Filomat 33:1 (2019), 289–306
https://doi.org/10.2298/FIL1901289S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximate Controllability of Retarded Impulsive Stochastic Integro-Differential Equations Driven by Fractional Brownian Motion

Abdeldjalil Slama[a], Ahmed Boudaoui[a]

[a] *Department of Mathematics and computer sciences, University of Adrar, Adrar, Algeria*

Abstract. In this paper, we investigate the approximate controllability of retarded impulsive stochastic integro-differential equations driven by fractional Brownian motion(fBm) in Hilbert space. With the help of the resolvent operators, the fixed-point theorem, stochastic analysis and methods of controllability theory, a new set of sufficient conditions for approximate controllability of the integro-differential equations are formulated and proven. An example is provided to illustrate the obtained theory.

1. Introduction

During the last decades, differential and integral equations have attracted great interest due to their applications in characterizing many problems in science, engineering, mathematical finance, and in almost all applied sciences. For some of these applications, one can see Kilbas et al.[21]. In particlar, the stochastic differential equations are important from the viewpoint of applications since they incorporate (natural) randomness into the mathematical description of the phenomena, and, therefore, provide a more accurate description of it [5].

Among the qualitative properties of differential equations, the controllability plays an important role both in deterministic and stochastic control theory. Controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls [39]. The controllability of nonlinear stochastic systems in infinite dimensional spaces has recently received a lot of attentions (see [2, 3, 8, 9, 19, 24, 25, 31, 39] and the references therein). Moreover, the approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. Approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications (see [2]). The approximate controllability of nonlinear systems represented by evolution processes (equations or inclusions) in abstract spaces has been extensively considered in many publications and monographs, an extensive list of these publications can be found in [2, 9, 10, 36–38, 40, 41] and references contained therein.

On the other hand, the theory of impulsive differential equations as well as the theory of neutral integro-differential equations has become an active area of investigation due to their applications in various fields

2010 *Mathematics Subject Classification*. Primary 34K50, 34K45, 93B05, 60G22, 60H20

Keywords. Impulsive stochastic integro-differential equations, Fixed point principle, Approximate controllability, fractional Brownian motion(fBm).

Received: 26 March 2018; Revised: 01 December 2018; Accepted: 10 January 2019

Communicated by Miljana Jovanovic

Email addresses: slama_dj@yahoo.fr (Abdeldjalil Slama), ahmedboudaoui@gmail.com (Ahmed Boudaoui)

such as in electronics, mechanical systems, biological systems and economics, etc. (see [6, 23] and the references therein). The impulsive effects exist in many evolution processes that abruptly change their state at a certain moment (see [6]). The impulsive neutral integro-differential equations with delay properties has been used for modeling the evolution of physical systems, in which the response of the system depends not only on the current state, but also on the past history of the system. Several authors have investigated the impulsive neutral integrodifferential equations with delay (see [19, 20, 28, 35, 43]) and references therein.

In recent years, stochastic differential equations driven by fractional Brownian motion (fBm) have attracted much attention due to its a wide applications in a variety of physical phenomena, such as in economic and finance (see [29]), in biology (see [13]) and communication networks (see [44]). The fractional Brownian motion was introduced within a Hilbert space framework by Kolmogorov in 1940 in [22], and later studied by Mandelbrot and Van Ness, who in 1968 provided in [30] a stochastic integral representation of this process in terms of a standard Brownian motion. There has been some recent interest in studying evolution equations driven by fractional Brownian motion. Recently, Lakhel [24] obtained controllability results of neutral stochastic delay partial functional integro-differential equations perturbed by fractional Brownian motion by using the theory of semigroup. Boudaoui and Lakhel [8] investigated the controllability results of impulsive neutral stochastic functional differential equations with infinite delay driven by fractional Brownian motion in a real separable Hilbert space. Huan et al. [18] established results concerning the approximate controllability for time-dependent impulsive neutral stochastic partial differential equations with memory in Hilbert space. Very recently, Lakhel [27] studied the controllability of an impulsive neutral stochastic integro-differential systems with infinite delay driven by fractional Brownian motion in separable Hilbert space. Many interesting works have been done on stochastic differential equationsd driven by fBm (see [24–27] and the references therein).

However, the study of the approximate controllability of neutral stochastic functional integro-differential equations driven by a fractional Brownian motion with impulsive effects has not been discussed in the standard literature. Motivated by the above consideration, the aim of this paper is to study the approximate controllability for the following impulsive neutral stochastic functional integrodifferential equations driven by a fractional Brownian motion:

$$\begin{cases} d[y(t) + g(t, y(t-r_1(t)))] = A[y(t) + g(t, y(t-r_1(t)))]dt + Bu(t)dt \\ + \int_0^t G(t-s)[y(s) + g(s,(s-r_1(s)))]dsdt + f(t, y(t-r_2(t)))dt \\ +\sigma(t)dB^H(t), t \in [0,T], t \neq t_k, \\ \Delta y(t_k) = y(t_k^+) - y(t_k) = I_k(y(t_k)), k = 1, ..., m, \\ y(t) = \varphi(t), -\tau \leq t \leq 0, \end{cases} \tag{1}$$

where A is the infinitesimal generator of a compact, analytic resolvent operator $R(t)$, $t \geq 0$ on a Hilbert space $\mathbb{H}$ with domain $D(A)$, $G(t)$ is a closed linear operator on $\mathbb{H}$ with domain $D(G) \supset D(A)$ which is independent of t, r_1, r_2, f, g and I_k are appropriate functions to be specified later, the initial data $\phi \in \mathcal{D}_1$ which will be defined later, the control function u(.) is given in $L^2([0,T], U)$, the Hilbert space of admissible control functions with U a Hilbert space. The symbol B stands for a bounded linear from U into $\mathbb{H}$. Also, B^H is a fractional Brownian motion on a real and separable Hilbert space $\mathbb{K}$, with Hurst parameter $H \in (1/2, 1)$, and with respect to a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ furnished with a family of right continuous and increasing σ-algebras $\{\mathcal{F}_t, t \geq 0\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. Also $\tau > 0$ is the maximum delay, and the impulse times t_k satisfy $0 = t_0 < t_1 < t_2 < \ldots, t_m < T$. As for y_t we mean the segment solution which is define in the usual way, that is, if $y(\cdot,\cdot) : [-\tau, T] \times \Omega \to \mathbb{H}$, then for any $t \geq 0$, $y_t(\cdot,\cdot) : [-\tau, 0] \times \Omega \to \mathbb{H}$ is given by

$$y_t(\theta, \omega) = y(t+\theta, \omega), \text{ for } \theta \in [-\tau, 0], \ \omega \in \Omega.$$

We need to introduce some spaces and notations.

Let $\mathcal{D}$ the Banach space defined by

$$\mathcal{D} = \{\phi\colon [-\tau,0] \to \mathbb{H},\ \phi \text{ is continuous everywhere except for a finite number of points } t \text{ at which } \phi(t^-) \text{ and } \phi(t^+) \text{ exist and satisfy } \phi(t^-) = \phi(t)\},$$

endowed with the L^2-norm

$$\|\phi\|^2 = \int_{-\tau}^{0} |\phi(t)|^2 dt.$$

Also, let $\mathcal{D}_1$ be the space of all piecewise continuous processes $\phi : [-\tau, 0] \times \Omega \to \mathbb{H}$ such that $\phi(\theta, \cdot)$ is $\mathcal{F}_0$-measurable for each $\theta \in [-\tau, 0]$ and $\sup_{\theta\in[-\tau,0]} E(|\phi(\theta)|^2) < \infty$.

Now, for a given $T > 0$, we define

$$\begin{aligned}\mathcal{D}_2 = \Big\{ y\colon\ & [-\tau, T] \times \Omega \to \mathbb{H},\ y_k \in C(J_k, \mathbb{H}) \text{ for } k = 1, \dots m,\ y(0) \in \mathcal{D}_1,\\ & \text{and there exist } y(t_k^-) \text{ and } y(t_k^+) \text{ with } y(t_k) = y(t_k^-),\ k = 1, \cdots, m\\ & \text{and } \sup_{t\in[-\tau,T]} E(|y(t)|^2) < \infty \Big\},\end{aligned}$$

endowed with the norm

$$\|y\|_{\mathcal{D}_2} = \sup_{t\in[-\tau,T]} (E(|y(t)|^2)^{\frac{1}{2}},$$

where y_k denotes the restriction of y to $J_k = (t_{k-1}, t_k]$, $k = 1, 2, \cdots, m$, and $J_0 = [-\tau, 0]$.

Let $\mathbb{K}$ be another real separable Hilbert and suppose that B^H is a $\mathbb{K}$-valued fractional Brownian motion with increment covariance given by a non-negative trace class operator Q, and let $\mathscr{L}(\mathbb{K}, \mathbb{H})$ be the space of all bounded, continuous and linear operators from $\mathbb{K}$ into $\mathbb{H}$. Further, $f : [0, T] \times \mathcal{D} \to \mathbb{H}$, $\sigma : [0, T] \longrightarrow \mathscr{L}_2^0(\mathbb{K}, \mathbb{H})$ are appropriate functions. Here, $\mathscr{L}_2^0(\mathbb{K}, \mathbb{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathbb{K}$ into $\mathbb{H}$.

Also, for the impulse functions we assume that $I_k \in C(\mathbb{H}, \mathbb{H})$ $(k = 1, ..., m)$, and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$.

The paper is organized as follows. In Section 2, we introduce some notations, concepts of resolvent operators, basic results about fractional Brownian motion, and Wiener integral over Hilbert spaces. In Section 3, we study the existence and uniqueness of mild solutions and the approximate controllability of for system (1). An example is given in Section 4 to illustrate the abstract results. In Section 5, concluding remarks are given.

2. Preliminaries

In this section, we introduce the fractional Brownian motion as well as the Wiener integral with respect to it. We also provide some important results which will be needed throughout this paper. For more details on this section, we refer the reader to [7, 34] and [14–17].

Fix a time interval $[0, T]$ and let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space.

Definition 2.1. *[33] Given $H \in (0, 1)$, a continuous centered Gaussian process $\beta^H(t)$, $t \in \mathbb{R}$, with covariance function*

$$R_H(s, t) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), t, s \in \mathbb{R},$$

is called a two-sided one-dimensional fractional Brownian motion (fBm), and H is the Hurst parameter.

Now, we are introducing the Wiener integral with respect to the one dimensional fBm β^H. Let $T > 0$ and denote by Λ the linear space of $\mathbb{R}$-valued step function on $[0, T]$, that is, $\phi \in \Lambda$ if

$$\phi(t) = \sum_{i=1}^{n-1} x_i 1_{[t_i,t_{i+1})}(t),$$

where $t \in [0, T], x_i \in \mathbb{R}$,and $0 = t_1 < t_2 < \cdots < t_n = T$. For $\phi \in \Lambda$, define its Wiener integral with respect to β^H as

$$\int_0^T \phi(s) d\beta^H(s) = \sum_{i=1}^{n-1} x_i(\beta^H(t_{i+1}) - \beta^H(t_i)).$$

Let $\mathbb{H}$ be the Hilbert space defined as the closure of Λ with respect to the scalar product

$$\langle 1_{[0;t]}, 1_{[0;s]}\rangle_{\mathbb{H}} = R_H(t,s).$$

Then the mapping

$$\phi = \sum_{i=1}^{n-1} x_i 1_{[t_i,t_{i+1})} \to \int_0^T \phi(s)d\beta^H(s)$$

is an isometry between Λ and the linear space $span\{\beta^H, t \in [0,T]\}$, which can be extended to an isometry between $\mathbb{H}$ and the first Wiener chaos of the fBm $\overline{span}^{L^2(\Omega)}\{\beta^H, t \in [0,T]\}$ (see [42]). The image of an element $\varphi \in \mathbb{H}$ by this isometry is called the Wiener integral of φ with respect to β^H. Our next aim is to give an explicit expression of this integral. To this end, consider the Kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{2}{3}} u^{H-\frac{1}{2}} du,$$

where

$$c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}},$$

with B representing the Beta function and $t \leq s$. It is easy to see that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H(\frac{t}{s})^{\frac{1}{2}-H}(t-s)^{H-\frac{2}{3}}.$$

Consider the linear operator $K_H^* : \Lambda \to L^2([0,T])$ given by

$$(K_H^*\varphi)(s) = \int_s^t \varphi(t)\frac{\partial K}{\partial t}(t,s)dt.$$

Then

$$K_H^* 1_{[0,t]}(s) = K_H(t,s)1_{[0;t]}(s),$$

and K_H^* is an isometry between Λ and $L^2([0,T])$ that can be extended to Λ(see [4] and references therein). Define $W = \{W(t), t \in [0,T]\}$ by

$$W(t) = \beta^H((K_H^*)^{-1}1_{[0,t]}),$$

it turns out that W is a Wiener process and β^H has the following Wiener integral representation

$$\beta^H(t) = \int_0^t K_H(t,s)dW(s).$$

In addition, for any $\varphi \in \Lambda$,

$$\int_O^T \varphi(s)d\beta^H(s) = \int_0^T (K_H^*\varphi)(t)dW(t)$$

if and only if $K_H^*\phi \in L^2([0,T])$

$$L_H^2([0,T]) = \{\varphi \in \Lambda, K_H^*\varphi \in L^2([0,T])\},$$

since $H > \frac{1}{2}$, we have see [32].

$$L^{1/H}([0,T]) \subset L^2_{\mathbb{H}}([0,T]). \tag{2}$$

Lemma 2.2. *[34] For $\varphi \in L^{1/H}([0,T])$, we have*

$$H(2H-1)\int_0^T\int_0^T |\varphi(r)||\varphi((u)||r-u|^{2H-2}drdu \leq c_H\|\varphi\|^2_{L^{1/H}}([0,T]).$$

Next, we are interested in considering an fBm with values in a Hilbert space and giving the definition of the corresponding stochastic integral. For more details, one can refer [11, 12].

Let $(\mathbb{H}, \|.\|_{\mathbb{H}}, (.,.)_{\mathbb{H}})$ and $(\mathbb{K}, \|.\|_{\mathbb{K}}, (.,.)_{\mathbb{K}})$ be separable Hilbert spaces, let $\mathscr{L}(\mathbb{K},\mathbb{H})$ denote the space of all bounded linear operator from $\mathbb{K}$ to $\mathbb{H}$, and let $Q \in L(\mathbb{K},\mathbb{H})$ be a non-negative self-adjoint operator. Denote by $\mathscr{L}^0_Q(\mathbb{K},\mathbb{H})$ the space of $\vartheta \in \mathscr{L}(\mathbb{K},\mathbb{H})$ such that $\vartheta Q^{1/2}$ is a Hilbert−Schmidt operator. The norm is given by

$$|\vartheta|^2_{\mathscr{L}^0_Q(\mathbb{K},\mathbb{H})} = |\vartheta Q^{1/2}|_{HS} = tr(\vartheta Q \vartheta^*).$$

Then ϑ is called a Q-Hilbert−Schmidt operator from $\mathbb{K}$ to $\mathbb{H}$. Let $\{\beta^H_n(t)\}_{n\in N}$ be a sequence of two−sided one−dimensional standard fractional Brownian motions mutually independent on $(\Omega, \mathscr{F}, \mathbb{P})$. Considers the following series:

$$\sum_{n=1}^{\infty} \beta^H_n(t)e^n, t \geq 0,$$

where $\{e_n\}_{n\in N}$ is a complete orthonormal bais in $\mathbb{H}$, this series does not necessarily converge in the space $\mathbb{K}$. Thus, we consider a $\mathbb{K}$ −valued stochastic process

$$B^H_Q = \sum_{n=1}^{\infty} \beta^H_n(t)Q^{1/2}e^n, t \geq 0.$$

If Q is a non-negative self−adjoint trace class operator, this series converges in the space $\mathbb{K}$, that is, it holds that $B^H_Q(t) \in L^2(\Omega,\mathbb{K})$. Then, we say that $B^H_Q(t)$ is a $\mathbb{K}$−valued Q−cylindrical fractional Brownian motion with covariance operator Q. For example, if $\{\sigma_n\}_{n\in N}$ is a bounded sequence of non−negative real numbers such that $Qe_n = \sigma_n e_n$, assuming that Q is a nuclear operator in $\mathbb{K}$ (that is,$\sum_{n=1}^{\infty}\sigma_n < \infty$), then the stochastic process

$$B^H_Q(t) = \sum_{n=1}^{\infty} \beta^H_n(t)Q^{1/2}e_n = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\beta^H_n(t)e_n, t \geq 0$$

is well defined as a $\mathbb{H}$-valued Q−cylindrical fractional Brownian motion. Let $\varphi : [0,T] \to L^0_Q(\mathbb{K},\mathbb{H})$ be such that

$$\sum_{n=1}^{\infty} \|K^*_H(Q^{1/2}e_n)\|_{\mathscr{L}^2([0,T];\mathbb{H})} < \infty. \tag{3}$$

Definition 2.3. *[12] Let $\varphi : [0,T] \to \mathscr{L}^0_H(\mathbb{K},\mathbb{H})$ satisfy equations (3). Then, its stochastic integral with respect to the fBm B^H_Q is defined, for $t \geq 0$, as follows:*

$$\begin{aligned}\int_0^t \varphi(s)dB^H_Q(s) &:= \sum_{n=1}^{\infty}\int_0^t \varphi(s)Q^{1/2}e^n\beta^H_n(s) \\ &= \sum_{n=1}^{\infty}\int_0^t (K^*_H(\varphi(s)Q^{1/2}e_n))dW(s). \end{aligned}\tag{4}$$

Note that if

$$\sum_{n=1}^{\infty} \|\varphi Q^{1/2}e_n\|_{(L^{1/H}([0,T];X)} < \infty,$$

then, in particularly, (3) holds, which follows immediately from (2). Now, we end this section by stating the following lemma which is fundamental to prove our result.

Lemma 2.4. *[11] If $\psi : [0, T] \to \mathscr{L}_2^0(\mathbb{K}, \mathbb{H})$ satisfies*

$$\int_0^T \|\psi\|_{\mathscr{L}_2^0} ds < \infty,$$

then the sum in (4) is well defined as an X−valued random variable, and we have

$$E\|\int_0^t \psi(s)dB^H(s)\|^2 \leq 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{\mathscr{L}_2^0} ds.$$

Further, we recollect some basic results related to resolvent operators. Regarding the theory of resolvent operators, we refer the reader to [16]. Let A and $B(t)$ are closed linear operators on $\mathbb{H}$ and $\mathbb{K}$ represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_{\mathbb{K}} := |Ay| + |y|, y \in \mathbb{K}.$$

The notations $C([0, +\infty); \mathbb{K})$ and $\mathscr{B}(\mathbb{K}, \mathbb{H})$ stand for the space of all continuous functions from $[0, +\infty)$ into $\mathbb{K}$ and the set of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$, respectively.

To obtain our results, we assume that the integro-differential abstract Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t G(t-s)v(s)ds, t \geq 0, \\ v(0) = v_0 \in \mathbb{H}. \end{cases} \tag{5}$$

has an associated resolvent operator of bounded linear operators $(R(t))_{t\geq 0}$ on $\mathbb{H}$.

Definition 2.5. *[16] A resolvent operator for (5) is a bounded linear operator valued function $R(t) \in \mathscr{L}(\mathbb{H})$ for $t \geq 0$, which satisfies the following properties:*

(i) $R(0) = I$ and $|R(t)| \leq Ne^{\beta t}$ for some constants N and β.

(ii) For each $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.

(iii) $R(t) \in \mathscr{L}(\mathbb{K})$ for $t \geq 0$. For $x \in \mathbb{K}$,

$$R(\cdot)x \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); \mathbb{K})$$

and

$$R'(t)x = AR(t)x + \int_0^t G(t-s)R(s)xds \tag{6}$$

$$= R(t)Ax + \int_0^t R(t-s)G(s)xds, t \geq 0. \tag{7}$$

The resolvent operators play an important role in obtaining variation of constants formula for nonlinear systems and in studying the existence of solutions [16]. For additions details related to resolvent of operator associated to integro-differential equations, see ([14–17]).

The following theorem will be used in this work to develop our main results. We assume that the following conditions hold.

(H1) A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{H}$.

(H2) For all $t \geq 0$, $G(t)$ is a closed linear operator from $D(A)$ to $\mathbb{H}$, and $G(t) \in \mathscr{B}(\mathbb{K}, \mathbb{H})$. For any $y \in \mathbb{K}$, the map $t \longmapsto G(t)y$ is bounded, differentiable, and the derivative $t \longmapsto G'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 2.6. *[16] Assume that* $(H1)$ *and* $(H2)$ *hold. Then there exists a unique resolvent operator for the Cauchy problem (5). In what follows, we establish some results for the existence of solutions of the following integro−differential equation:*

$$\begin{cases} v'(t) = Av(t) + \int_0^t G(t-s)v(s)ds, t \geq 0, \\ v(0) = v_0 \in \mathbb{H}, \end{cases} \tag{8}$$

where $q : [0, +\infty) \to \mathbb{H}$ *is a continuous function.*

Definition 2.7. *A continuous function* $v : [0, +\infty) \to \mathbb{H}$ *is said to be a strict solution of (8) if*

(i) $v \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); \mathbb{K})$,

(ii) *v satisfies (8) for $t \geq 0$.*

From Definition 2.7 , we deduce that $v(t) \in D(A)$, and the function $G(t-s)v(s)$ is integrable for all $t \geq 0$ and $s \in [0, t]$.

Theorem 2.8. *[16] Assume that* $(H1)$ *and* $(H2)$ *hold. If v is a strict solution of (8), then*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds, t \geq 0. \tag{9}$$

Now, we have the following definition for mild solution of (8).

Definition 2.9. *A function* $v : [0, +\infty) \to \mathbb{H}$ *is called a mild solution of (8) if v satisfies the variation of constants formula (9) for* $v_0 \in \mathbb{H}$.

The next theorem provides sufficient conditions for the regularity of solutions of (8). Namely, we establish a sufficient condition ensuring when a mild solution is a strict one.

Theorem 2.10. *[16] Let* $q \in C^1([0, +\infty); \mathbb{H})$, *and let v be defined by (9). If* $v_0 \in D(A)$, *then v is a strict solution of (8).*

3. Approximate controllability

In this section, we prove the result on approximate controllability of nonlinear stochastic systems. To do this, we first prove the existence of solutions using Banach fixed point theorem. Second, we show that under certain assumptions, the approximate controllability of (1) is implied by the approximate controllability of the corresponding linear system.

We improve the following hypotheses to prove our results:

(H3) The function $f : [0, T] \times \mathcal{D} \to \mathbb{H}$ satisfies the following conditions: there exist positive constants C_1 and C_2 such that, for all $t \in [0, T]$ and $x, y \in \mathcal{D}$,

$$\|f(t, x) - f(t, y)\| \leq C_1\|x - y\|,$$

$$\|f(t, x)\|^2 \leq C_2(1 + \|x\|^2).$$

(H4) The function $g : [0, T] \times \mathcal{D} \to \mathbb{H}$ satisfies the following conditions: there exist positive constants C_3 and C_4 such that, for all $t \in [0, T]$ and $x, y \in \mathcal{D}$,

$$\|g(t, x) - g(t, y)\| \leq C_3\|x - y\|,$$

$$\|g(t, x)\|^2 \leq C_4(1 + \|x\|^2).$$

(H5) The function g is continuous in the quadratic mean sense: for all $x \in \mathcal{D}_3([0,T], L^2(\Omega,\mathbb{H}))$,

$$\lim_{t\to s} \mathbb{E}\|g(t,x(t)) - g(s,x(s))\|^2 = 0.$$

(H6) The function $\sigma : [0,T] \to \mathscr{L}_2^0(\mathbb{K},\mathbb{H})$ satisfies

$$\int_0^t \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds < \infty, \quad \forall T > 0.$$

(H7) There exist some positive numbers q_k , $k = 1, ..., m$ such that

$$\|I_k(x) - I_k(y)\| \leq q_k\|x - y\|,$$

for all $x, y \in \mathbb{H}$.

We now introduce the concept of mild solution of (1).

Definition 3.1. *An $\mathbb{H}$-valued process $\{y(t), t \in [-\tau, T]\}$ is called a mild solution of (1) if $y \in \mathcal{D}_1$, $y(t) = \varphi(t)$ for $t \in [-\tau, 0]$, and, for $t \in [0,T]$, satisfies*

$$y(t) + g(t, y(t - r_1(t))) = R(t)[\varphi(0) - g(0, \varphi(-r_1(0)))] + \int_0^t R(t-s)f(s, y(s - r_2(s)))ds$$
$$+ \int_0^t R(t-s)Bu(s)ds + \int_0^t R(t-s)\sigma(s)dB^H(s) + \sum_{0<t_k<t} R(t,t_k)I_k(y(t_k)),\ t \in [0,T] \tag{10}$$

In order to study the approximate controllability for the system (1), we introduce the following linear differential system:

$$\begin{cases} dy(t) = Ay(t)dt + Bu(t)dt; \quad t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{11}$$

The controllability operator associated with (11) is defined by

$$\Gamma_0^T = \int_0^T R(T-s)BB^*R^*(T-s)ds,$$

where B^* and R^* denote the adjoint of B and R, respectively

Let $x_T(x_0; u)$ be the state value of (1) at terminal time T corresponding to the control u and the initial value $x_0 = \varphi$. Introduce the set

$$\Re(T, x_0) = \left\{x_b(x_0, u)(0) : u(.) \in L^2(J, U)\right\}$$

which is called the reachable set of system (1) at terminal time T, its closure in X is denoted by $\overline{\Re(T, x_0)}$

Definition 3.2. *The system (1) is said to be approximately controllable on the interval J if $\overline{\Re(T, x_0)} = L^2(\Omega,\mathbb{H})$.*

Lemma 3.3. *(see [36]) The system (1) is said to be approximately controllable on the interval $[0,T]$ if and only if $z(zI + \Gamma_0^T)^{-1} \to 0$ strongly as $z \to 0^+$;*

Lemma 3.4. *For any $\overline{x}_T \in L^2(\Omega,\mathbb{H})$ there exists $\overline{\varphi} \in L^2(\Omega; L^2([0,T], L_2^0)$; such that*

$$\overline{x}_T = E\overline{x}_T + \int_0^T \overline{\varphi}(s)dB^H(s).$$

Now for any $\delta > 0$ and $\overline{x}_T \in L^2(\Omega, \mathbb{H})$, we define the control function in the following form:

$$\begin{aligned}
u^\delta(t,x) &= B^*R^*(T-t)(zI+\Gamma_0^T)^{-1} \\
&\quad \times\Big[E\overline{x}_T - R(T)\left[\varphi(0) - g(0,\varphi(-r_1(0)))\right] + g(T,x_T)\Big] \\
&\quad +B^*R^*(T-t)\int_0^t (zI+\Gamma_0^T)^{-1}\overline{\varphi}(s)dB^H(s) \\
&\quad -B^*R^*(T-t)\int_0^t (zI+\Gamma_0^T)^{-1}R(T-s)f(s,x(s-r_2(s)))ds \\
&\quad -B^*R^*(T-t)\int_0^t (zI+\Gamma_0^T)^{-1}R(T-s)\sigma(s)dB^H(s) \\
&\quad -B^*R^*(T-t)\sum_{0<t_k<T}(zI+\Gamma_0^T)^{-1}R(T-t_k)I_k(x(t_k^-)).
\end{aligned}$$

Lemma 3.5. *There exists a positive real constant M_C such that, for all $x, y \in \mathcal{D}_2$, we have*

$$E|u^\delta(s,y) - u^\delta(s,x)|^2 \leq \frac{M_C}{z^2}\int_0^t E|y(s)-x(s)|^2 ds, \tag{12}$$

$$E|u^\delta(s,x)|^2 \leq \frac{M_C}{z^2}\left(1+\int_0^t E|x(s)|^2 ds\right). \tag{13}$$

Proof: The proof of this lemma is similar to the proof of the Lemma 2.5 in [38].

Theorem 3.6. *Suppose that (H1)-(H7) hold and that*

$$4\Big[C_3^2 + N^2C_1^2e^{2\beta T}T^2 + N^2e^{2\beta T}\|B\|^2T\frac{M_C}{z^2} + N^2e^{2\beta T}\sum_{k=1}^m q_k^2\Big] < 1 \tag{14}$$

then the system (1) has a unique mild solution on $[-\tau, T]$.

Proof:
Transform the problem (1) into a fixed point problem.
For any $\delta > 0$, consider the operator:

$$\Psi : \mathcal{D}_2 \to \mathcal{D}_2$$

defined by:

$$\Psi_\delta(y)(t) = \begin{cases} \varphi(t), & \text{if} \quad t \in [-\tau, 0]; \\ R(t)[\varphi(0) - g(0,\varphi(-r_1(0)))] - g(t, y(t-r_1(t))) & \\ +\int_0^t R(t-s)f(s,y(s-r_2(s)))ds + \int_0^t R(t-s)\sigma(s)dB^H(s) & \\ +\int_0^t R(t-s)Bu^\delta(s,x)ds + \sum_{0<t_k<t} R(t,t_k)I_k(y(t_k)). & \text{if} \quad t \in [0,T] \end{cases}$$

Further, the problem of finding the solution of problem (1) is reduced to finding the solution of the operator equation $\Psi_\delta(y)(t) = y(t)$, $t \in [-r, T]$.

Now, we will show that by using the Banach fixed point theorem that, for all $\delta > 0$, the operator Ψ_δ has a fixed point. This fixed point is then a solution of equation (1). To prove this result, we divide the subsequent proof into two steps.

Step 1 For arbitrary $y \in \mathcal{D}_2$, let us prove that $t \to \Psi_\delta(y)(t)$ is continuous on the interval $[0, T]$ in the $L^2(\Omega, \mathbb{H})$-sense. Let $0 < t < T$, and let $|h|$ be sufficiently small. Then, for any fixed $y \in \mathcal{D}_2$, we have

$$\begin{aligned}
&\|\Psi_\delta(y)(t+h) - \Psi_\delta(y)(t)\| \\
&\leq \|(R(t+h) - R(t))[\varphi(0) - g(0, \varphi(-r_1(0)))]\| \\
&+ \|g(t+h, y(t+h-r_1(t+h))) - g(t, y(t-r_1(t)))\| \\
&+ \|\int_0^{t+h} R(t+h-s) f(s, y(s-r_2(s)))ds - \int_0^t R(t-s) f(s, y(s-r_2(s)))ds\| \\
&+ \|\int_0^{t+h} R(t+h-s)\sigma(s)dB^H(s) - \int_0^t R(t-s)\sigma(s)dB^H(s)\| \\
&+ \|\int_0^{t+h} R(t+h-s)Bu^\delta(s,x) - \int_0^t R(t-s)Bu^\delta(s,x)\| \\
&+ \|\sum_{0<t_k<t+h} R(t+h-t_k)I_k(y(t_k)) - \sum_{0<t_k<t} R(t-t_k)I_k(y(t_k))\| \\
&=: \sum_{1\leq i\leq 6} \eta_i(h).
\end{aligned}$$

Using property (ii) of Definition (2.5), we obtain that

$$\lim_{h\to 0}(R(t+h) - R(t))(\varphi(0) - g(0, \varphi(-r(0)))) = 0.$$

Without loss of generality, we can assume that $\beta > 0$. Using property (i) of definition 2.5, we get

$$\begin{aligned}
&\|(R(t+h) - R(t))(\varphi(0) - g(0, \varphi(-r_1(0))))\| \\
&\leq [Ne^{\beta(t+h)} + Ne^{\beta t}]\|\varphi(0) - g(0, \varphi(-r_1(0)))\|.
\end{aligned}$$

Then, by the Lebesgue majorant theorem, we conclude that $\lim_{h\to 0} \mathbb{E}|\eta_1(h)|^2 = 0$. Moreover, assumption (H5) ensures that $\lim_{h\to 0} \mathbb{E}|\eta_2(h)|^2 = 0$. For the third term $\eta_3(h)$, we suppose $h > 0$ (similar estimates hold for $h < 0$). Then, we have

$$\begin{aligned}
\eta_3(h) \leq& \|\int_0^t (R(t+h-s) - R(t-s)) f(s, y(s-r_2(s)))ds\| \\
&+ \|\int_t^{t+h} R(t+h-s) f(s, y(s-r_2(s)))ds\| \\
&=: \eta_{31}(h) + \eta_{32}(h).
\end{aligned}$$

Thanks to Hölder's inequality, we have

$$\mathbb{E}|\eta_{31}(h)|^2 \leq t\mathbb{E}\int_0^t \|(R(t+h-s) - R(t-s)) f(s, y(s-r_2(s)))\|^2 ds.$$

Again exploiting properties (i) and (ii) of Definition 2.5, for each $s \in [0, t]$, we have

$$\lim_{h\to 0}(R(t+h-s) - R(t-s)) f(s, y(s-r_2(s))) = 0$$

and

$$\|(R(t+h-s) - R(t-s)) f(s, y(s-r_2(s)))\| \leq \widetilde{N}\|f(s, u(s-r_2(s)))\|,$$

where

$$\widetilde{N} = 2N^2 e^{2\beta(t+h)} + 2N^2 e^{2\beta t}.$$

Also, by the Lebesgue majorant theorem, we obtain

$$\lim_{h\to 0} E|\eta_{31}(h)|^2 = 0.$$

Next, using property (ii) of Definition 2.5 and Hölder's inequality, it follows that

$$E|\eta_{32}(h)|^2 \leq C_2 h N^2 e^{2\beta T} \int_0^T (E\|y(s - r_2(s))\|^2 + 1)ds,$$

and we have,

$$\lim_{h\to 0} E|\eta_{32}(h)|^2 = 0.$$

Now, for the term $\eta_4(h)$, we have

$$\begin{aligned}\eta_4(h) \leq& \|\int_0^t (R(t+h-s) - R(t-s))\sigma(s)dB^H(s)\| \\ &+ \|\int_t^{t+h} R(t+h-s)\sigma(s)dB^H(s)\| \\ =:& \eta_{41}(h) + \eta_{42}(h).\end{aligned}$$

Also, it follows from Lemma 2.4 that

$$E|\eta_{41}(h)|^2 \leq 2Ht^{2H-1} \int_0^t \|(R(t+h-s) - R(t-s))\sigma(s)\|^2_{\mathscr{L}_2^0} ds.$$

Since

$$\lim_{h\to 0} \|(R(t+h-s) - R(t-s))\sigma(s)\|^2_{\mathscr{L}_2^0} = 0$$

and

$$\|(R(t+h-s) - R(t-s))\sigma(s)\|_{\mathscr{L}_2^0} \leq [N^2 e^{2\beta(t+h)} + N^2 e^{2\beta t}]\|\sigma(s)\|^2_{\mathscr{L}_2^0},$$

the Lebesgue majorant theorem implies $\lim_{h\to 0} E|\eta_{41}(h)|^2 = 0$. Again by Lemma 2.4, we obtain

$$\mathbb{E}|\eta_{42}(h)|^2 \leq 2HN^2 e^{\beta(t+h)} h^{2H-1} \int_t^{t+h} \|\sigma(s)\|_{\mathscr{L}_2^0} ds \to 0, h \to 0.$$

and, for the terme $\eta_5(h)$, we have

$$\begin{aligned}\eta_5(h) \leq& \|\int_0^t (R(t+h-s) - R(t-s))Bu^\delta(s,x)ds\| \\ &+ \|\int_t^{t+h} R(t-s)Bu^\delta(s,x)ds\| \\ =:& \eta_{51}(h) + \eta_{52}(h).\end{aligned}$$

from lemma (3.5), we have

$$E|\eta_{51}(h)|^2 \leq t \int_0^t E\|(R(t+h-s) - R(t-s))Bu^\delta(s,x)\|^2 ds$$

and

$$E|\eta_{52}(h)|^2 \leq hN^2 e^{2\beta T}\|B\|^2 \int_t^{t+h} E\|u^\delta(s,x)\|^2 ds$$

Using the strong continuity of $R(t)$ and Lebesgue¢s dominated convergence theorem, we conclude that

$$\lim_{h\to 0} E|\eta_5(h)|^2 = 0$$

Now, for the term $\eta_6(h)$, we have

$$\begin{aligned}\eta_6(h) \leq & \|\sum_{0<t_k<t}(R(t+h-t_k)-R(t-t_k))I_k(y(t_k))\| \\ & +\|\sum_{t<t_k<t+h}R(t+h-t_k)I_k(y(t_k))\|\end{aligned} \tag{15}$$

By using the Hölder's inequality, we have

$$\mathbb{E}|\eta_{61}(h)|^2 \leq \sum_{0<t_k<t}\mathbb{E}\|(R(t+h-t_k)-R(t-t_k))I_k(y(t_k))\|^2 ds.$$

Again exploiting properties (i) and (ii) of Definition 2.5, for each $s\in[0,t]$, we have

$$\lim_{h\to 0}(R(t+h-t_k)-R(t-t_k))I_k(y(t_k)) = 0,$$

and

$$\|(R(t+h-t_k)-R(t-t_k))I_k(y(t_k))\| \leq \widetilde{N_1}\|I_k(y(t_k))\|,$$

where

$$\widetilde{N_1} = 2N^2e^{2\beta(t+h)} + 2N^2e^{2\beta t}$$

Then, by the Lebesgue majorant theorem, we obtain

$$\lim_{h\to 0} E|\eta_{61}(h)|^2 = 0.$$

Next, using property (ii) of Definition 2.5 and Hölder's inequality, it follows that

$$E|\eta_{62}(h)|^2 \leq N^2e^{2\beta(t+h)}\sum_{t<t_k<t+h}E\|I_k(y(t_k))\|^2$$

and then

$$\lim_{h\to 0} E|\eta_{62}(h)|^2 = 0.$$

The above arguments show that

$$\lim_{h\to 0} E\|\Psi(y)(t+h)-\Psi(y)(t)\|^2 = 0.$$

Hence, we conclude that the function $t\to\Psi(y)(t)$ is continuous on $[0,T]$ in the L^2-sense.

Step 2 Now, we prove that Ψ_δ is a contracting mapping in $\mathcal{D}_2$.

For every $x, y\in\mathcal{D}_2$ and $t\in[0,T]$, we obtain

$$\begin{aligned}\|\Psi(x)(t)-\Psi(y)(t)\|^2 \leq & 4\|g(t,x(t-r_1(t)))-g(t,y(t-r_1(t)))\|^2 \\ & +4\|\int_0^t R(t-s)(f(s,x(s-r_2(s)))-f(s,y(s-r_2(s))))ds\|^2 \\ & +4\|\int_0^t R(t-s)B\left[u^\delta(s,y)-u^\delta(s,x)\right]\|^2 \\ & +4\|\sum_{0<t_k<t}R(t-t_k)[I_k(x(t_k))-I_k(y(t_k))]\|^2\end{aligned}$$

Owing to the Lipschitz properties of f and g combined with Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\|\Psi(x)(t)-\Psi(y)(t)\|^2 \leq & 4C_3^2\mathbb{E}\|x(t-r_1(t))-y(t-r_1(t))\|^2 \\ & + 4N^2C_1^2e^{2\beta T}T\int_0^t \mathbb{E}\|x(s-r_2(s))-y(s-r_2(s))\|^2 ds. \\ & + 4N^2e^{2\beta T}\|B\|^2\frac{M_C}{z^2}\int_0^t E\|x(s)-y(s)\|^2 ds \\ & + 4N^2e^{2\beta T}\sum_{k=1}^m q_k^2\mathbb{E}\|x(t)-y(t)\|^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\sup_{s\in[-\tau,T]} E\|\Psi(x)(t)-\Psi(y)(t)\|^2 \leq & 4\Big[C_3^2 + N^2C_1^2e^{2\beta T}T^2 + N^2e^{2\beta T}\|B\|^2 T\frac{M_C}{z^2} \\ & + N^2e^{2\beta T}\sum_{k=1}^m q_k^2\Big]\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)-y(s)\|^2.\end{aligned}$$

Hence Ψ is a contraction mapping on $\mathcal{D}_2$ and therefore Ψ has a unique fixed point, which is a mild solution of (1) on $[-\tau, T]$. This completes the proof.

Theorem 3.7. *Assume that (H1)-(H6) are satisfied. Further, if the functions f and g are uniformly bounded, and $R(t)$ is compact, then the system (1) is approximately controllable on* $[0, T]$.

Proof. Let x_δ be a fixed point of Ψ_δ . By using the stochastic Fubini theorem, it can easily be seen that

$$\begin{aligned}x_\delta(T) &= \overline{x}_T - z(zI+\Gamma_0^T)^{-1}\Big\{E\overline{x}_T - R(T)\big[\phi(0)-g(0,\varphi(-r_1(0)))\big] \\ &\quad +g(T,x_\delta(T)) + \int_0^T \overline{\varphi}(s)dB^H(s)\Big\} \\ &\quad +z\int_0^T (zI+\Gamma_0^T)^{-1}R(T-s)f(s,x_{r_2}(s-v(s)))ds \\ &\quad +z\int_0^T (zI+\Gamma_0^T)^{-1}R(T-s)\sigma(s)dB^H(s) \\ &\quad + \sum_{0<t_k<T} z(zI+\Gamma_0^T)^{-1}R(T-t_k)I_k(x_\delta(t_k^-)).\end{aligned}$$

□

It follows from the assumption on f and g that there exists $\overline{D} > 0$ such that

$$\|f(x,x_\delta(s_v(s)))\|^2 + \|g(x,x_\delta(s_r(s)))\|^2 \leq \overline{D} \tag{16}$$

for all $(s, w) \in [0, T]\times\Omega$. Then there is a subsequence still denoted by $\{f(s, x_\delta(s-v(s))), g(s, x_\delta(s-r(s)))\}$ which converges weakly to, say, $\{f(s), g(s)\}$ in $\mathbb{H}\times L_2^0$.

From the above equation, we have

$$\begin{aligned} E\|x_\delta(T)-\overline{x}_T\|^2 \leq\ & 5E\Big\|z(zI+\Gamma_0^T)^{-1}\Big\{E\overline{x}_T - R(T)\Big[\phi(0)-g(0,\varphi(-r_1(0)))\Big] \\ & +g(T,x_\delta(T))+\int_0^T \overline{\varphi}(s)dB^H(s)\Big\}\Big\|^2 \\ & +5E\left(\int_0^T \Big\|z(zI+\Gamma_0^T)^{-1}R(T-s)[f(s,x_\delta(s-r_2(s)))-f(s)]\Big\|\,ds\right)^2 \\ & +5E\left(\int_0^T \Big\|z(zI+\Gamma_0^T)^{-1}R(T-s)f(s)\Big\|\right)^2 ds \\ & +10HT^{2H-1}\int_0^T \|z(zI+\Gamma_0^T)^{-1}R(T-s)\sigma(s)\|^2_{\mathscr{L}_2^0}ds \\ & +5E\Big\|\sum_{0<t_k<T} z(zI+\Gamma_0^T)^{-1}R(T-t_k)I_k(x_\delta(t_k^-))\Big\|^2. \end{aligned}$$

On the other hand, by Lemma 3.3, the operator $z(zI+\Gamma_0^T)^{-1} \to 0$ strongly as $z \to 0^+$ for all $0 \leq s \leq T$, and, moreover, $\|z(zI+\Gamma_0^T)^{-1}\| \leq 1$. Thus, by the Lebesgue dominated convergence theorem the compactness of $R(t)$ implies that $E\|x_\delta(T)-\overline{x}_T\|^2 \to 0$ as $z \to 0^+$. This gives the approximate controllability of (1).

Remark 3.8. *If the system (1) is without impulse, that is $q_k = 0, k = 1, ..., m$. The system (1) becomes in the form of the following neutral stochastic integrodifferential equation:*

$$\begin{cases} d[y(t)+g(t,y(t-r_1(t)))] = A[y(t)+g(t,y(t-r_1(t)))]dt + Bu(t)dt \\ +\int_0^t G(t-s)[y(s)+g(s,(s-r_1(s)))]dsdt + f(t,y(t-r_2(t)))dt \\ +\sigma(t)dB^H(t), t\in[0,T], \\ y(t)=\varphi(t), -\tau\leq t\leq 0, \end{cases} \tag{17}$$

where the operators A, B, G, the functions $r_1, r_2, f, g, \sigma, u$ are defined as same as before. Here $C = \{y : [-\tau, T] \to \mathbb{H} : y(t)$ is continuous$\}$, the Banach space of all stochastic processes $y(t)$ from $[-\tau, T]$ into $\mathbb{H}$, endowed with the norm $\|\phi\|_C^2 = \sup_{\theta\in[-\tau,T]} \|\phi(\theta)\|^2$, for $\phi \in C$. By using the same technique in Theorem 3.6 and Theorem 3.7, we can easily deduce the following corollary.

Corollary 3.9. *Suppose that (H1)-(H2) and (H3)-(H6) hold. Then, the system (17) is approximately controllable on $[0, T]$, provide that the condition (14) is satisfied.*

Remark 3.10. *The use of a nonlocal condition has a better effect on the solution and is more precise for physical measurements than the classical condition $y(0) = y_0$ alone [37]. For example, for the nonlocal condition $y(0) + g(y) = \phi$, where $g : C([0, T], \mathbb{H}) \to \mathbb{H}$ is a given function which satisfies some appropriate conditions and $\phi \in \mathcal{D}_1$. The function g can be written as*

$$g(y) = \sum_{k=1}^{m} c_k y(t_k),$$

where c_k, for $k = 1, 2, \cdots, n$; are given constants and $0 < t_1 < \cdots < t_n \leq T$.

Approximate Controllability problems with non local conditions for different kinds of dynamical systems have been studied by several authors (see [1, 10, 40, 41]) and references therein. However, the approximate controllability of neutral impulsive stochastic integro-differential systems driven by a fBm is an untreated topic in the literature so far. Upon making some appropriate assumption on system functions, by adapting the techniques and ideas established in this paper with suitable modifications, one can prove the approximate controllability of impulsive neutral stochastic functional integro-differential equations driven by a fractional Brownian motion (1) with non local conditions.

4. Example

In this section, we present an example to illustrate our main result. we consider the following stochastic impulsive partial neutral functional integro-differential equation with finite delays r_1 and r_2 ($\infty > \tau > r_i \geq 0, i = 1, 2$) driven by a fractional Brownian motion:

$$\begin{cases} \frac{\partial}{\partial t}[y(t,\xi) + \hat{G}(t, y(t-r_1,\xi))] \\ = \frac{\partial^2}{\partial \xi^2}[y(t,\xi) + \hat{G}(t, y(t-r_1,\xi))] + \mu(t,\xi) \\ + \int_0^t b(t-s)\frac{\partial^2}{\partial \xi^2}[y(s,\xi) + \hat{G}(s, y(s-r_1,\xi))]ds \\ +\hat{F}(t, y(t-r_2,\xi)) + \sigma(t)\frac{dB^H}{dt}(t), \\ I_k(y(t_k^-,\xi)) = \int_0^\pi K(t_k,\xi,x)y(t_k,x)dx, \quad k = 1,\cdots,m \\ y(t,0) + \hat{G}(t, y(t-r_1,0)) = 0, t \geq 0, \\ y(t,\pi) + \hat{G}(t, y(t-r_1,\pi)) = 0, t \geq 0, \\ y(\theta,\xi) = \varphi(\theta,\xi), -\tau \leq \theta \leq 0 \ \ a.s. \end{cases} \tag{18}$$

where $B^H(t)$ denotes a fractional Brownian motion, $\hat{G}, \hat{F} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions and $\varphi : [-\tau, 0] \times [0, \pi] \to \mathbb{R}$ is a given continuous function such that $\varphi(s, .) \in L^2[0, T]$ is measurable and satisfies $E\|\varphi\|^2 < \infty$.

We rewrite (18) into abstract form of (1), let $\mathbb{H} = L^2([0, \pi])$. Define the operator $A : D(A) \subset \mathbb{H} \longrightarrow \mathbb{H}$ given by $Ay = \frac{\partial^2}{\partial \xi^2}$ with domain

$$D(A) = H^2([0,\pi]) \cap H_0^1([0,\pi]),$$

then we get

$$Ay = \sum_{n=1}^{\infty} n^2 \langle y, e_n \rangle e_n, \forall y \in\in D(A),$$

where $e_n := \sqrt{\frac{2}{\pi}} \sin ny$, $n = 1, 2, \cdots$. is an orthogonal set of eigenvector of $-A$.
It is well known that A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)_{t\geq 0}$ on $\mathbb{H}$, thus, (H1) is true. Furthermore Furthermore, $T(t)_{t\geq 0}$ is given

$$T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t} \langle y, e_n \rangle e_n,$$

for $y \in \mathbb{H}$ and $t \geq 0$, that satisfies $\|T(t)\| \leq e^{-\pi^2 t}$ for every $t \geq 0$.

Define an infinite-dimensional space $\mathbb{U}$ by $\mathbb{U} = \left\{ u : u = \sum_{n=2}^{\infty} u_n w_n \quad \text{with } \sum_{n=2}^{\infty} u_n^2 < \infty \right\}$. The norm in $\mathbb{U}$ is defined by $\|u\|_{\mathbb{U}}^2 = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$. Now, define a continuous linear mapping B from $\mathbb{U}$ into $\mathbb{H}$ as $Bu = 2u_2 w_1 + \sum_{n=2}^{\infty} u_n w_n$ for $u = \sum_{n=2}^{\infty} u_n w_n \in \mathbb{U}$.
We assume that the following conditions hold:

(i) Let bounded linear operator $B : \mathbb{U} \to \mathbb{H}$ be defined by $Bu(t)(\xi) = \mu(t,\xi)$, $0 \leq \xi \leq \pi$.

(ii) For $t \in [0, T]$, $\hat{F}(t, 0) = \hat{G}(t, 0) = 0$.

(iii) There exist positive constants C_1, and C_3, such that

$$|\hat{F}(t,\xi_1) - \hat{F}(t,\xi_2)| \leq C_1|\xi_1 - \xi_2|, \text{ for } t \in [0,T] \text{ and } \xi_1, \xi_2 \in \mathbb{R},$$

$$|\hat{G}(t,\xi_1) - \hat{G}(t,\xi_2)| \leq C_3|\xi_1 - \xi_2|, \text{ for } t \in [0,T] \text{ and } \xi_1, \xi_2 \in \mathbb{R}.$$

(iv) There exist positive constants C_2 and C_4, such that

$$|\hat{F}(t,\xi)| \leq C_2(1+|\xi|^2), \text{ for } t \in [0,T] \text{ and } \xi \in \mathbb{R},$$

$$|\hat{G}(t,\xi)| \leq C_4(1+|\xi|^2), \text{ for } t \in [0,T] \text{ and } \xi \in \mathbb{R}.$$

(v) The function $\sigma : [0,T] \longrightarrow \mathscr{L}_2^0(\mathbb{K},\mathbb{H})$ satisfies

$$\int_0^t \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds < \infty, \quad \forall T > 0.$$

(vi) $K(t,\xi,y) : J \longrightarrow L^2([0,\pi]\times[0,\pi])$ is measurable and continuous, thus bounded. $l_k := \int_0^\pi \int_0^\pi |K(t_k,\xi,x)|^2 d\xi dx, k = 1,2,\cdots,m$, we have

$$\|I_k(\xi_1) - I_k(\xi_2)\| \leq q_k \|\xi_1 - \xi_2\|, \text{ and } \xi_1, \xi_2 \in \mathbb{R},$$

where $q_k = \pi l_k$.

Define the operators $f, g : \mathbb{R}^+ \longrightarrow L^2([0,\pi]) \longrightarrow L^2([0,\pi])$ by

$$f(t,\phi)(\xi) = \hat{F}(t,\phi(-\tau_1)(\xi)) \text{ for } \xi \in [0,\pi] \text{ and } \phi \in L^2([0,\pi]),$$

$$g(t,\phi)(\xi) = \hat{G}(t,\phi(-\tau_2)(\xi)) \text{ and } \phi \in L^2([0,\pi]),$$

and

$$I_k(\psi)(\xi) = \int_0^\pi K(t_k,\xi,y)\psi(x)dx, \ \ \varphi \in \mathbb{H},$$

If we put

$$\begin{cases} y(t)(\zeta) = y(t,\zeta), \ \ t \in [0,T] & \text{and; } \zeta \in [0,\pi] \\ y(t)(\zeta) = \varphi(t,\zeta), \ \ t \in [-\tau,0] & \text{and; } \zeta \in [0,\pi] \end{cases}$$

Thus the problem (18) can be written in the abstract form

$$\begin{cases} d[y(t) + g(t,y(t-r_1(t)))] = A[y(t) + g(t,y(t-r_1(t)))]dt + Bu(t)dt \\ + \int_0^t G(t-s)[y(s) + g(s,(s-r_1(s)))]dsdt + f(t,y(t-r_2(t)))dt \\ +\sigma(t)dB^H(t), t \in [0,T], t \neq t_k, \\ \Delta y(t_k) = y(t_k^+) - y(t_k) = I_k(y(t_k)), k = 1,\dots,m, \\ y(t) = \varphi(t), -\tau \leq t \leq 0, \end{cases} \tag{19}$$

Moreover, if b is bounded and C^1 such that b' is bounded and uniformly continuous, then (H.2) is satisfied, hence equation (18) has a resolvent operator $(R(t))_{t\geq 0}$ on $\mathbb{H}$. Besides, the continuity of $\hat{F}$ and $\hat{G}$ and assumption (*ii*) it ensues that f and g are continuous. In accordance with assumption (*vi*) we obtain

$$\|f(t,\phi_1) - f(t,\phi_2)\|_{L^2([0,\pi])} \leq C_1\|\phi_1 - \phi_2\|_{L^2([0,\pi])},$$
$$\|g(t,\phi_1) - g(t,\phi_2)\|_{L^2([0,\pi])} \leq C_3\|\phi_1 - \phi_2\|_{L^2([0,\pi])}.$$

Furthermore, by Assumption (*iv*), it follows that

$$\|f(t,\phi_1)\|_{L^2([0,\pi])} \leq C_2(1+\|\phi\|^2) \text{ and } \|g(t,\phi_1)\|_{L^2([0,\pi])} \leq C_4(1+\|\phi\|^2),$$

By condition (*vi*) we have

$$\|I_k(\phi_1) - I_k(\phi_2)\|_{L^2([0,\pi])} \leq q_k\|\phi_1 - \phi_2\|_{L^2([0,\pi])}.$$

Moreover, it is possible to choose the constants in such way that:

$$4\Big[C_3^2 + N^2 C_1^2 e^{2\beta T} T^2 + N^2 e^{2\beta T} ||B||^2 T \frac{M_C}{z^2} + N^2 e^{2\beta T} \sum_{k=1}^{m} q_k^2\Big] < 1.$$

Thus, all the assumptions of Theorem 3.7 are fulfilled. Consequently, the system (18) is approximately controllable on $[0, T]$.

5. Conclusion

In this paper, we study the approximate controllability of retarded impulsive stochastic integro-differential equations driven by fractional Brownian motion in Hilbert space. We give sufficient conditions ensuring the existence and uniqueness of a mild solution and the approximate controllability to the considered system by using the fixed point approach. Our future work will be focused on investigate the approximate and complete controllability for impulsive stochastic inclusions driven by a fractional Brownian motion with infinite delay.

Acknowledgements
The authors would like to thank the Editor and the anonymous referees for their careful reading and suggestions that led to a substantial improvement of the presentation of the early paper.

References

[1] H. M. Ahmed, Semilinear neutral fractional stochastic integro-differential equations with nonlocal conditions,*J. Theoret. Probab.* 28(2015), no. 2, 667-680.
[2] H. M. Ahmed, Approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space, *Advances in Difference Equations*, 2014, 113:1-11.
[3] H. M. Ahmed, Controllability of impulsive neutral stochastic differential equations with fractional Brownian motion, *IMA Journal of Mathematical Control and Information*, 2014, 1-14
[4] Alos E, Mazet O, Nualart D. Stochastic calculus with respect to Gaussian processes.*Ann Probab*, 1999, 29: 766-801.
[5] P. Balasubramaniam, J.Y. Park , A. Vincent Antony Kumar, Existence of solutions for semilinear neutral stochastic functional differential equations with nonlocal conditions. *Nonlinear Analysis* , 71 (2009) 1049-1058
[6] Benchohra, M., Henderson, J. & Ntouyas, S. K. *Impulsive Differential Equations and Inclusions*. (2006), New York: Hindawi Publishing Corporation.
[7] F. Biagini, Y. Hu, B. ksendal, T. Zhang *Stochastic Calculus for Fractional Brownian Motion and Applications*, Springer-Verlag (2008)
[8] Boudaoui, A., Lakhel, E., Controllability of Stochastic Impulsive Neutral Functional Differential Equations Driven by Fractional Brownian Motion with Infinite Delay *E. Differ Equ Dyn Syst* (2017).
[9] Boudaoui, A., Slama, A., Approximate controllability of stochastic impulsive integro differential systems with infinite delay, *Advances in Differential Equations and Control Processes*, Volume 13, 2014, 1-19
[10] Boudaoui, A., Slama, A., Approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay, *Nonlinear Dynamics and Systems Theory*, 16 (1) (2016) 35-48
[11] Boufoussi B, Hajji S. Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space. *Statist Probab Lett*, 2012, 82: 1549-1558.
[12] Caraballo T, Garrido-Atienza M J, Taniguchi T. The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion. *Nonlinear Anal*, 2011, 74: 3671-3684.
[13] de la Fuente I M, Perez-Samartin A L, Martinez L, Garcia M A, Vera-Lopez A. Long-range correlations in rabbit brain neural activity. *Ann Biomed Eng*, 2006, 34(2): 295-299
[14] Grimmer R.and F. Kappel, Series expansions for resolvents of Volterra integrodifferential equations in Banach space, SIAM Journal on Mathematical Analysis, 15 (1984), 595-604.
[15] Grimmer R. and J. Pruss, On linear Volterra equations in Banach spaces. Hyperbolic partial differential equations, II. Computers and Mathematics with Applications, 11 (1985), 189-205.
[16] Grimmer R. Resolvent operators for integral equations in a Banach space.*Trans Amer Math Soc*, 1982, 273: 333-349.
[17] R. Grimmer, A.J. Pritchard, Analytic resolvent operators for integral equations in a Banach space, *J. Differential Equations* 50 (1983) 234-259.
[18] Huan, D. D. , R. P. Agarwal and H. Gao, Approximate Controllability for Time-Dependent Impulsive Neutral stochastic partial differential equations with memory. *Filomat* 31:11 , 3433-3442, 2017.
[19] Huan DD, Gao H. Controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with delay and Poisson jumps.*Cogent Engineering*. 2015;2:16.

[20] Huang, Hao, et al. Existence and controllability of second-order neutral impulsive stochastic evolution integro-differential equations with state-dependent delay. *Journal of Fixed Point Theory and Applications* 20.1 (2018): 9.
[21] Kilbas, A. A., Srivastava H. M., and Trujillo J. J., *Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies*, vol. 204, Elsevier, Amsterdam, 2006.
[22] Kolmogorov, A.N.: Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum.*C.R.(Doklady) Acad. URSS (N.S)* 26, 115-118, 1940.
[23] Lakshmikantham, V., Bainov, D., Simeonov, P., 1989. *Theory of Impulsive Differential Equations*. World Scientific Press, Singapore.
[24] Lakhel, E., Controllability of neutral stochastic functional integro-differential equations driven by fractional Brownian motion, *Stoch. Anal. Appl.* 34 (3) (2016), pp. 427-440.
[25] Lakhel, E., Controllability of Neutral Functional Differential Equations Driven by Fractional Brownian Motion with Infinite Delay, *Nonlinear Dynamics and Systems Theory*, Volume 17, (3), (2017), 291-302.
[26] Lakhel, E., McKibben M A., Existence of solutions for fractional neutral functional differential equations driven by fBm with infinite delay. *Stochastics*, 2018, 90(3): 313-329.
[27] Lakhel, E., Controllability of impulsive neutral stochastic integro-differential systems driven by FBM with unbounded delay. *Le Matematiche* 73.2 (2018): 319-339.
[28] Lin, A. , Ren, Y., Xia, N., On neutral impulsive stochastic integro-differential equations with infinite delays via fractional operators *Math. Comput. Modelling*, 51 (5-6) (2010), 413-424
[29] Lo, A.W., Long-term memory in stock market prices. *Econometrica* 59, 1279-1313 (1991)
[30] Mandelbrot, B. B.and Van Ness, J.W., Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* 10, 422-437, 1968.
[31] Mao, X., *Stochastic Differential Equations and Applications*, Horwood, Chichestic, UK, 1997.
[32] Mishura Y., *Stochastic Calculus for Fractional Brownian Motion and Related Topics*. Lecture Notes in Mathematics, Vol 1929. Berlin: Springer-Verlag, 2008.
[33] Nguyen Tien D,Neutral stochastic differential equations driven by a fractional Brownian motion with impulsive effects and varying-time delays,*Journal of the Korean Statistical Society* ,2014.
[34] Nualart D. *The Malliavin Calculus and Related Topics. 2nd ed*. Berlin: Springer-Verlag, 2006 760.
[35] Park, J. Y., Balachandran, K., Annapoorani, N. Existence results for impulsive neutral functional integrodifferential equations with infinite delay. *Nonlinear Analysis*, 71, 3152-3162, 2009.
[36] Sakthivel, R., Ganesh, R., Ren, Y., Anthoni, SM., Approximate controllability of nonlinear fractional dynamical systems. *Commun. Nonlinear Sci. Numer. Simul.* 18, 3498-3508, 2013.
[37] Sakthivel, R.; Ganesh, R.; Anthoni, S.M.: Approximate controllability of fractional nonlinear differential inclusions. *Appl.Math. Comput.* 225, 708-717 (2013)
[38] Sakthivel, R. , Approximate controllability of impulsive stochastic evolution equations. *Funkc. Ekvacioj* 52, 381-393, 2009.
[39] Sakthivel,R., Ren, Y., Complete controllability of stochastic evolution equations with jumps. *Reports on mathematical physics*, Vol. 68, 163-174, 2011.
[40] Slama, A., Boudaoui,A., Approximate controllability of fractional impulsive neutral stochastic integro-differential equations with nonlocal conditions and infinite delay, *Annals of Differential Equation* 2 127-139. 2015.
[41] Slama, A., Boudaoui,A., Approximate controllability of fractional nonlinear neutral stochastic differential inclusion with nonlocal conditions and infinite delay, *Arabian Journal of Mathematics*, 1-24, 2017.
[42] Tindel S, Tudor C, Viens F. Stochastic evolution equations with fractional Brownian motion. *Probab Theory Related Fields*, 127(2): 186-204, 2003.
[43] Yan, Z., and Lu, F., The optimal control of a new class of impulsive stochastic neutral evolution integro-differential equations with infinite delay.*International Journal of Control*, 89.8: 1592-1612, 2016.
[44] Willinger W., Leland W, Taqqu M, Wilson D. On self-similar nature of ethernet traffic.*IEEE/ACM Trans Networking*, 2: 1-15, 1994.