Sigma^{\alpha,\lambda}$ given by the Taylor-Maclaurin series expansion (1.1). We first state the following theorem.

Theorem 2. *Let f given by* (1.1) *be in the class* $\mathcal{N}_\Sigma^{\alpha,\lambda}$ $(0 \leqq \alpha < 1$ *and* $\lambda \geqq 0)$. *Then*

$$|a_2| \leqq \begin{cases} \sqrt{\dfrac{2(1-\alpha)}{3(1+2\lambda)}}, & 0 \leqq \alpha < \dfrac{1+2\lambda-2\lambda^2}{3(1+2\lambda)} \\[2ex] \dfrac{1-\alpha}{1+\lambda}, & \dfrac{1+2\lambda-2\lambda^2}{3(1+2\lambda)} \leqq \alpha < 1 \end{cases} \tag{3.1}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)}{3(1+2\lambda)}. \tag{3.2}$$

Proof. If we set $n = 2$ by and $n = 3$ in (2.9) and (2.10), respectively, we obtain

$$2(1+\lambda)a_2 = (1-\alpha)c_1, \tag{3.3}$$

$$3(1+2\lambda)a_3 = (1-\alpha)c_2, \tag{3.4}$$

$$-2(1+\lambda)a_2 = (1-\alpha)d_1 \tag{3.5}$$

and

$$3(1+2\lambda)(2a_2^2 - a_3) = (1-\alpha)d_2. \tag{3.6}$$

Upon dividing both sides of (3.3) or (3.5) by $2(1+\lambda)$, if we take their moduli and apply the Carathéodory Lemma, we find that

$$|a_2| \leqq \frac{(1-\alpha)\,|c_1|}{2(1+\lambda)} = \frac{(1-\alpha)\,|d_1|}{2(1+\lambda)} \leqq \frac{1-\alpha}{1+\lambda}. \tag{3.7}$$

Now, by adding (3.4) to (3.6), we have

$$6(1+2\lambda)a_2^2 = (1-\alpha)(c_2+d_2), \tag{3.8}$$

that is,

$$a_2^2 = \frac{(1-\alpha)(c_2+d_2)}{6(1+2\lambda)}. \tag{3.9}$$

Another application of the Carathéodory Lemma followed by taking the square roots in this last equation (3.9) yields

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)}{3(1+2\lambda)}}, \tag{3.10}$$

which proves the first assertion (3.1) of Theorem 2.

Next, for

$$\frac{1+2\lambda-2\lambda^2}{3(1+2\lambda)} \leqq \alpha < 1,$$

we note that

$$\frac{1-\alpha}{1+\lambda} \leqq \sqrt{\frac{2(1-\alpha)}{3(1+2\lambda)}}. \tag{3.11}$$

Thus, upon dividing both sides of (3.4) by $3(1+2\lambda)$, if we take the modulus of each side and apply the Carathéodory Lemma once again, we get

$$|a_3| \leqq \frac{2(1-\alpha)}{3(1+2\lambda)}, \tag{3.12}$$

which completes the proof of the second assertion (3.2) of Theorem 2. □

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