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Non-Linear Multi-Point Flux Approximation in the Near-Well Region

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Abstract. We consider non-linear multi-point flux approximations (MPFA) scheme for flow simulations in a model of anisotropic porous medium that includes wells. The hydraulic head varies logarithmically and its gradient changes rapidly in the well vicinity. Due to this strong non-linearity of the near-well flow, use of the MPFA scheme in the near well region results in a completely wrong total well flux and an inaccurate hydraulic head distribution. In this article we propose correction of the MPFA scheme. The outcome is a scheme that is second-order accurate even in the well vicinity for anisotropic medium. Solution obtained with this scheme respects minimum and maximum principle, and also, it is non-oscillating.

1. Introduction

Groundwater flow movement is modeled by unsaturated groundwater flow equation (Richards equation), which is obtained from the conservation law, Boussinesq approximation and Darcy's law [16]

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (k_{\mathrm{r}}(h)\mathbb{K}(\mathbf{x})\nabla h) + g_s,$$

where θ is the water content, k_{r} is relative conductivity, $\mathbb{K}(\mathbf{x})$ is symmetric and positive definite hydraulic conductivity tensor, h is the hydraulic head, and term g_s describes sources and sinks.

In this paper we will examine stationary version of saturated groundwater flow, which is described with

$$-\nabla \cdot (\mathbb{K}\nabla h) = g_s. \tag{1}$$

Together with this equation we examine Dirichlet boundary condition

$$h = g_{\mathrm{D}}, \quad \text{on} \quad \Gamma_{\mathrm{D}}.$$

Hydraulic head varies logarithmically and its gradient changes sharply in the well vicinity. Thus, linear approximation of hydraulic head is inappropriate and numerical methods based on it are inaccurate in the

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near-well region. Local grid refinement can alleviate the problem, but this comes at a computational cost. In the reservoir engineering, accurate well modeling is crucial for reliable fluid flow simulations. Flow in the entire reservoir is induced mainly by wells, therefore poor near-well modeling results in accuracy loss throughout the model.

In this paper we are examining non-linear MPFA (multi-point flux approximation) scheme [5, 7, 11–13, 15]. Flux approximation in this scheme is obtained as a linear combination of one-sided flux approximations, such that obtained scheme preserves discrete maximum and minimum principles. The non-linear MPFA scheme is second order accurate [5]. Nevertheless, linear approximation is employed and therefore the accuracy is lost if a well is present. In this paper we are examining a way to enhance this scheme in the near-well region.

Methods for well modeling have been widely discussed in the literature [1–4, 9, 10, 14]. Commonly used method is the Peaceman model [1, 14] which greatly improves flow rate, but it does not improve order of accuracy in the near-well region.

Near-well correction (NWC) scheme presented in [3, 4] for non-linear two-point (TPFA) scheme is also applicable for multi-point scheme. Idea of this scheme is similar to those presented in [2], i.e. to split flux in the well vicinity into a linear part and a part that is due to the influence of the well. The NWC scheme is further generalized in [9, 10] for polyhedral grids and arbitrary wells. This correction changes only the approximation of one-sided linear fluxes, but use the same logic for their combining as the non-linear multi-point scheme.

Obtained results indicate that NWC scheme gives not only improved well extraction rate, but also obtained hydraulic head is second order accurate.

2. Exact solution

Exact solution of equation (1) when a well is present in a domain exist only in a limited number of cases. For a isotropic ($\mathbb{K} = K\mathbb{I}$) circular domain with a well in center of it, exact solution is given with

$$h = C_0 \ln \rho(\mathbf{x}) + C_1, \tag{2}$$

where C_0 and C_1 are constants dependent on boundary conditions, and

$$\rho(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_{\mathrm{w}}\|,$$

is the Euclidean distance from the well center $\mathbf{x}_{\mathrm{w}}$.

In the anisotropic case solution can be found using coordinate transformation

$$\overline{\mathbf{x}} = S(\mathbf{x} - \mathbf{x}_{\mathrm{w}}), \tag{3}$$

where S is coordinate transformation obtained from tensor $\mathbb{K}$ as in [4] such that $\mathbb{K} = S^{-1}(S^{-1})^{\mathrm{T}}$. Applying this transformation to equation (1) gives

$$-\nabla \cdot (\mathbb{K}\nabla h) = -\overline{\nabla} \cdot (S\mathbb{K}S^{\mathrm{T}}\overline{\nabla} h) = \overline{\nabla} \cdot (\overline{\nabla} h) = g_s,$$

where $\overline{\nabla}$ is the nabla operator in the new coordinate system. Therefore, exact solution is

$$h = C_0 \ln \overline{\rho}(\mathbf{x}) + C_1,$$

where

$$\overline{\rho}(\mathbf{x}) = \|S(\mathbf{x} - \mathbf{x}_{\mathrm{w}})\|,$$

and $\|\cdot\|$ denotes standard Euclidean norm. Therefore, $\overline{\rho}(\mathbf{x})$ is the distance of point $\mathbf{x}$ from the well center $\mathbf{x}_{\mathrm{w}}$ in the new coordinate system.

3. Discretization

Integrating (1) over each cell T of the mesh $\mathcal{M}$ gives

$$-\int_T \nabla\cdot(\mathbb{K}\nabla h)\mathrm{d}T = \int_T g_s.$$

Applying divergence theorem yields

$$\sum_{f\in\partial T} u_{f,T} = \int_T g_s \mathrm{d}T, \quad \text{where} \quad u_{f,T} = \int_f (-\mathbb{K}\nabla h)\cdot \mathbf{n}_{f,T}\mathrm{d}s. \tag{4}$$

Term $u_{f,T}$ denotes the flux through face f, an outer unit vector normal of cell T to face f is denoted with $\mathbf{n}_{f,T}$. For inner face $f = \partial T^+ \cap \partial T^-$ we demand local conservation, i.e.

$$u_{f,+} = -u_{f,-}.$$

Sign difference comes from the difference in normals sign $\mathbf{n}_{f,+} = -\mathbf{n}_{f,-}$.

We want a scheme that satisfies discrete maximum and minimum principle even for a very anisotropic tensor. It has been proved in [8] that on square meshes with a very anisotropic tensor linear nine-point finite volume scheme can not preserve maximum and minimum principle. Therefore, in order to fulfill this request we look for a non-linear MPFA scheme. We also want that scheme is accurate in the near-well region. Thus, we will develop correction of the MPFA scheme in the near-well region.

Scheme will preserve maximum and minimum principle [6] if the flux approximation (4) is given in the form

$$u_{f,T} = \sum_i \nu_i (h_T - h_i), \tag{5}$$

where $\nu_i > 0$ if $f = \partial T \cap \partial i$, else $\nu_i \geq 0$.

The hydraulic head varies logarithmically in the well vicinity therefore we will approximate it as the sum of linear and logarithmic function

$$h \approx L + C_0 \ln \overline{\rho}(\mathbf{x}),$$

where L is linear function and C_0 is an arbitrary constant. Therefore, approximation of the hydraulic head gradient is

$$\nabla h \approx \mathbf{G} + C_0 \nabla(\ln \overline{\rho}(\mathbf{x})),$$

where $\mathbf{G}$ is the gradient of the linear function L, thus $\mathbf{G}$ is constant. Therefore, flux $u_{f,T}$ in (4) can be written

$$u_{f,T} = -\int_f (-\mathbb{K}\nabla h)\cdot \mathbf{n}_{f,T}\mathrm{d}s \approx -|f|(\mathbb{K}\mathbf{G}_{f,T})\cdot \mathbf{n}_{f,T} - \int_f C_0(\mathbb{K}\nabla(\ln\overline{\rho}(\mathbf{x})))\cdot \mathbf{n}_{f,T}\mathrm{d}s.$$

Using coordinate transformation (3) for the integral, this approximation becomes

$$u_{f,T} \approx -|f|(\mathbb{K}\mathbf{G}_f)\cdot \mathbf{n}_{f,T} - C_0|\det(S^{-1})| \int_f (\overline{\nabla}(\ln\overline{\rho}(\mathbf{x})))\cdot \overline{\mathbf{n}}_{f,T}\mathrm{d}\overline{s}. \tag{6}$$

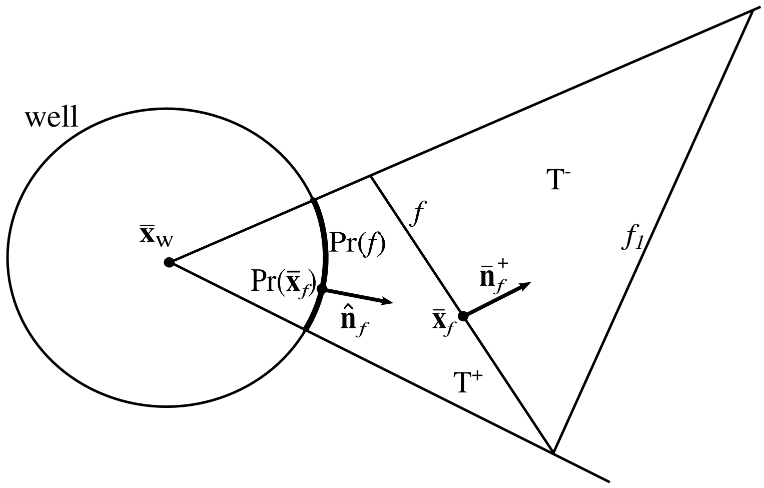

Figure 1: Radial projection onto the well.

Let us denote with Pr(f) the central projection of f onto the set $\|\bar{\mathbf{x}}\|^2 = r^2$ from $\bar{\mathbf{x}}_W$ (see Fig. 1), where r is the radius of the well. The radial flow component of the fluxes through faces f and f_1 is the same as through Pr(f)

$$C_0|\det(S^{-1})| \int_f (\overline{\nabla}(\ln \overline{\rho}(\mathbf{x}))) \cdot \overline{\mathbf{n}}_{f,T} \mathrm{d}\bar{s} = C_0|\det(S^{-1})|\sigma_f \int_{\mathrm{Pr}(f)} (\overline{\nabla}(\ln \overline{\rho}(\mathrm{Pr}(\mathbf{x}))) \cdot \hat{\mathbf{n}}_f \mathrm{d}\bar{s},$$

where $\hat{\mathbf{n}}_f$ is the outer unit normal on the set $\|\bar{\mathbf{x}}\|^2 = r^2$ at point $\mathrm{Pr}(\bar{\mathbf{x}}_f)$ in the new coordinate system, and $\sigma_f = -1$ if $\overline{\mathbf{n}}_{f,T}$ is in the half-space defined by f and $\bar{x}_W$, or $\sigma_f = 1$ otherwise. Note that

$$\overline{\nabla}(\ln \overline{\rho}(\mathrm{Pr}(\mathbf{x}))) \cdot \hat{\mathbf{n}}_f = \frac{1}{r}$$

Therefore, approximation (6) becomes

$$u_{f,T} \approx -|f|(\mathbb{K}\mathbf{G}_{f,T}) \cdot \mathbf{n}_{f,T} - C_0|\det(S^{-1})|\sigma_f \frac{|\mathrm{Pr}(f)|_S}{r}. \tag{7}$$

In this flux approximation, for any inner face $f = \partial T^+ \cap \partial T^-$, flux approximation unknowns

$$\mathbf{X} = \begin{bmatrix} \mathbf{G}_{f,+} \\ C_0 \end{bmatrix}$$

are found from a system of linear equations

$$\mathbf{AX} = \mathbf{b}, \tag{8}$$

which consist of three (in two dimensional problems) or four (in three dimensional problems) equations of the following type

$$\mathbf{G}_{f,+} \cdot (\mathbf{x}_i - \mathbf{x}_+) + C_0 \ln \frac{\overline{\rho}(\mathbf{x}_i)}{\overline{\rho}(\mathbf{x}_+)} = h_i - h_+, \quad \text{where} \quad i \in \mathcal{M},$$

$$\mathbf{G}_{f,+} \cdot (\mathbf{x}_n - \mathbf{x}_+) + C_0 \ln \frac{\overline{\rho}(\mathbf{x}_n)}{\overline{\rho}(\mathbf{x}_+)} = g_D(\mathbf{x}_n) - h_+, \quad \text{where} \quad \mathbf{x}_n \in \Gamma_D,$$

where one equation must be for $i = T^-$. Equations for the system (8) should be chosen in such a way that the matrix $\mathbf{A}$ is invertible.

Each coordinate of the unknown vector $\mathbf{X}$ has the solution in the form

$$\sum_k a_k^+(h_k - h_+) + a_-^+(h_- - h_+)$$

where a_i is the corresponding element in the inverse matrix $\mathbf{A}^{-1}$, while h_k denotes both hydraulic head in some cell and boundary value $g_D(\mathbf{x})$.

Substituting solution of the linear system (8) into (7) gives flux approximation

$$u_{f,+} \approx \sum_k \alpha_k^+(h_+ - h_k) + \alpha_-^+(h_+ - h_-). \tag{9}$$

In the same manner we can derive flux approximation depending on the hydraulic head in cell T^-

$$u_{f,-} \approx \sum_l \alpha_l^-(h_- - h_l) - \alpha_+^-(h_- - h_+). \tag{10}$$

We demand that $\alpha_i^\pm \geq 0$ and $\alpha_\mp^\pm > 0$, if this is not case, other set of equations should be chosen to form system (8).

Approximations (9) and (10) are in form (5) but are not equal. Thus, using (9) for flux approximation in cell T^+ and (10) for flux approximation in cell T^- would result in losing a local conservation. Therefore, we will use their convex combination for the flux approximation of face f for both cells

$$\begin{aligned} &u_f \approx \mu_+ u_{f,+} + \mu_-(-u_{f,-}), \\ &\mu_+ + \mu_- = 1, \quad \mu_\pm \geq 0. \end{aligned} \tag{11}$$

The aim is to choose these coefficients $\mu_\pm$ in such a way that the obtained approximation is in the form (5).

Equations (9) and (10) can be written as

$$u_{f,+} = \gamma^+ + \beta_+(h_+ - h_-), \quad u_{f,-} = \gamma^- + \beta_-(h_- - h_+), \tag{12}$$

where

$$\gamma^+ = \sum_k \alpha_k^+(h_+ - h_k) + (\alpha_-^+ - \beta_+)(h_+ - h_-),$$

$$\gamma^- = \sum_j \alpha_j^-(h_- - h_j) + (\alpha_+^- - \beta_-)(h_- - h_+).$$

The way for choosing constants $\beta_\pm > 0$ will be explained later.

Besides condition that the obtained approximation is in the form (5), it would be ideal to choose coefficients $\mu_\pm$ so we could lose all contributions except those from cells $T^\pm$ whenever it is possible. This can be achieved by demanding

$$\mu_+\gamma^+ - \mu_-\gamma^- = 0. \tag{13}$$

By solving system (11) and (13), we obtain

$$\mu_\pm = \frac{\gamma^\mp}{\gamma^+ + \gamma^-}.$$

However, if we determine $\mu_\pm$ in this way, condition $\mu_\pm \geq 0$ is not always satisfied as $\gamma^\pm$ do not have to be of the same sign. Nevertheless, we can choose $\mu_\pm$ as

$$\mu_\pm = \frac{|\gamma^\mp|}{|\gamma^+| + |\gamma^-|}$$

or $\mu_\pm = 0.5$, if $|\gamma^+| = |\gamma^-| = 0$.

In this way we have arrived to approximation

$$\begin{aligned} u_f &\approx (\mu_+\beta_+ + \mu_-\beta_-)(h_+ - h_-) + \mu_+\gamma^+ - \mu_-\gamma^- \\ &= \frac{|\gamma^-|\beta_+ + |\gamma^+|\beta_-}{|\gamma^+| + |\gamma^-|}(h_+ - h_-) + \frac{|\gamma^-|\gamma^+ - |\gamma^+|\gamma^-}{|\gamma^+| + |\gamma^-|}. \end{aligned} \tag{14}$$

Note that we will lose all contributions except those from cells $T^\pm$ whenever γ^+ and γ^- are of same sign. Let us show that this approximation satisfies (5). Depending on the sign of $\gamma^+\gamma^-$ we have two cases

- 1) $\boldsymbol{\gamma^+\gamma^-} \geq \mathbf{0}$, flux approximation reduces to

$$u_f \approx \frac{|\gamma^-|\beta_+ + |\gamma^+|\beta_-}{|\gamma^+| + |\gamma^-|}(h_+ - h_-). \tag{15}$$

  Because $\beta_\pm > 0$, condition (5) is always satisfied.

- 2) $\boldsymbol{\gamma^+\gamma^-} < \mathbf{0}$, then $|\gamma^-|\gamma^+ - |\gamma^+|\gamma^- = 2|\gamma^-|\gamma^+ = -2\gamma^-|\gamma^+|$. Therefore flux approximation through face f for cell $\mathcal{T}^+$ is

$$\begin{aligned} u_{f,+} = u_f &\approx \frac{|\gamma^-|\beta_+ + |\gamma^+|\beta_-}{|\gamma^+| + |\gamma^-|}(h_+ - h_-) + \frac{2|\gamma^-|\gamma^+}{|\gamma^+| + |\gamma^-|} \\ &= \frac{|\gamma^-|\beta_+ + |\gamma^+|\beta_-}{|\gamma^+| + |\gamma^-|}(h_+ - h_-) + \frac{2|\gamma^-|}{|\gamma^+| + |\gamma^-|}\left(\sum_i \alpha_i^+(h_+ - h_i) + (\alpha_-^+ - \beta_+)(h_+ - h_-)\right) \\ &= \frac{(h_+ - h_-)}{|\gamma^+| + |\gamma^-|}\left(|\gamma^-|(2\alpha_-^+ - \beta_+) + |\gamma^+|\beta_-\right) + \frac{2|\gamma^-|}{|\gamma^+| + |\gamma^-|}\left(\sum_i \alpha_i^+(h_+ - h_i)\right), \end{aligned} \tag{16}$$

  and for cell $\mathcal{T}^-$

$$\begin{aligned} u_{f,-} = -u_f &\approx \frac{|\gamma^-|\beta_+ + |\gamma^+|\beta_-}{|\gamma^+| + |\gamma^-|}(h_- - h_+) + \frac{2|\gamma^+|\gamma^-}{|\gamma^+| + |\gamma^-|} \\ &= \frac{|\gamma^-|\beta_+ + |\gamma^+|\beta_-}{|\gamma^+| + |\gamma^-|}(h_- - h_+) + \frac{2|\gamma^+|}{|\gamma^+| + |\gamma^-|}\left(\sum_j \alpha_j^-(h_- - h_i) + (\alpha_+^- - \beta_-)(h_- - h_+)\right) \\ &= \frac{(h_- - h_+)}{|\gamma^+| + |\gamma^-|}\left(|\gamma^+|(2\alpha_+^- - \beta_-) + |\gamma^-|\beta_+\right) + \frac{2|\gamma^+|}{|\gamma^+| + |\gamma^-|}\left(\sum_j \alpha_j^-(h_- - h_i)\right). \end{aligned} \tag{17}$$

  Approximations (16) and (17) are equal, therefore we have local conservation $u_{f,+} = -u_{f,-}$. If $0 < \beta_+ \leq 2\alpha_-^+$ and $0 < \beta_- \leq 2\alpha_+^-$, approximations (16) and (17) satisfies (5). We choose $\beta_+ = \alpha_-^+$ and $\beta_- = \alpha_+^-$.

Boundary face can be seen as a ghost cell with known hydraulic head in its centroid. Therefore, flux in boundary faces can be calculated in the same way as for inner faces.

Variables γ^+ and γ^- depend on hydraulic head, therefore this kind of discretization yields non-linear system of equations

$$A(\mathbf{h})\mathbf{h} = \mathbf{h},$$

which can be solved using Picard or Newton method.

In constructing scheme it was required that $\alpha_i^\pm \geq 0$, $\alpha_\mp^\pm$, $0 < \beta_+ \leq 2\alpha_-^+$, and $0 < \beta_- \leq 2\alpha_+^-$. Therefore, the off-diagonal entries of the matrix $A(\mathbf{h})$ are negative. Furthermore, for all cells, diagonal entries are greater or equal than the sum of off-diagonals entries for every cell without boundary face and strictly greater for every cell with boundary face. Thus, obtained solution respects minimum and maximum principle, and also, it is non-oscillating [5, 11].

The proposed NWC scheme is used within near-well region, and outside of it the non-linear MPFA scheme without correction is used. A near-well region can be of any shape as long as it includes at least the cells nearest to the well. Near-well regions belonging to different wells must not overlap.

4. Numerical tests

We are solving two problems with known analytical solution in order to verify proposed method. In each example we compare analytical solution to the results obtained with non-linear MPFA scheme without correction in the near-well region and also with the NWC method.

Parameter $\mathfrak{h}$ denotes length of the largest cell edge. Meshes are independently generated i.e. they are not hierarchically related.

The weighted discrete L_2 and maximum norms are used to evaluate relative hydraulic head errors:

$$\epsilon_2^h = \left[\frac{\sum_T (h(\mathbf{x}_T) - h_T)^2 |T|}{\sum_T (h(\mathbf{x}_T))^2 |T|} \right]^{1/2},$$

$$\epsilon_{\max}^h = \frac{\max_T |h(\mathbf{x}_T) - h_T|}{\left[\sum_T (h(\mathbf{x}_T))^2 |T| \Big/ \sum_T |T| \right]^{1/2}},$$

where $|T|$ stands for the area or volume of the cell T. The exact hydraulic head at the centroid of cell T is denoted by $h(\mathbf{x}_T)$, while the calculated hydraulic head in this cell is denoted by h_T.

the relative error of the total well flux is computed as:

$$\epsilon_Q = \frac{Q - Q_A}{Q_A},$$

where Q is the numerical obtained well flux and Q_A is the analytical flux.

4.1. Example 1.

Consider a circular domain with radius $R = 200$, and a well of radius $r = 0.05$ in ite center. Hydraulic conductivity in the domain is set to be homogeneous and isotropic $\mathbb{K} = K\mathbb{I}$, where $K = 0.0001$. Dirichlet boundary condition is prescribed at the outer boundary of the domain with $h_R = 100$ and on the inner boundary of the domain with $h_r = 60$.

Exact solution of this problem is given with equation (2). Coefficients C_1 and C_2 are obtained from boundary conditions

$$C_0 = \frac{h_R - h_r}{\ln \frac{R}{r}}, \quad C_1 = \frac{h_r \ln R - h_R \ln r}{\ln \frac{R}{r}}.$$

Exact flux through the well is given with

$$Q = AK \frac{h_R - h_r}{r \ln \frac{R}{r}},$$

where A is the circumference of the well screen.

Table 1: Absolute errors of the hydraulic head in the Example 1.

$\mathfrak{h}$	$32\sqrt{2}$	$16\sqrt{2}$	$8\sqrt{2}$	$4\sqrt{2}$	$2\sqrt{2}$	$\sqrt{2}$
MPFA scheme without correction						
ϵ_2^h	3.35e-02	3.37e-02	3.17e-02	2.90e-02	2.35e-02	1.53e-02
$\epsilon_{\max}^h$	9.76e-02	1.24e-01	1.55e-01	1.63e-01	1.57e-01	1.21e-01
ϵ_Q	-9.71e-01	-9.48e-01	-8.85e-01	-8.04e-01	-6.50e-01	-4.21e-01
NWC scheme						
ϵ_2^h	8.19e-04	2.09e-04	5.60e-05	1.01e-05	2.27e-06	5.92e-07
$\epsilon_{\max}^h$	3.74e-03	1.31e-03	6.25e-04	9.68e-05	2.29e-05	5.81e-06
ϵ_Q	6.33e-03	4.74e-04	1.41e-04	2.52e-05	8.74e-06	2.14e-06

The errors obtained with the uncorrected MPFA scheme and with the NWC scheme are presented in Table 1. Results shows that the uncorrected MPFA scheme is inconsistent in the maximum norm and the flow rate through the well is completely wrong. This is expected as the hydraulic head gradient changes sharply in the well vicinity. Therefore, the hydraulic head error is larger near the well (Fig 2, left). On the other hand, the NWC scheme is second-order accurate for hydraulic head. Results for NWC scheme are obtained using a near-well region of radius 50. The largest errors are located just outside of the near-well region (Fig 2, right).

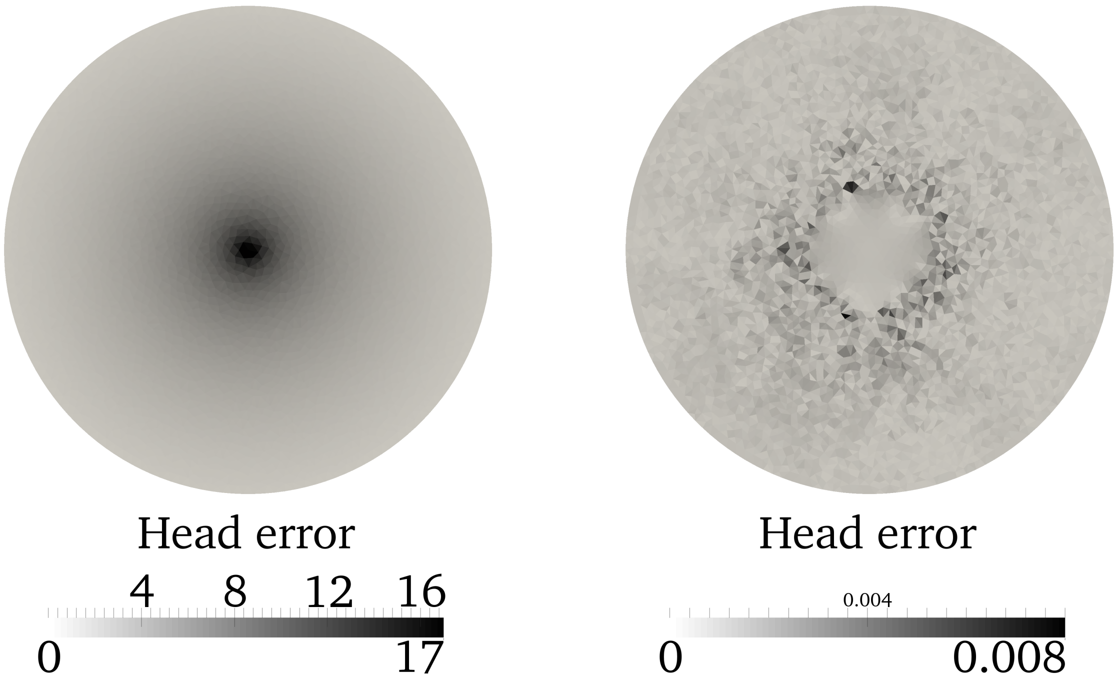

Figure 2: Absolute hydraulic head error using non-linear MPFA scheme without correction (left) and using NWC scheme (right) in the Example 1.

Example 2. In this example we consider a rectangular reservoir with corners $(\pm 300, \pm 150)$. Two wells with radii $r_l = 0.5$ and $r_r = 0.6$ are specified at $\mathbf{x}_l = (-150, 0)$ and $\mathbf{x}_r = (150, 0)$, respectively. Hydraulic conductivity is

$$\mathbb{K} = 10^{-4} \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}.$$

An analytical solution is obtained by superposing two solutions of form (2)

$$\begin{aligned} h &= \frac{h_{R_l} - h_{r_l}}{\ln \frac{R_l}{r_l}} \ln \|S(\mathbf{x} - \mathbf{x}_l)\| + \frac{h_{r_l} \ln R_l - h_{R_l} \ln r_l}{\ln \frac{R_l}{r_l}} \\ &+ \frac{h_{R_r} - h_{r_r}}{\ln \frac{R_r}{r_r}} \ln \|S(\mathbf{x} - \mathbf{x}_r)\| + \frac{h_{r_r} \ln R_r - h_{R_r} \ln r_r}{\ln \frac{R_r}{r_r}}. \end{aligned}$$

We take $h_{R_l} = h_{R_r} = 20$, $h_{r_l} = 5$, $h_{r_r} = 10$, $R_r = R_l = 1200$. Note that in this case h_{R_l}, h_{R_r}, h_{r_l}, h_{r_r}, R_r, and R_l are just formal parameters. In engineering practice these parameters are obtained when one of the wells is turned off.

Dirichlet boundary conditions are imposed on the outer boundary, in the left well, and in the right well using the exact hydraulic head.

Table 2: Absolute errors of the hydraulic head in the Example 2.

$\mathfrak{h}$	32	16	8	4	2
MPFA scheme without correction					
ϵ_2^h	5.92e-03	3.27e-03	2.83e-03	1.99e-03	1.58e-03
$\epsilon_{\max}^h$	4.64e-02	3.04e-02	2.60e-02	2.01e-02	2.91e-02
ϵ_{Q_l}	-2.53e-01	1.89e-01	2.73e-01	1.69e-01	1.28e-01
ϵ_{Q_r}	5.72e-02	2.64e-01	1.18e-01	1.02e-01	9.33e-02
NWC scheme					
ϵ_2^h	1.32e-03	5.17e-04	1.67e-04	6.23e-05	1.47e-05
$\epsilon_{\max}^h$	6.92e-03	3.48e-03	1.48e-03	9.61e-04	2.06e-04
ϵ_{Q_l}	1.35e-03	9.21e-04	6.32e-04	1.41e-04	3.47e-05
ϵ_{Q_r}	4.69e-03	2.42e-03	4.45e-04	7.91e-05	3.42e-05

As in the previous example, the MPFA scheme without correction gives a very inaccurate well flow rate in both wells (see Table 2) and hydraulic head errors are larger near the well. With NWC scheme obtained results implies that the hydraulic head is second order accurate.Near-well regions for the NWC scheme are circular with radius 100. Distribution of the absolute hydraulic head errors are shown in Figure 3.

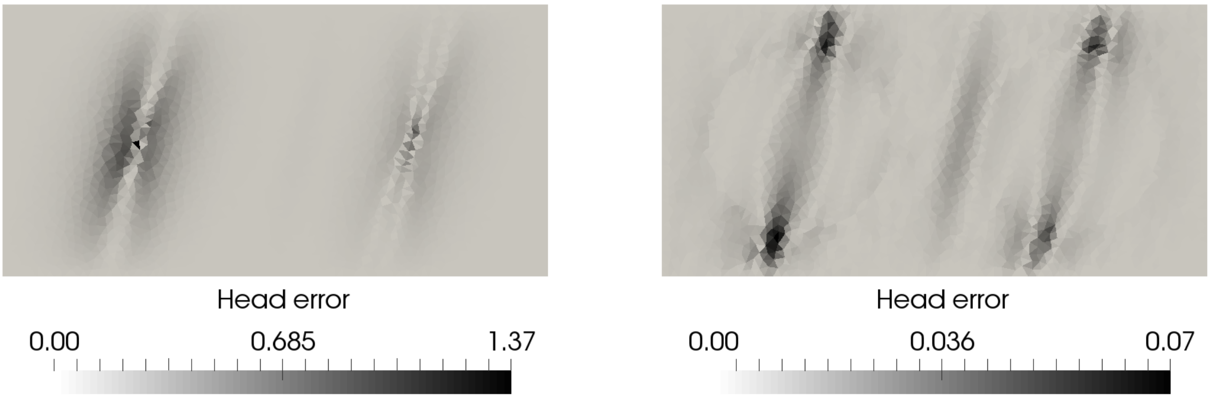

Figure 3: Absolute hydraulic head error using non-linear MPFA scheme without correction (left) and using NWC scheme (right) in the Example 2.

5. Conclusion

Numerical schemes based on the linear approximation produce inaccurate results in the near-well region. Local grid refinement can alleviate this problem, but it is not always feasible because of computational cost.

Papers [3, 4] introduce schemes that give second-order accurate hydraulic head when the wells are present in the domain. However, the NWC scheme in these papers are based on a nonlinear TPFA scheme, therefore, it does not respect both minimum and maximum principle.

NWC scheme developed in this paper represents non-linear MPFA scheme with correction in the near-well region. Obtained scheme respects minimum and maximum principle and it is second-order accurate, even for anisotropic tensors and wells in the model. This property is obtained at a price, i.e. we are solving system of non-linear equations even if the partial differential equation is linear. In practice this is not a problem as we are solving Richards equation which is non-linear partial differential equation.

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