Filomat 29:8 (2015), 1795–1809
DOI 10.2298/FIL1508795K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Rank One Perturbations of Complex Symmetric Operators

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Abstract. In this paper we study the decomposability of rank one perturbations of complex symmetric operators $R = T + u \otimes v$. Also we investigate some conditions for which R satisfies a-Weyl's theorem. Finally, we characterize some conditions for R to be hyponormal. As consequences, we provide several cases for such operators.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ and let $\mathcal{K}(\mathcal{H})$ be the ideal of all compact operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\rho(T)$, $\sigma(T)$, $\sigma_{su}(T)$, $\sigma_a(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$, $\sigma_{se}(T)$, and $\sigma_{es}(T)$ for the resolvent set, the spectrum, the surjective spectrum, the approximate point spectrum, the essential spectrum, the left essential spectrum, the right essential spectrum, the semi-regular spectrum, and the essentially semi-regular spectrum of T, respectively.

A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. For any conjugation C, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [9] for more details). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. In this case, we say that T is complex symmetric with conjugation C. This concept is due to the fact that T is a complex symmetric operator if and only if it is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an l^2-space of the appropriate dimension (see [9]). All normal operators, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators, and some Volterra integration operators are included the class of complex symmetric operators. We refer the reader to [9]-[11] for more details.

If u and v are nonzero vectors in $\mathcal{H}$, we write $u \otimes v$ for the operator of *rank one* defined by

$$(u \otimes v)x = \langle x,\ v\rangle u,\ x \in \mathcal{H},$$

where $\langle\ ,\ \rangle$ denotes the inner product of the Hilbert space $\mathcal{H}$.

2010 *Mathematics Subject Classification*. Primary 47A11, 47A53, Secondary 47B38, 47B35.
Keywords. Rank one perturbation; Complex symmetric operators; Decomposable; a-Weyl's theorem; Hyponormal.
Received: 20 January 2014; Accepted: 11 February 2015
Communicated by Dragan Djordjevic
The research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2012R1A2A2A02008590). The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012R1A1A3006841)
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Let $\{e_n\}_{n=0}^{\infty}$ denote an orthonormal basis for $\mathcal{H}$ which will remain fixed throughout this paper. *Throughout the paper we suppose that u and v are nonzero vectors in $\mathcal{H}$* and their expansions with respect to the orthonormal basis $\{e_n\}_{n=0}^{\infty}$ are

$$u = \sum_{n=0}^{\infty} a_n e_n \quad \text{and} \quad v = \sum_{n=0}^{\infty} b_n e_n$$

where a_n and b_n are nonzero coefficients for all nonnegative integer n. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if T and T^* commute. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$ where T^* is the adjoint of T. It is known that the class of hyponormal operators is a larger class containing normal operators.

We say that an operator $R \in \mathcal{L}(\mathcal{H})$ is *a rank one perturbation of an operator* $T \in \mathcal{L}(\mathcal{H})$ if there exist vectors u and v (defined above) in $\mathcal{H}$ such that $R = T + u \otimes v$. In 2001, E. Ionascu has studied several properties of rank one perturbations of diagonal operators (see [15]). It was shown from [9] that every normal operator is complex symmetric. Moreover, S. R. Garcia and W. R. Wogen [11] proved that the rank one perturbations of normal operator is also complex symmetric. In the model space $\mathcal{K}_u$, the compressed shift and Clark unitary operator have the forms which are the rank one perturbations of complex symmetric operators (see [6], [9], [23], and [21] for more details). In view of this, it is natural to consider the rank one perturbations of complex symmetric operators.

In this paper, we study the decomposability of rank one perturbations of complex symmetric operators $R = T + u \otimes v$. Also we investigate some conditions for which R satisfies a-Weyl's theorem. Finally, we characterize some conditions for R to be hyponormal. As consequences, we provide several cases for such operators.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T - \lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for a vector $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{H})$ by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on G such that $(T - \lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, we get that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\mathcal{H} = \mathcal{X} + \mathcal{Y}, \sigma(T|_{\mathcal{X}}) \subset \overline{U}, \text{ and } \sigma(T|_{\mathcal{Y}}) \subset \overline{V}.$$

It is well known that

Bishop's property (β) $\Rightarrow$ Dunford's property (C) $\Rightarrow$ SVEP.

Any of the converse implications does not hold, in general (see [19] for more details).

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if T has closed range and $\mathit{dim}\,\mathit{ker}(T) < \infty$, and $T \in \mathcal{L}(\mathcal{H})$ is called *lower semi-Fredholm* if T has closed range and $\mathit{dim}(\mathcal{H}/\mathit{ran}(T)) < \infty$. When T is upper semi-Fredholm or lower semi-Fredholm, T is said to be *semi-Fredholm*. The *index of a semi-Fredholm operator* $T \in \mathcal{L}(\mathcal{H})$, denoted $\mathit{ind}(T)$, is given by

$$\mathit{ind}(T) = \mathit{dim}\,\mathit{ker}(T) - \mathit{dim}(\mathcal{H}/\mathit{ran}(T))$$

and this value is an integer or $\pm\infty$. Also an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Fredholm* if it is both upper and lower semi-Fredholm. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Weyl* if it is Fredholm of index zero. If there is a nonnegative integer m such that $ker(T^m) = ker(T^{m+1})$, then T is said to have *finite ascent*. If there is a nonnegative integer n satisfying $ran(T^n) = ran(T^{n+1})$, then T is said to have *finite descent*. We say that $T \in \mathcal{L}(\mathcal{H})$ is *Browder* if it has finite ascent and finite descent. We define the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.$$

It is evident that $\sigma_e(T) \subset \sigma_w(T) \subset \sigma_b(T)$. We say that *Weyl's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T), \text{ or equivalently, } \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < dim\, ker(T - \lambda) < \infty\}$ and $iso\Delta$ denotes the set of all isolated points of Δ. We say that *Browder's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma_b(T) = \sigma_w(T)$.

We define the following subsets of the essential spectrum of an operator $T \in \mathcal{L}(\mathcal{H})$:

$$\sigma_{ea}(T) := \cap\{\sigma_a(T + K) : K \in \mathcal{K}(\mathcal{H})\}$$

is the *essential approximate point spectrum*, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \in \mathcal{K}(\mathcal{H})\}$$

is the *Browder essential approximate point spectrum*. For $T \in \mathcal{L}(\mathcal{H})$, we say that *a-Weyl's theorem holds for T* if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$$

where $\pi_{00}^a(T) = \{\lambda \in iso\, \sigma_a(T) : 0 < dimker(T - \lambda) < \infty\}$, while *a-Browder's theorem holds for T* if $\sigma_{ea}(T) = \sigma_{ab}(T)$. It is known that

$$\begin{array}{ccc} \text{Browder's theorem} & \Longleftarrow & a\text{-Browder's theorem} \\ \Uparrow & & \Uparrow \\ \text{Weyl's theorem} & \Longleftarrow & a\text{-Weyl's theorem.} \end{array}$$

We refer the reader to [1], [13], [8], and [16] for more details.

Let L^2 be the Lebesque (Hilbert) space on the unit circle, and let L^∞ be the Banach space of all essentially bounded functions on $\partial\mathbb{D}$. The Hardy-Hilbert space, denoted by H^2, consists of all analytic functions $f(z) = \sum_{n=0}^\infty a_n z^n$ on $\mathbb{D}$ such that $\|f\|_2 := (\sum_{n=0}^\infty |a_n|^2)^{\frac{1}{2}} < \infty$, or equivalently, with $\sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\right) < \infty$.

For $\varphi \in L^\infty$, the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by

$$T_\varphi f = P(\varphi f)$$

for $f \in H^2$ where P denotes the orthogonal projection of L^2 onto H^2. For $u \in H^2$ with power series representation $u(z) = \sum_{n=0}^\infty a_n z^n$, it is well known that $\lim_{r\to 1^-} u(rz)$ exists for almost every $z \in \partial\mathbb{D}$, and so one defines $\widetilde{u}(e^{i\theta}) := \sum_{n=0}^\infty a_n e^{in\theta}$ for almost every $\theta \in [0, 2\pi)$. A function $u \in H^2$ is called *inner* if $|\widetilde{u}(e^{i\theta})| = 1$ for almost every $\theta \in [0, 2\pi)$. For a nonconstant inner function u, the *model space* is given by $\mathcal{K}_u := H^2 \ominus uH^2$ (see [9] and [23] for more details). For an inner function u and $\varphi \in L^2$, the *truncated Toeplitz operator* $A_\varphi^u : \mathcal{K}_u \to \mathcal{K}_u$ is the compressed operator of T_φ to the space $\mathcal{K}_u$, that is,

$$A_\varphi^u := P_u T_\varphi P_u$$

where P_u denotes the orthogonal projection of L^2 onto $\mathcal{K}_u$. It is evident that A_φ^u is bounded on $\mathcal{K}_u$ whenever $\varphi \in L^\infty$. We denote the truncated Toeplitz operator on $\mathcal{K}_u$ corresponding to the unilateral shift T_z merely by S_u, i.e., $S_u := A_z^u$. For $\lambda \in \mathbb{D}$, let K_λ denote the *reproducing kernel* for H^2, i.e., K_λ has the property that $\langle f, K_\lambda\rangle = f(\lambda)$ for all $f \in H^2$ where $\langle\cdot,\cdot\rangle$ stands for the inner product inducing the canonical norm $\|\cdot\|_2$ on H^2. In fact, it is easy to check that $K_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$ for any $\lambda \in \mathbb{D}$. If defining $k_\lambda^u := P_u k_\lambda$, since $P_u = P - M_u P M_{\overline{u}}$, we have $k_\lambda^u = k_\lambda - \overline{u(\lambda)} u k_\lambda$, i.e.,

$$k_\lambda^u(z) = P_u\Big(\frac{1}{1-\overline{\lambda}z}\Big) = P_u\Big(\frac{\overline{u(\lambda)}u(z)}{1-\overline{\lambda}z} + \frac{1-\overline{u(\lambda)}u(z)}{1-\overline{\lambda}z}\Big) = \frac{1-\overline{u(\lambda)}u(z)}{1-\overline{\lambda}z}$$

and we call k_λ^u the *reproducing kernel* for $\mathcal{K}_u$. Note that the kernel function k_λ^u belongs to $\mathcal{K}_u^\infty$ which is dense in $\mathcal{K}_u$. Define an antilinear operator C on $\mathcal{K}_u$ by $Cf = \overline{zf}u$. It is known from [9] that $\overline{zf}u \in \mathcal{K}_u$ for all $f \in \mathcal{K}_u$ and C is a conjugation operator on $\mathcal{K}_u$. It is easy to see that

$$\widetilde{k_\lambda^u}(z) := (Ck_\lambda^u)(z) = \frac{u(z)-u(\lambda)}{z-\lambda}$$

for $\lambda \in \mathbb{D}$ (see [9] and [23]). For $\alpha \in \mathbb{D}$, the *compressed shifts* are defined by

$$U_\alpha := A_z^u + \frac{\alpha}{1-\alpha u(0)}(k_0^u \otimes Ck_0^u).$$

If $\alpha \in \mathbb{D}$, then U_α is unitarily equivalent to A_z^u and is a completely non-unitary contraction which is related to the result of Sz.-Nagy and C. Foias (see [21] for more details). If $\alpha \in \partial\mathbb{D}$, then U_α is the Clark unitary operators defined in [6].

3. Main Results

In this section, we consider the decomposability of rank one perturbations of complex symmetric operators. For example, let S be the unilateral shift in $\mathcal{L}(\mathcal{H})$ and let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis of $\mathcal{H}$. If

$$T = \begin{pmatrix} S & e_0\otimes e_0 \\ 0 & S^* \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & -e_0\otimes e_0 \\ 0 & 0 \end{pmatrix}$$

are in $\mathcal{L}(\mathcal{H}\oplus\mathcal{H})$, then we have $T^*T = I = TT^*$. Therefore, T is unitary and P is a rank one operator. Furthermore, in this case, we know that T and P are decomposable by [19]. However, $T + P = S \oplus S^*$ is not decomposable even if $S \oplus S^*$ is complex symmetric. So, it is natural to consider the decomposability of rank one perturbations of complex symmetric operators $R = T + u \otimes v$. We start our program with the following lemma.

Lemma 3.1. If $T \in \mathcal{L}(\mathcal{H})$ is complex symmetric with the conjugation C and $\lambda \in \mathbb{C}$ is a nonzero eigenvalue of T corresponding to an eigenvector Cv for any vector $v \in \mathcal{H}$, then $R = T + u \otimes v$ is complex symmetric with the same conjugation C and $T(u\otimes v) = (u\otimes v)T$ where $u = \lambda Cv$ with a nonzero complex number λ.

Proof. If $R = T + u\otimes v$, then $R^* = T^* + v\otimes u$. Since $T^* = CTC$, it follows that the relation $CR^*C = R$ is equivalent to $u \otimes v = Cv \otimes Cu$. If $u = \lambda Cv$ with a nonzero complex number λ, then we have

$$\begin{aligned} u\otimes v - Cv\otimes Cu &= \lambda Cv\otimes v - Cv\otimes C(\lambda Cv) \\ &= \lambda Cv\otimes v - Cv\otimes \overline{\lambda}C^2 v = 0. \end{aligned}$$

Hence R is complex symmetric with the conjugation C.

Since T is complex symmetric, λ is a nonzero eigenvalue of T with respect to Cv, and $u = \lambda Cv$, we thus obtain

$$T(u\otimes v) - (u\otimes v)T = Tu\otimes v - u\otimes T^*v = Tu\otimes v - u\otimes CTCv$$

$$\begin{aligned} &= Tu \otimes v - u \otimes C(\lambda Cv) = Tu \otimes v - u \otimes \overline{\lambda} C^2 v \\ &= T\lambda Cv \otimes v - \lambda^2 Cv \otimes v = 0. \end{aligned}$$

Then we have $T(u \otimes v) = (u \otimes v)T$. $\square$

We next consider the decomposability of the rank one perturbations of complex symmetric operators.

Theorem 3.2. Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric with the conjugation C, let $\lambda \in \mathbb{C}$ be a nonzero eigenvalue of T corresponding to an eigenvector Cv for any vector $v \in \mathcal{H}$, and let $u = \lambda Cv$ for a nonzero complex number λ. Then the following statements are equivalent:
(i) $T + u \otimes v$ is decomposable.
(ii) T has the property (β).
(iii) T^* has the property (β).
(iv) $T^* + v \otimes u$ is decomposable.

Proof. Let $R = T + u \otimes v$ and let $F = u \otimes v$. Then $F^2 - \langle u, v\rangle F = 0$. Since $F = u \otimes v$ is an algebraic operator of order 2, there exists a non-constant polynomial $p(z) = z(z - \langle u, v\rangle)$ such that $p(F) = 0$.

(i) $\Leftrightarrow$ (ii) Suppose that T has the property (β). Let D be an open set in $\mathbb{C}$ and let $f_n : D \to \mathcal{H}$ be a sequence of analytic functions such that

$$\lim_{n\to\infty} \|(R - z)f_n(z)\|_K = 0 \tag{1}$$

for every compact set K in D, where $\|f\|_K$ denotes $\sup_{z\in K} \|f(z)\|$ for an $\mathcal{H}$-valued function $f(z)$. Then we get from (1) that

$$\lim_{n\to\infty} \|(T - z)f_n(z) + Ff_n(z)\|_K = 0. \tag{2}$$

Since $p(F) = 0$ and $TF = FT$ by Lemma 3.1, it follows that

$$\lim_{n\to\infty} \|(T + \langle u, v\rangle - z)Ff_n(z)\|_K = 0.$$

Since $T + \langle u, v\rangle$ has the property (β), we have

$$\lim_{n\to\infty} \|Ff_n(z)\|_K = 0. \tag{3}$$

The equations (2) and (3) imply that

$$\lim_{n\to\infty} \|(T - z)f_n(z)\|_K = 0.$$

Since T has the property (β), we get that $\lim_{n\to\infty} \|f_n(z)\|_K = 0$. Hence R has the property (β). Since T^* and $(u\otimes v)^* = v\otimes u$ have the property (β) by [16, Theorem 2.1] with $T^* v\otimes u = v\otimes T^* u$, we know from the previous argument that $R^* = T^* + v \otimes u$ also has the property (β). Hence we conclude from [19, Theorems 1.2.29 and 2.5.5] that R is decomposable.

Conversely, assume that R is decomposable. Let D be an open set in $\mathbb{C}$ and let $f_n : D \to \mathcal{H}$ be a sequence of analytic functions such that

$$\lim_{n\to\infty} \|(T - z)f_n(z)\|_K = 0 \tag{4}$$

for every compact set K in D, where $\|f\|_K$ denotes $\sup_{z\in K} \|f(z)\|$ for an $\mathcal{H}$-valued function $f(z)$. Then we get from (4) that

$$\lim_{n\to\infty} \|(R - z)f_n(z) - Ff_n(z)\|_K = 0. \tag{5}$$

Since $p(F) = 0$ and $TF = FT$ by Lemma 3.1, it follows that

$$\lim_{n\to\infty} \|(R - \langle u, v\rangle - z)Ff_n(z)\|_K = 0.$$

Since $R - \langle u, v\rangle$ has the property (β), we obtain

$$\lim_{n\to\infty} \|Ff_n(z)\|_K = 0. \tag{6}$$

The relations (5) and (6) imply that

$$\lim_{n\to\infty} \|(R-z)f_n(z)\|_K = 0.$$

Since R has the property (β), we get that $\lim_{n\to\infty} \|f_n(z)\|_K = 0$. Hence T has the property (β). So we complete the proof.

(ii) $\Leftrightarrow$ (iii) Since T is complex symmetric, the proof follows from [16].

(iii) $\Leftrightarrow$ (iv) We get this result by using a similar method of (i) $\Leftrightarrow$ (ii). $\square$

From Theorem 3.2, we provide several examples of the rank one perturbations of complex symmetric operators which is decomposable.

Example 3.3. If N is a normal operator on $\mathcal{H}$, then N is a complex symmetric operator by [9]. So there exists a conjugation operator C such that $CNC = N^*$. If $NCv = \lambda Cv$ for some nonzero complex number λ, then $N + \lambda Cv \otimes v$ is decomposable from Theorem 3.2.

Example 3.4. Let T be a compact and complex symmetric operator with conjugation C and let u_k be any element of an orthonormal basis $\{u_n\}_{n=0}^{\infty}$ of $\mathcal{H}$. Then we know from [4, Page 33] that T has the property (β). If λ is a nonzero eigenvalue of T corresponding to Cu_k, then $T + \lambda Cu_k \otimes u_k$ is decomposable by Theorem 3.2.

Example 3.5. Let T be the multiplication operator on a Lebesgue space $L^2(\mu)$ where μ is a planar positive Borel measure with compact support. Then it is clear that T is normal and it is complex symmetric with respect to the conjugation $Cf = \overline{f}$ for all $f \in L^2(\mu)$ by [9]. Consequently, if $\lambda \in \mathbb{C}$ is a nonzero eigenvalue of T with respect to $\overline{f}$, we get that $T + \lambda \overline{f} \otimes f$ is decomposable by Theorem 3.2.

Example 3.6. Let u be the inner function, let $\varphi \in L^2$, and let $\mathcal{T}_u$ be the set of bounded truncated Toeplitz operators on $\mathcal{K}_u$. Assume that $A_\varphi^u \in \mathcal{T}_u$ satisfies one of the following assertions:
(i) There is $\alpha \in \partial\mathbb{D}$ so that A_φ^u belongs to Sedlock classes $\mathcal{B}_u^\alpha$ where $\mathcal{B}_u^\alpha$ is the collection of A_φ^u in $\mathcal{T}_u$ with $\varphi = \alpha S_u \overline{C\varphi} + c$ for $\varphi \in \mathcal{K}_u$ and $c \in \mathbb{C}$ (see [25]).
(ii) A_φ^u is a linear combination of a self-adjoint truncated Toeplitz operator and the identity.
Then we know that A_φ^u is normal from [5, Theorem 6.2]. Hence we conclude that $A_\varphi^u + \lambda Ck_0^u \otimes k_0^u$ is decomposable where λ is a nonzero eigenvalue of A_φ^u corresponding to Ck_0^u by Theorem 3.2 and Example 3.3.

Even if T is not a complex symmetric operator in Theorem 3.2, the commutativity preserves the property (β) under the rank one perturbation.

Theorem 3.7. Assume that $T \in \mathcal{L}(\mathcal{H})$ is not a complex symmetric operator. If T commutes with $u \otimes v$, then $T + u \otimes v$ has the property (β) if and only if T does.

Proof. Let $R = T + u \otimes v$. Suppose that T has the property (β). Let D be an open set in $\mathbb{C}$ and let $f_n : D \to \mathcal{H}$ be a sequence of analytic functions such that

$$\lim_{n\to\infty} \|(R-z)f_n(z)\|_K = 0 \tag{7}$$

for every compact set K in D, where $\|f\|_K$ denotes $\sup_{z\in K} \|f(z)\|$ for an $\mathcal{H}$-valued function $f(z)$. Set $F = u \otimes v$. Then we obtain from (7) that

$$\lim_{n\to\infty} \|(T-z)f_n(z) + Ff_n(z)\|_K = 0. \tag{8}$$

Since $TF = FT$ and $F^2 - \langle u, v\rangle F = 0$, it follows that

$$\lim_{n\to\infty} \|(T + \langle u, v\rangle - z)Ff_n(z)\|_K = 0.$$

Since $T + \langle u, v\rangle$ has the property (β), it ensures that

$$\lim_{n\to\infty} \|Ff_n(z)\|_K = 0. \tag{9}$$

The relations (8) and (9) give that

$$\lim_{n\to\infty} \|(T - z)f_n(z)\|_K = 0.$$

Since T has the property (β), we get that $\lim_{n\to\infty} \|f_n(z)\|_K = 0$. Hence R has the property (β).

Conversely, assume that R has the property (β). Let D be an open set in $\mathbb{C}$ and let $f_n : D \to \mathcal{H}$ be a sequence of analytic functions such that

$$\lim_{n\to\infty} \|(T - z)f_n(z)\|_K = 0 \tag{10}$$

for every compact set K in D, where $\|f\|_K$ denotes $\sup_{z\in K} \|f(z)\|$ for an $\mathcal{H}$-valued function $f(z)$. Then we get from (10) that

$$\lim_{n\to\infty} \|(R - z)f_n(z) - Ff_n(z)\|_K = 0. \tag{11}$$

Since $TF = FT$ and $F^2 - \langle u, v\rangle F = 0$, it follows that

$$\lim_{n\to\infty} \|(R - \langle u, v\rangle - z)Ff_n(z)\|_K = 0.$$

Since $R - \langle u, v\rangle$ has the property (β), we get

$$\lim_{n\to\infty} \|Ff_n(z)\|_K = 0. \tag{12}$$

The equations (11) and (12) yield that

$$\lim_{n\to\infty} \|(R - z)f_n(z)\|_K = 0.$$

Since R has the property (β), we obtain that $\lim_{n\to\infty} \|f_n(z)\|_K = 0$. Hence T has the property (β). □

Corollary 3.8. If $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator commuting with $u \otimes v$, then $T + u \otimes v$ has the property (β).

Proof. If T is normal, then this result holds by Theorem 3.2. If T is nonnormal and hyponormal, then T is not complex symmetric by [26, Lemma 3.1] and has the property (β) Hence the proof follows from Theorem 3.7. □

In the following theorem, we study the single-valued extension property of the rank one perturbations of complex symmetric operators.

Theorem 3.9. If T satisfies the hypotheses as in Theorem 3.2, then the following statements are equivalent:
(i) $T + u \otimes v$ has the single-valued extension property.
(ii) T has the single-valued extension property.
(iii) $T^* + v \otimes u$ has the single-valued extension property.
(iv) T^* has the single-valued extension property.

Proof. Let $R = T + u \otimes v$ with the hypotheses as in Theorem 3.2.
(i) $\Leftrightarrow$ (ii) Suppose that T has the single-valued extension property. Let G be an open set in $\mathbb{C}$ and let $f : G \to \mathcal{H}$ be an analytic function such that $(R - z)f(z) \equiv 0$ on G. This gives that $(T - z)f(z) + Ff(z) = 0$. Since $p(F) = 0$ and $TF = FT$ by Lemma 3.1, it follows that $(T + \langle u, v\rangle - z)Ff(z) = 0$. Since $T + \langle u, v\rangle$ has the single-valued extension property, we have $Ff(z) = 0$ and so $(T - z)f(z) = 0$. Since T has the single-valued extension property, we get that $f(z) = 0$. Hence R has the single-valued extension property.

Conversely, assume that R has the single-valued extension property. Let G be an open set in $\mathbb{C}$ and let $f : G \to \mathcal{H}$ be an analytic function such that $(T - z)f(z) \equiv 0$ on G. Then we have $(R - z)f(z) - Ff(z) = 0$. Since $p(F) = 0$ and $TF = FT$ by Lemma 3.1, it follows that $(R - \langle u, v\rangle - z)Ff(z) = 0$. Since $R - \langle u, v\rangle$ has the single-valued extension property, we have $Ff(z) = 0$. From this, we obtain $(R - z)f(z) = 0$. Since R has the single-valued extension property, we get that $f(z) = 0$. Hence T has the single-valued extension property.

(ii) $\Leftrightarrow$ (iv) Since T is complex symmetric, the proof follows from [17, Lemma 3.5].

(iii) $\Leftrightarrow$ (iv) We get this result by using a similar method of (i) $\Leftrightarrow$ (ii). □

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *quasitriangular* if T can be written as sum $T = T_0 + K$, where T_0 is a triangular operator (i.e., there exists an orthonormal basis for $\mathcal{H}$ with respect to which the matrix for T_0 has upper triangular form) and $K \in \mathcal{K}(\mathcal{H})$. We say that $T \in \mathcal{L}(\mathcal{H})$ is *biquasitriangular* if both T and T^* are quasitriangular (see [22] for more details). For an operator $T \in \mathcal{L}(\mathcal{H})$, the *quasinilpotent part* of T is defined by

$$H_0(T) := \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Then $H_0(T)$ is a linear (not necessarily closed) subspace of $\mathcal{H}$. We remark from [2] that if T has the single-valued extension property, then

$$H_0(T - \lambda) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\} = H_T(\{\lambda\})$$

for all $\lambda \in \mathbb{C}$. It is well known from [1] and [2] that if $H_0(T-\lambda) = \{0\}$ for all $\lambda \in \mathbb{C}$, then T has the single-valued extension property. The *analytical core* $K(T)$ of T is the set of all $x \in \mathcal{H}$ with the property that there is a sequence $\{u_n\} \subset \mathcal{H}$ and a constant $\delta > 0$ such that $x = u_0$, $Tu_{n+1} = u_n$, and $\|u_n\| \le \delta^n \|x\|$ for every integer $n \ge 0$ (see [1] for more details).

Corollary 3.10. Let $R = T + u \otimes v$ be an operator in $\mathcal{L}(\mathcal{H})$ with the same hypotheses as in Theorem 3.2. If T has the single-valued extension property, then the following properties hold:
(i) $\sigma(R) = \sigma_{su}(R) = \sigma_a(R) = \sigma_{se}(R)$.
(ii) $\sigma_{es}(R) = \sigma_b(R) = \sigma_w(R) = \sigma_e(R)$.
(iii) $\lambda_0 \in \sigma_e(R)$ if and only if λ_0 is a cluster point of $\sigma(R)$ or $\lambda_0 \in iso\sigma(R)$ for which $K(\lambda_0 - R)$ is infinite codimensional, or $H_0(\lambda_0 - R)$ is infinite codimensional.
(iv) $H_0(R - \lambda) = H_R(\{\lambda\})$ and $H_{R^*}(\{\lambda\}) = H_0(R^* - \lambda)$ for all $\lambda \in \mathbb{C}$.
(v) R is biquasitriangular.

Proof. Since T has the single-valued extension property, the operator $R = T + u \otimes v$ and R^* have the single-valued extension property from Theorem 3.9. Hence the proof follows from [1, Corollaries 2.45 and 3.53], [2], [20], and [22]. □

As some applications of Theorems 3.2 and 3.9, we get the following corollary.

Corollary 3.11. Let $R = T + u \otimes v$ be an operator in $\mathcal{L}(\mathcal{H})$ with the same hypotheses as in Theorem 3.2. If T has the property (β), then the following assertions hold:
(i) R and R^* have the property (β), Dunford's property (C), and the single-valued extension property.
(ii) If $\sigma(R)$ has nonempty interior, then R has a nontrivial invariant subspace.
(iii) $H_R(F)$ is a hyperinvariant subspace for R.
(iv) If f is any function analytic on a neighborhood of $\sigma(R)$, then both Weyl's and Browder's theorems hold for $f(R)$ and

$$\sigma_w(f(R)) = \sigma_b(f(R)) = f(\sigma_w(R)) = f(\sigma_b(R)).$$

Proof. (i) It is well known from [19] that R is decomposable if and only if R and R^* have the property (β). Hence we complete our proof.

(ii) Since R has the property (β) and $\sigma(R)$ has nonempty interior, the proof follows from [7].

(iii) Since R is decomposable from Theorem 3.2, it ensures that $H_R(F)$ is a spectral maximal space of R by [4]. Hence the proof follows from [4].

(iv) Since $f(R)$ is decomposable from [19], it follows that $f(R)$ is clearly subscalar. Hence Weyl's theorem holds for $f(R)$ by [1]. Moreover, since $f(R)$ has the single-valued extension property, Browder's theorem holds for $f(R)$ and given equations are satisfied from [1]. □

Next, we provide several spectral relations between $R = T + u \otimes v$, T, and $u \otimes v$ as in Theorem 3.2.

Proposition 3.12. Let $R = T + u \otimes v$ be an operator in $\mathcal{L}(\mathcal{H})$ with the same hypotheses as in Theorem 3.2. Then the following properties hold:
(i) $\sigma_p(R) \subset \sigma_p(T) \cup \sigma_p(u \otimes v)$, $\Gamma(R) \subset \Gamma(T) \cup \Gamma(u \otimes v)$, and

$$\begin{aligned} \sigma(R) &= \sigma_a(R) = \sigma_{su}(R) = \cup\{\sigma_R(x) : x \in \mathcal{H}\} \\ &\subset \sigma_a(T) \cup \sigma_a(u \otimes v) = \sigma(T) \cup \sigma(u \otimes v). \end{aligned}$$

(ii) $\sigma_{le}(R) \subset \sigma_{le}(T + \langle u, v\rangle)$, $\sigma_{re}(R) \subset \sigma_{re}(T + \langle u, v\rangle)$, and

$$\sigma_e(T) = \sigma_w(R) = \sigma_{ea}(R) = \sigma_e(R) \subset \sigma_e(T + \langle u, v\rangle).$$

(iii) $\pi_{00}^a(T + u \otimes v) \subset$ iso $\sigma_a(T) \cup \rho(T)$.
(iv) $\sigma_R((u \otimes v)x) \subset \sigma_{T+\langle u,v\rangle}(x)$ and $\sigma_{T+\langle u,v\rangle}((u \otimes v)x) \subset \sigma_R(x)$ for all $x \in \mathcal{H}$.
(v) $(u \otimes v)H_{T+\langle u,v\rangle}(F) \subset H_R(F)$ and $(u \otimes v)H_R(F) \subset H_{T+\langle u,v\rangle}(F)$ where F is any subset of $\mathbb{C}$.

Proof. Assume that $R = T + u \otimes v$. Note that if $S \in \mathcal{L}(\mathcal{H})$ is complex symmetric, then $\sigma(S) = \sigma_a(S) \cup \sigma_a(S^*)^* = \sigma_a(S)$ from [12, Corollary, page 222] and [16, Lemma 4.1].

(i) Since T commutes with $(u \otimes v)$ by Lemma 3.1, it ensures that

$$\sigma_a(R) = \sigma_a(T + (u \otimes v)) \subseteq \sigma_a(T) + \sigma_a(u \otimes v)$$

by [19]. Since T, $u \otimes v$, and R are complex symmetric by [11] and Lemma 3.1, we conclude from [18, Lemma 3.22] that

$$\begin{aligned} \sigma(R) &= \sigma_a(R) = \sigma_{su}(R) = \cup\{\sigma_R(x) : x \in \mathcal{H}\} \\ &\subset \sigma_a(T) \cup \sigma_a(u \otimes v) = \sigma(T) \cup \sigma(u \otimes v). \end{aligned}$$

By the similar method, we get that $\sigma_p(R) \subset \sigma_p(T) \cup \sigma_p(u \otimes v)$. On the other hand, since $\Gamma(S)^* = \sigma_p(S^*)$ for any $S \in \mathcal{L}(\mathcal{H})$, $\Gamma(T) = \Gamma(T^*)^*$ by [16], and the previous result, we conclude that $\Gamma(R) \subset \Gamma(T) \cup \Gamma(u \otimes v)$ because R, T, and $u \otimes v$ are complex symmetric.

(ii) Since R is complex symmetric, we know that $\sigma_{re}(R)^* = \sigma_{re}(R^*)$ and $\sigma_{le}(R)^* = \sigma_{le}(R^*)$ from [16, Lemma 4.1]. Since $\sigma_{re}(S)^* = \sigma_{le}(S^*)$ and $\sigma_e(S) = \sigma_{le}(S) \cup \sigma_{re}(S)$ for any $S \in \mathcal{L}(\mathcal{H})$, it suffices to prove $\sigma_{le}(R) \subset \sigma_{le}(T + \langle u, v\rangle)$. If $\lambda \in \sigma_{le}(R)$, then there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\{x_n\}$ weakly converges to 0 and $\lim_{n\to\infty} \|(R - \lambda)x_n\| = 0$ for any $R \in \mathcal{L}(\mathcal{H})$. Put $y_n = (u \otimes v)x_n$. Since T commutes with $(u \otimes v)$ by Lemma 3.1, we obtain that

$$\lim_{n\to\infty} \|(u \otimes v)(R - \lambda)x_n\| = \lim_{n\to\infty} \|(T + \langle u, v\rangle - \lambda)(u \otimes v)x_n\| = 0.$$

In addition, if $\{x_n\}$ weakly converges to 0, then $\{(u \otimes v)x_n\}$ clearly weakly converges to 0 and so $\{y_n\}$ clearly weakly converges to 0. Therefore $\lambda \in \sigma_{le}(T + \langle u, v\rangle)$. Since $\sigma_e(T + K) = \sigma_e(T)$ for all compact operator K, it ensures that $\sigma_e(T) = \sigma_e(R)$. The remaining relations come from [18, Lemma 3.22]. So, we complete the proof.

(iii) Let $\lambda \in \pi_{00}^a(R)$. Then $\lambda \in \text{iso}\sigma_a(R)$ is an eigenvalue of R with finite multiplicity. By (i), we know that $\lambda \in \text{iso}\sigma(R)$ is an eigenvalue of R with finite multiplicity. This gives from [14, Proposition 2.1] that $\lambda \in \sigma(T) \cup \rho(T)$. Since T is complex symmetric, we conclude that

$$\pi_{00}^a(R) \subset \text{ iso } \sigma_a(T) \cup \rho(T)$$

where $\rho(T)$ denotes the resolvent set of T.

(iv) Suppose that $\lambda_0 \in \rho_{T+\langle u,v\rangle}(x)$. Then there is an $\mathcal{H}$-valued analytic function $f(\lambda)$ in a neighborhood D of λ_0 such that $(T + \langle u, v\rangle - \lambda) f(\lambda) = x$ for every $\lambda \in D$. Since T commutes with $(u \otimes v)$ by Lemma 3.1, it follows that

$$(R - \lambda)(u \otimes v) f(\lambda) = (T + u \otimes v - \lambda)(u \otimes v) f(\lambda) \equiv (u \otimes v)x \text{ on } D.$$

Since $(u \otimes v) f(\lambda)$ is analytic on D, we get $\lambda_0 \in \rho_R((u \otimes v)x)$. Hence $\rho_R((u \otimes v)x) \supset \rho_{T+\langle u,v\rangle}(x)$ and so $\sigma_R((u \otimes v)x) \subset \sigma_{T+\langle u,v\rangle}(x)$.

On the other hand, we assume $\lambda_0 \in \rho_R(x)$. Then there is an $\mathcal{H}$-valued analytic function $f(\lambda)$ in a neighborhood D of λ_0 such that $(R - \lambda) f(\lambda) = x$ for every $\lambda \in D$. Since $T(u \otimes v) = (u \otimes v)T$ by Lemma 3.1, it follows that

$$(T + \langle u, v\rangle - \lambda)(u \otimes v) f(\lambda) \equiv (u \otimes v)x \text{ on } D.$$

Since $(u \otimes v) f(\lambda)$ is analytic on D, we get $\lambda_0 \in \rho_{T+\langle u,v\rangle}((u \otimes v)x)$. Hence we have $\rho_R(x) \subset \rho_{T+\langle u,v\rangle}((u \otimes v)x)$ and so $\sigma_{T+\langle u,v\rangle}((u \otimes v)x) \subset \sigma_R(x)$.

(v) Let $x \in H_{T+\langle u,v\rangle}(F)$. Then $\sigma_{T+\langle u,v\rangle}(x) \subset F$. Hence $F^c \subset \rho_{T+\langle u,v\rangle}(x)$. Therefore, there is an $\mathcal{H}$-valued analytic function f defined on F^c such that

$$(T + \langle u, v\rangle - \lambda) f(\lambda) = x, \quad \lambda \in F^c.$$

Since T commutes with $(u \otimes v)$ by Lemma 3.1, it follows that

$$(R - \lambda)(u \otimes v) f(\lambda) = (u \otimes v)(T + \langle u, v\rangle - \lambda) f(\lambda) = (u \otimes v)x.$$

Since $(u \otimes v) f$ is analytic on F^c, it ensures that $F^c \subset \rho_R((u \otimes v)x)$. Hence $\sigma_R((u \otimes v)x) \subset F$ which means $(u \otimes v)x \in H_R(F)$. Hence we conclude that $(u \otimes v)H_{T+\langle u,v\rangle}(F) \subset H_R(F)$.

On the other hand, if $x \in H_R(F)$, then $\sigma_R(x) \subset F$ and so $F^c \subset \rho_R(x)$. Therefore, there is an $\mathcal{H}$-valued analytic function f defined on F^c such that

$$(R - \lambda) f(\lambda) = x, \quad \lambda \in F^c.$$

Since $T(u \otimes v) = (u \otimes v)T$ by Lemma 3.1, it follows that

$$(T + \langle u, v\rangle - \lambda)(u \otimes v) f(\lambda) = (u \otimes v)x.$$

Since $(u \otimes v) f$ is analytic on F^c, it ensures that $F^c \subset \rho_{T+\langle u,v\rangle}((u \otimes v)x)$. Hence $\sigma_{T+\langle u,v\rangle}((u \otimes v)x) \subset F$ which means $(u \otimes v)x \in H_{T+\langle u,v\rangle}(F)$. Therefore we have $(u \otimes v)H_R(F) \subset H_{T+\langle u,v\rangle}(F)$. Hence we obtain $(u \otimes v)H_{T+\langle u,v\rangle}(F) \subset H_R(F)$ and $(u \otimes v)H_R(F) \subset H_{T+\langle u,v\rangle}(F)$ where F is any subset of $\mathbb{C}$. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *finite-isoloid* if iso $\sigma(T) \subset \pi_{0f}(T)$ where $\pi_{0f}(T)$ is the set of the eigenvalue of finite multiplicity. Next, we consider a-Browder's or a-Weyl's theorems hold for the rank one perturbations of complex symmetric operators.

Theorem 3.13. If T satisfies the hypotheses as in Theorem 3.2, then the following arguments hold:
(i) If T has the property (β), then a-Weyl's theorem holds for $T + u \otimes v$ and $T^* + v \otimes u$.
(ii) If T has the property (C) or if $H_0(T - \lambda)$ is closed for every $\lambda \in \mathbb{C}$, then $T + u \otimes v$ and $T^* + v \otimes u$ satisfy a-Browder's theorem.
(iii) If T is finite-isoloid and satisfies Weyl's theorem, then $T + u \otimes v$ and $T^* + v \otimes u$ also satisfy a-Weyl's theorem.

Proof. Let $R = T + u \otimes v$ with the hypotheses as in Theorem 3.2.
(i) If T is complex symmetric with the property (β), then R and R^* have the property (β) by Theorem 3.2. Hence R is subscalar by [17] and [19]. Therefore R has the following property by [1, Page 175]; for each $\lambda \in \mathbb{C}$, there exists $q_\lambda \in \mathbb{N}$ such that

$$H_0(R-\lambda) = \ker(R-\lambda)^{q_\lambda}.$$

Let $p_\lambda := q_{\overline{\lambda}}$ for $\lambda \in \mathbb{C}$. Since $H_0(R^*-\lambda) \supset \ker(R^*-\lambda)^{p_\lambda}$ for any $\lambda \in \mathbb{C}$, it suffices to show the converse inclusion. Let $x \in H_0(R^*-\lambda)$. Then, since $CRC = R^*$ for some conjugation C by Lemma 3.1, we obtain that

$$\|(R-\overline{\lambda})^n Cx\|^{\frac{1}{n}} = \|C(R^*-\lambda)^n x\|^{\frac{1}{n}} = \|(R^*-\lambda)^n x\|^{\frac{1}{n}} \to 0 \text{ as } n \to \infty.$$

Since $Cx \in H_0(R-\overline{\lambda}) = \ker(R-\overline{\lambda})^{p_\lambda}$, we have $(R^*-\lambda)^{p_\lambda}x = C(R-\overline{\lambda})^{p_\lambda}Cx = 0$. Thus we have $H_0(R^*-\lambda) \subset \ker(R^*-\lambda)^{p_\lambda}$. Hence we get that for each $\lambda \in \mathbb{C}$

$$H_0(R^*-\lambda) = \ker(R^*-\lambda)^{p_\lambda},$$

which implies that Weyl's theorem holds for $T + u \otimes v$ by [1, Theorem 3.99]. Since R is complex symmetric by Lemma 3.1, it follows that $\sigma_w(R) = \sigma_{ea}(R)$ by Proposition 3.12 (iii). Therefore we get that

$$\pi_{00}^a(R) = \pi_{00}(R) = \sigma(R) \setminus \sigma_w(R) = \sigma_a(R) \setminus \sigma_{ea}(R).$$

Hence a-Weyl's theorem holds for $T+u\otimes v$. Similarly, we can show that a-Weyl's theorem holds for $T^*+v\otimes u$.

(ii) Since T has the property (C) and $T(u \otimes v) = (u \otimes v)T$ by Lemma 3.1, this ensures from Theorem 3.9 that R has the single-valued extension property. Hence R satisfies Browder's theorem by [1]. Moreover, since R is complex symmetric by Lemma 3.1, we get that $\sigma(R) = \sigma_a(R)$ and $\sigma_w(R) = \sigma_{ea}(R)$ by Proposition 3.12. Hence [1, Theorem 3.65] implies that

$$\sigma_b(R) = \sigma_w(R) \cup acc(\sigma(R)) = \sigma_{ea}(R) \cup acc(\sigma_a(R)) = \sigma_{ab}(R)$$

where $acc(\Delta)$ is the set of all accumulation points of $\Delta \subset \mathbb{C}$. Since $\sigma_b(R) = \sigma_w(R)$, we have $\sigma_{ea}(R) = \sigma_w(R) = \sigma_{ab}(R)$. Hence a-Browder's theorem holds for R. On the other hand, we verify that $R^* = T^* + v \otimes u$ also satisfies a-Browder's theorem by a similar fashion. If $H_0(T-\lambda)$ is closed for every complex number λ, then, by a similar way, we get from [1, Page 336] this result.

(iii) This result follows from Proposition 3.12, [18], and [14, Proposition 2.2]. □

As some applications of Theorem 3.13, we get the following corollary.

Corollary 3.14. Suppose that T satisfies the hypotheses as in Theorem 3.2. Then following assertions hold:
(i) If T is normal, compact, or hyponormal, then $T + u \otimes v$ and $T^* + v \otimes u$ satisfy a-Weyl's theorem.
(ii) If T is quasinilpotent, then $T + u \otimes v$ and $T^* + v \otimes u$ satisfy a-Browder's theorem.

Proof. (i) Since T is normal or compact, it follows from [4] or [19] that T has the property (β). Hence the result follows from Theorem 3.13.

If T is hyponormal, it ensures that T is isoloid. Moreover, since T is complex symmetric, we know that T is normal by [26]. Hence T satisfies Weyl's theorem and finitely isoloid because $iso\sigma(T) \subset \pi_{00}(T) \subset \pi_{0f}(T)$. Therefore, the proof follows from Theorem 3.13.

(ii) Since T is quasinilpotent, it ensures from [1] and Theorem 3.13 that $T + u \otimes v$ and $T^* + v \otimes u$ satisfy a-Browder's theorem. □

Finally, we investigate the hyponormality of the rank one perturbations of complex symmetric operators.

Theorem 3.15. Let $T \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator with the conjugation C. If $R = T + u \otimes v$ where u and v are unit vectors, then R is hyponormal if and only if

$$||Tx||^2 - ||T^*x||^2 \geq 2Re(\langle x,u\rangle\langle Tv,x\rangle - \langle v,x\rangle\langle TCu,Cx\rangle) + |\langle x,u\rangle|^2 - |\langle x,v\rangle|^2 \tag{13}$$

for all $x \in \mathcal{H}$. In particular, if u and v are linearly dependent, R is hyponormal if and only if

$$||Tx||^2 - ||T^*x||^2 \geq 2Re(\langle x,u\rangle\langle Tv,x\rangle - \langle v,x\rangle\langle TCu,Cx\rangle) \tag{14}$$

for all $x \in \mathcal{H}$. Moreover, if u and v be linearly independent, then R is hyponormal if and only if for all $x = y + z$ with $y \in \vee\{u,v\}$ and $z \in (\vee\{u,v\})^\perp$

$$Re\langle t,x\rangle \leq \frac{1}{2}(||Tx||^2 - ||CTCx||^2 - |\langle y,u\rangle|^2 + |\langle y,v\rangle|^2) \tag{15}$$

holds where $t = \langle x,u\rangle Tv - \langle TCu,Cx\rangle v$.

Proof. Since $||u|| = ||v|| = 1$, we get that

$$\begin{cases} R^*R = T^*T + T^*u\otimes v + v\otimes T^*u + v\otimes v, \text{ and} \\ RR^* = TT^* + Tv\otimes u + u\otimes Tv + u\otimes u. \end{cases}$$

Since $T^* = CTC$ for some conjugation C, it follows that

$$\begin{aligned}
&\langle (R^*R - RR^*)x,x\rangle \\
=\ &\langle (T^*T - TT^*)x,x\rangle + \langle x,v\rangle\langle T^*u,x\rangle + \langle x,T^*u\rangle\langle v,x\rangle \\
&+\langle x,v\rangle\langle v,x\rangle - \langle x,u\rangle\langle Tv,x\rangle - \langle x,Tv\rangle\langle u,x\rangle - \langle x,u\rangle\langle u,x\rangle \\
=\ &\langle (T^*T - TT^*)x,x\rangle + \langle x,v\rangle\langle CTCu,x\rangle + \langle x,CTCu\rangle\langle v,x\rangle \\
&+\langle x,v\rangle\langle v,x\rangle - \langle x,u\rangle\langle Tv,x\rangle - \langle x,Tv\rangle\langle u,x\rangle - \langle x,u\rangle\langle u,x\rangle \\
=\ &\langle (T^*T - TT^*)x,x\rangle + \langle x,v\rangle\langle Cx,TCu\rangle + \langle TCu,Cx\rangle\langle v,x\rangle \\
&+\langle x,v\rangle\langle v,x\rangle - \langle x,u\rangle\langle Tv,x\rangle - \langle x,Tv\rangle\langle u,x\rangle - \langle x,u\rangle\langle u,x\rangle.
\end{aligned}$$

for all $x \in \mathcal{H}$. Hence we obtain that R is hyponormal if and only if

$$||Tx||^2 - ||T^*x||^2 \geq 2Re(\langle x,u\rangle\langle Tv,x\rangle - \langle v,x\rangle\langle TCu,Cx\rangle) + |\langle x,u\rangle|^2 - |\langle x,v\rangle|^2 \tag{16}$$

for all $x \in \mathcal{H}$.

In particular, if u and v are linearly dependent, then there exists $a \in \mathbb{C}$ such that $u = av$ and this implies $|\langle x,u\rangle| = |\overline{a}\langle x,v\rangle| = |\langle x,v\rangle|$. Therefore, the inequality (16) yields that R is hyponormal if and only if

$$||Tx||^2 - ||T^*x||^2 \geq 2Re(\langle x,u\rangle\langle Tv,x\rangle - \langle v,x\rangle\langle TCu,Cx\rangle)$$

for all $x \in \mathcal{H}$.

On the other hand, let u and v are linearly independent. Set $t = \langle x,u\rangle Tv - \langle TCu,Cx\rangle v$. Then the relation (16) ensures that R is hyponormal if and only if for all $x = y + z$ with $y \in \vee\{u,v\}$ and $z \in (\vee\{u,v\})^\perp$,

$$Re\langle t,x\rangle \leq \frac{1}{2}(||Tx||^2 - ||TCx||^2 - |\langle y,u\rangle|^2 + |\langle y,v\rangle|^2)$$

holds where $t = \langle x,u\rangle Tv - \langle TCu,Cx\rangle v$. The reverse implication is clearly holds. Hence we complete the proof. □

From Theorem 3.15, then we obtain the following results.

Corollary 3.16. Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric and let $R = T + u \otimes v$ be hyponormal on $\mathcal{H}$. If $Re\langle t,x\rangle \geq -\frac{1}{2}(|\langle x,u\rangle|^2 - |\langle x,v\rangle|^2)$ holds where $t = \langle x,u\rangle Tv - \langle TCu,Cx\rangle v$, then T is normal.

Proof. Assume that R is hyponormal. If $Re\langle t, x\rangle \geq -\frac{1}{2}(|\langle x, u\rangle|^2 - |\langle x, v\rangle|^2)$ holds where $t = \langle x, u\rangle Tv - \langle TCu, Cx\rangle v$, then (13) gives $\|Tx\|^2 \geq \|T^*x\|^2$ for all x. Therefore T is hyponormal. Since T is complex symmetric, we conclude that T is normal by [26]. □

Corollary 3.17. Let φ be in L^∞ and let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis of H^2 with $e_n(z) = z^n$. If a Toeplitz operator T_φ is complex symmetric with the conjugation C, then the following statements hold:
(i) If $T_\varphi + (e_0 \otimes e_1) \in \mathcal{L}(H^2)$ where e_0 and e_1 are elements of the basis $\{e_n\}_{n=0}^\infty$, then $T_\varphi + (e_0 \otimes e_1)$ is hyponormal if and only if

$$Re(\hat{f}(0)\langle T_\varphi e_1, f\rangle - \overline{\hat{f}(1)}\langle f, T_\varphi^* e_0\rangle) \leq \frac{1}{2}(|\langle g, e_1\rangle|^2 - |\langle g, e_0\rangle|^2) \tag{17}$$

for all $f = g + h \in \vee\{e_0, e_1\} \oplus (\vee\{e_0, e_1\})^\perp = H^2$.
(ii) If $T_\varphi + (e_0 \otimes e_0) \in \mathcal{L}(H^2)$ where e_0 is element of the basis $\{e_n\}_{n=0}^\infty$, then $T_\varphi + (e_0 \otimes e_0)$ is hyponormal if and only if

$$Re(\hat{f}(0)\langle T_\varphi e_0, f\rangle - \overline{\hat{f}(0)}\langle f, T_\varphi^* e_0\rangle) \leq 0 \tag{18}$$

for all $f \in H^2$.

Proof. Let $\varphi \in L^\infty$ and let T_φ be complex symmetric with the conjugation C on H^2. Then it holds that $CT_\varphi C = T_\varphi^*$. Therefore, $\|T_\varphi Cf\| = \|T_\varphi^* f\|$ for all $f \in H^2$.

(i) Since $\langle T_\varphi Ce_0, Cf\rangle = \langle CT_\varphi^* e_0, Cf\rangle = \langle f, T_\varphi^* e_0\rangle$, it follows that

$$\begin{aligned}
& Re(\langle f, e_0\rangle\langle T_\varphi e_1, f\rangle - \langle e_1, f\rangle\langle T_\varphi Ce_0, Cf\rangle) \\
= \; & Re(\langle f, e_0\rangle\langle T_\varphi e_1, f\rangle - \langle e_1, f\rangle\langle CT_\varphi^* e_0, Cf\rangle) \\
= \; & Re(\langle f, e_0\rangle\langle T_\varphi e_1, f\rangle - \langle e_1, f\rangle\langle f, T_\varphi^* e_0\rangle) \\
= \; & Re(\hat{f}(0)\langle T_\varphi e_1, f\rangle - \overline{\hat{f}(1)}\langle f, T_\varphi^* e_0\rangle)
\end{aligned} \tag{19}$$

for all $f \in H^2$. Since e_0 and e_1 are linearly independent, it follows from Theorem 3.15 that we conclude that R is hyponormal if and only if for all $f = g + h$ with $g \in \vee\{e_0, e_1\}$ and $h \in (\vee\{e_0, e_1\})^\perp$,

$$Re(\hat{f}(0)\langle T_\varphi e_1, f\rangle - \overline{\hat{f}(1)}\langle f, T_\varphi^* e_0\rangle) \leq \frac{1}{2}(|\langle g, e_1\rangle|^2 - |\langle g, e_0\rangle|^2)$$

holds.

(ii) By a similar method of (i) and (14) in Theorem 3.15, we obtain that $T_\varphi + (e_0 \otimes e_0)$ is hyponormal if and only if

$$Re(\hat{f}(0)\langle T_\varphi e_0, f\rangle - \overline{\hat{f}(0)}\langle f, T_\varphi^* e_0\rangle) \leq 0$$

for all $f \in H^2$. □

Corollary 3.18. Let $\varphi = \alpha\psi + \beta$ be in L^∞ where ψ is a real-valued function in L^∞ with α and $\beta \in \mathbb{C}$. If $T_\varphi + (e_0 \otimes e_1)$ and $T_\varphi + (e_0 \otimes e_0)$ satisfy the conditions (17) and (18), respectively, then $T_\varphi + (e_0 \otimes e_1)$ and $T_\varphi + (e_0 \otimes e_0)$ have a nontrivial invariant subspace.

Proof. Since T_φ is normal by [3], it follows from [11, Theorem 3] that $T_\varphi + e_0 \otimes e_1$ and $T_\varphi + e_0 \otimes e_0$ are complex symmetric. If (17) and (18) hold for T_φ, respectively, we obtain that $T_\varphi + (e_0 \otimes e_1)$ and $T_\varphi + (e_0 \otimes e_0)$ are hyponormal by Corollary 3.17. Since $T_\varphi + (e_0 \otimes e_1)$ and $T_\varphi + (e_0 \otimes e_0)$ are complex symmetric by [11], they must be normal by [26]. Hence $T_\varphi + (e_0 \otimes e_1)$ and $T_\varphi + (e_0 \otimes e_0)$ have a nontrivial invariant subspace. □

As an application of Theorem 3.15, we consider the hyponormality of the rank one perturbations of truncated Toeplitz operators.

Corollary 3.19. Let u be an inner function and let $\varphi = \varphi_1 + \overline{\varphi_2} \in L^2$ where φ_1 and φ_2 are in $\mathcal{K}_u$. Suppose that $A^u_\varphi \in \mathcal{T}_u$ is a truncated Toeplitz operator on $\mathcal{K}_u$. If $R = A^u_\varphi + Ck^u_0 \otimes k^u_0$ where $Cf = \overline{zf}u$ for all $f \in \mathcal{K}_u$, then R is hyponormal if and only if

$$2Re(\langle[\varphi_1 + \overline{\varphi_2(0)}k^u_0 - \overline{u(0)}S_u C\varphi_2], [(\overline{f(0)} - f(0))Cf]\rangle) \leq 0 \tag{20}$$

for all $f \in \mathcal{K}_u$. In particular, if $u(0) = 0$, then R is hyponormal if and only if

$$2Re(\langle[\varphi_1 + \overline{\varphi_2(0)}], [(\overline{f(0)} - f(0))Cf]\rangle) \leq 0 \tag{21}$$

for all $f \in \mathcal{K}_u$.

Proof. Let $\varphi = \varphi_1 + \overline{\varphi_2} \in L^2$ where φ_1 and φ_2 are in $\mathcal{K}_u$. Since every truncated Toeplitz operator is complex symmetric by [9], we get that

$$\langle f, Ck^u_0\rangle = \langle k^u_0, Cf\rangle = \overline{Cf(0)}, \text{ and } \langle f, k^u_0\rangle = f(0) \tag{22}$$

for all $f \in \mathcal{K}_u$. Since $S_u Cf = P_u(u\overline{f})$ for $f \in \mathcal{K}_u$, we know from [25, Proposition 3.2] that

$$\begin{aligned} A^u_\varphi k^u_0 &= P_u\{(\varphi_1 + \overline{\varphi_2})(1 - \overline{u(0)}u)\} \\ &= P_u(\varphi_1 + \overline{\varphi_2} - \overline{u(0)}\varphi_1 u - \overline{u(0)\varphi_2}u) \\ &= \varphi_1 + P_u(\overline{\varphi_2(0)}) - \overline{u(0)}P_u(u\overline{\varphi_2}) \\ &= \varphi_1 + \overline{\varphi_2(0)}k^u_0 - \overline{u(0)}S_u C\varphi_2. \end{aligned} \tag{23}$$

Combining (22), (23), and (13) in Theorem 3.15, we get that R is hyponormal if and only if

$$\begin{aligned} \|A^u_\varphi f\|^2 - \|A^u_\varphi Cf\|^2 &\geq 2Re(\langle f, Ck^u_0\rangle\langle A^u_\varphi k^u_0, f\rangle - \langle k^u_0, f\rangle\langle A^u_\varphi k^u_0, Cf\rangle) \\ &= 2Re(\langle[\varphi_1 + \overline{\varphi_2(0)}k^u_0 - \overline{u(0)}S_u C\varphi_2], [C\overline{f(0)}f - f(0)Cf]\rangle) \\ &= 2Re(\langle[\varphi_1 + \overline{\varphi_2(0)}k^u_0 - \overline{u(0)}S_u C\varphi_2], [(\overline{f(0)} - f(0))Cf]\rangle) \end{aligned}$$

for all $f \in \mathcal{K}_u$. Assume that R is hyponormal. Since

$$CRC - R^* = CA_{\varphi^u}C + k^u_0 \otimes Ck^u_0 - A^*_{\varphi^u} - k^u_0 \otimes Ck^u_0 = 0,$$

it follows that R is complex symmetric. From [11], we know that R should be normal. The condition (20) clearly holds. The converse implication trivially holds. In particular, if $u(0) = 0$, then $k^u_0 = 1$. So we can get this result. □

References

[1] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Academic Pub. 2004.
[2] P. Aiena, M. L. Colasante, and M. Gonza'lez, *Operators which have a closed quasi-nilpotent part*, Proc. Amer. Math. Soc. 2002.
[3] A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine angew. Math., **213**(1963-1964), 89-102.
[4] I. Colojoara and C. Foias, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.
[5] I. Chalendar, and D. Timotin, *Commutation relation for Truncated Toeplitz operators*, Operators and Matrices. to appear.
[6] D. N. Clark, *One dimensional perturbations of restricted shifts*, J. Analyse Math. **25**(1972), 169-191.
[7] J. Eschmeier, *Invariant subspaces for subscalar operators*, Arch. Math. **52**(1989), 562-570.
[8] S. V. Djordjević and Y. M. Han, *Browder's theorems and spectral continuity*, Glasgow Math. J. **42** (2000), 479–486.
[9] S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358**(2006), 1285-1315.
[10] ______________, *Complex symmetric operators and applications* II, Trans. Amer. Math. Soc. **359**(2007), 3913-3931.
[11] S. R. Garcia and W. R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362**(2010), 6065-6077.

[12] P. R. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag Berlin Heidelberg New York, 1980.
[13] R. E. Harte and W. Y. Lee, *Another note on Weyl's theorem*, Trans. Amer. Math. Soc. **349**(1997), 2115–2124.
[14] Y. M. Han and W. Y. Lee, *Weyl spectra and Weyl's theorem*, Studia Math. **148**(2001), 193-206.
[15] E. Ionascu, *Rank-one perturbations of diagonal operators*, Integral Equations and Operator Theory. **39**(2001), 421-440.
[16] S. Jung, E. Ko, M. Lee, and J. Lee, *On local spectral properties of complex symmetric operators*, J. Math. Anal. Appl. **379**(2011), 325-333.
[17] S. Jung, E. Ko, and J. Lee, *On scalar extensions and spectral decompositions of complex symmetric operators*, J. Math. Anal. Appl. **382**(2011), 252-260.
[18] ______________, *On complex symmetric operator matrices*, J. Math. Anal. Appl. **406**(2013), 373-385.
[19] K. Laursen and M. Neumann, *An introduction to local spectral theory*, Clarendon Press, Oxford, 2000.
[20] R. Lange and S. Wang, *New approaches in spectral decomposition*, Contemp. Math. **128**, Amer. Math. Soc., 1992.
[21] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland Publishing Co., Amsterdam-London, 1970.
[22] C. Pearcy, *Some recent developments in operator theory*, C.B.M.S. Regional Conference Series in Mathematics No **36**, Amer. Math. Soc. Providence, 1978.
[23] D. Sarason, *Algebraic properties of truncated Toeplitz operators*, Operators and Matrices. **1**(2007), 419-526.
[24] N. A. Sedlock, *Algebras of truncated Toeplitz operators*, Oper. Matrices, **5**(2011), 309-326.
[25] N. A. Sedlock, *Multiplication of truncated Toeplitz operators*, preprint.
[26] X. Wang and Z. Gao, *A note on Aluthge transforms of complex symmetric operators and applications*, Int. Eq. Op. Th. **65**(2009), 573-580.