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Slant Ruled Surfaces and Slant Developable Surfaces of Spacelike Curves in Lorentz-Minkowski 3-space

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Abstract. In this paper, by means of the Lorentzian Frenet frame along a spacelike curve in Lorentz-Minkowski 3-space, we construct slant ruled surfaces and slant developable surfaces with different director curves which belong to one-parameter families of the pseudo-spheres in this space. Moreover, for each slant ruled surface with each director curve, we search if this slant ruled surface has any singularities or not. Furthermore, for the cases in which the singularities appear, we determine the singularities of non-lightlike and non-cylindrical slant developable surfaces and also investigate the singularities of slant ruled surfaces.

1. Introduction

It is known that a ruled surface is defined by a one-parameter family of lines while a developable surface is a ruled surface whose regular part's Gauss curvature is identically zero. Ruled surfaces and developable surfaces are of great interest in classical differential geometry. Indeed, these surfaces have been studied intensively in Euclidean space and Lorentz-Minkowski space from different viewpoints (See, for instance, [1], [4], [6], [7], [9]-[18], [23], [25], [26], [28], [30]-[32], [35]-[38], [40], [41].). We point out that some of these papers use the singularity theory techniques given in [2] and [5].

A *ruled surface* in $\mathbb{R}^3$ is parametrized by

$$\begin{array}{rccl} F_{(\gamma,\mathbb{N})} & : & I\times J & \longrightarrow & \mathbb{R}^3 \\ & & (s,u) & \longmapsto & \gamma(s)+u\mathbb{N}(s) \end{array}$$

such that $\gamma : I \to \mathbb{R}^3$ and $\mathbb{N} : I \to S^2$ are smooth mappings, where I and J are open intervals in $\mathbb{R}$ or unit circles S^1. Here, γ is said to be a *base curve*. Without loss of generality, we may assume that γ is parametrized by arc length s. Moreover, $\mathbb{N}$ is said to be a *director curve* and the straight lines $u \to \gamma(s) + u\mathbb{N}(s)$ are said to be *rulings*. Since

$$\frac{\partial F_{(\gamma,\mathbb{N})}}{\partial s}(s,u) = \gamma'(s) + u\mathbb{N}'(s) \quad \text{and} \quad \frac{\partial F_{(\gamma,\mathbb{N})}}{\partial u}(s,u) = \mathbb{N}(s),$$

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we have the following equation for the normal vector of $F_{(\gamma,\mathbb{N})}$ at any $(s,u) \in I \times J$:

$$\frac{\partial F_{(\gamma,\mathbb{N})}}{\partial s}(s,u) \times \frac{\partial F_{(\gamma,\mathbb{N})}}{\partial u}(s,u) = \gamma'(s) \times \mathbb{N}(s) + u\mathbb{N}'(s) \times \mathbb{N}(s).$$

So, (s_0, u_0) is a singular point of $F_{(\gamma,\mathbb{N})}$ if and only if

$$\gamma'(s_0) \times \mathbb{N}(s_0) + u_0\mathbb{N}'(s_0) \times \mathbb{N}(s_0) = 0$$

(See [17] for the details.).

A ruled surface $F_{(\gamma,\mathbb{N})}$ is called *cylindrical* if $\mathbb{N}(s) \times \mathbb{N}'(s) \equiv 0$. Moreover, it is called *non-cylindrical* if $\mathbb{N}(s) \times \mathbb{N}'(s) \neq 0$ (Cf. [17].).

Let σ be a curve on $F_{(\gamma,\mathbb{N})}$ such that $\langle \sigma'(s), \mathbb{N}'(s) \rangle = 0$. Then, it is said to be *the line of striction of* $F_{(\gamma,\mathbb{N})}$. It is known that the singular points of $F_{(\gamma,\mathbb{N})}$ are located on the line of striction on which the Gauss curvature is zero. At regular points of $F_{(\gamma,\mathbb{N})}$, its Gauss curvature denoted by K satisfies $K \leq 0$ and K is zero only along the rulings which meet the line of striction at a singular point (See [17] for the details.).

It was shown in [17] that *the cuspidal edge* $C \times \mathbb{R}$, *the swallowtail* SW and *the cuspidal cross cap* CCR, which are respectively defined by

$$C \times \mathbb{R} = \{(x_1, x_2) \,|\, x_1^2 = x_2^3\} \times \mathbb{R},$$

$$SW = \{(x_1, x_2, x_3) \,|\, x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$$

and

$$CCR = \{(x_1, x_2, x_3) \,|\, x_1 = u^3, x_2 = u^3 v^3, x_3 = v^2\},$$

appear as the singularities of the developable surfaces in $\mathbb{R}^3$. Moreover, we refer [15] and [17] for the singularities of the general ruled surfaces in $\mathbb{R}^3$.

In this paper, by means of the Lorentzian Frenet frame along a spacelike base curve γ which is parametrized by arc length s in Lorentz-Minkowski 3-space, we deal with the ruled surfaces having different director curves which belong to one-parameter families of the pseudo-spheres (depending on a parameter $\phi \in [0, \pi/2]$) in this space. These one-parameter families of the pseudo-spheres were given in [22]. The geometry related with this parameter ϕ is said to be *slant geometry* (See [3], [21] and [22] for the details.). Since we are interested in the ruled (respectively, developable) surfaces depending on ϕ, we call these surfaces *slant ruled* (respectively, *slant developable*) surfaces. In this study, for each slant ruled surface with each director curve, we first search if this slant ruled surface has any singularities or not. Moreover, for the cases in which the singularities appear, we determine the singularities of non-lightlike and non-cylindrical slant developable surfaces and also investigate the singularities of slant ruled surfaces. Here we remark that, for our purpose, we used the tools and the techniques which were given in [17], [33] and [39]. We also emphasize that $\phi = 0$ case was studied in [14].

Throughout the whole paper, we assume that all of the manifolds and maps are of class C^∞.

2. Basic notions

In this section, we give some basic notions related with Lorentz-Minkowski 3-space. Let $\mathbb{R}^3 = \{(x_0, x_1, x_2) \,|\, x_i \in \mathbb{R},\ i = 0, 1, 2\}$ be a 3-dimensional real vector space. For any vectors $\boldsymbol{x} = (x_0, x_1, x_2)$ and $\boldsymbol{y} = (y_0, y_1, y_2)$ in $\mathbb{R}^3$, the *pseudo-scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{2} x_i y_i$. The space $(\mathbb{R}^3, \langle , \rangle)$ is said to be *Lorentz-Minkowski 3-space* and denoted by $\mathbb{R}^3_1$ briefly. A vector $\boldsymbol{x} \in \mathbb{R}^3_1 \setminus \{\boldsymbol{0}\}$ is called *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0, respectively. Also, the signature of $\boldsymbol{x}$ is given by

$$sign(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \text{ is spacelike,} \\ 0 & \text{if } \boldsymbol{x} \text{ is lightlike,} \\ -1 & \text{if } \boldsymbol{x} \text{ is timelike.} \end{cases}$$

Moreover, the *norm* of a vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$ (Cf. [29].). Furthermore, for any vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_1$, the vector $\boldsymbol{x} \times \boldsymbol{y}$ is defined by

$$\boldsymbol{x} \times \boldsymbol{y} = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ is the orthonormal basis of $\mathbb{R}^3_1$ (See [8].). It is obvious that

$$\langle \boldsymbol{z}, \boldsymbol{x} \times \boldsymbol{y} \rangle = \det(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{y}),$$

so that $\boldsymbol{x} \times \boldsymbol{y}$ is pseudo-orthogonal to $\boldsymbol{x}$ and $\boldsymbol{y}$.

It is known that *Hyperbolic* 2*-space* $H^2(-1)$, *de Sitter* 2*-space* S^2_1 and 2*-dimensional (open) lightcone* LC^* are three kinds of pseudo-spheres in $\mathbb{R}^3_1$ which are defined respectively by

$$H^2(-1) = \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

$$S^2_1 = \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

and

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^3_1 \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

For $\phi \in [0, \pi/2]$, $H^2(-\sin^2\phi)$ (respectively, $S^2_1(\sin^2\phi)$) is said to be ϕ*-hyperbolic* 2*-space* (respectively, ϕ*-de Sitter* 2*-space*) (Cf. [3], [21] and [22].). Here, we remark that $H^2(-\sin^2 0) \setminus \{\boldsymbol{0}\} = S^2_1(\sin^2 0) \setminus \{\boldsymbol{0}\} = LC^*$. Throughout the remainer part of this paper, we write S^2_1 instead of $S^2_1(1)$ and for $\phi = 0$, we deal with only LC^*.

Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ be a spacelike curve parametrized by arc length s, where $I \subset \mathbb{R}$. In this case, at any $s \in I$, the tangent vector of $\boldsymbol{\gamma}$ denoted by $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ is always spacelike, where $\boldsymbol{\gamma}'(s) = \frac{d\boldsymbol{\gamma}}{ds}(s)$. Since $\boldsymbol{\gamma}$ is spacelike, the normal plane of $\boldsymbol{\gamma}$ at any $s \in I$ is always timelike (See [29].).

The *curvature* of $\boldsymbol{\gamma}$ at any $s \in I$ is defined by $k(s) = \sqrt{|\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}''(s) \rangle|}$. Throughout this paper, we assume that $k(s) \neq 0$ for any $s \in I$. Then, the unit principal normal vector $\boldsymbol{n}(s)$ of $\boldsymbol{\gamma}$ at any $s \in I$ is given by $\boldsymbol{n}(s) = \boldsymbol{\gamma}''(s)/k(s)$. On the other hand, the unit binormal vector $\boldsymbol{b}(s)$ of $\boldsymbol{\gamma}$ at any $s \in I$ is defined by $\boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s)$. Since $\boldsymbol{t}(s)$ is spacelike, it is clear that $sign\big(\boldsymbol{b}(s)\big) = -\delta\big(\boldsymbol{\gamma}(s)\big)$, where $\delta\big(\boldsymbol{\gamma}(s)\big) = sign\big(\boldsymbol{n}(s)\big)$. It can be easily seen that $\boldsymbol{n}(s) = \boldsymbol{t}(s) \times \boldsymbol{b}(s)$ and $\boldsymbol{t}(s) = -\delta\big(\boldsymbol{\gamma}(s)\big)\boldsymbol{n}(s) \times \boldsymbol{b}(s)$. Moreover, in terms of the frame $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ which is said to be *Lorentzian Frenet frame* along $\boldsymbol{\gamma}$, we have the following Frenet-Serret type equations for any $s \in I$:

$$\begin{aligned} \boldsymbol{t}'(s) &= k(s)\boldsymbol{n}(s), \\ \boldsymbol{n}'(s) &= -\delta\big(\boldsymbol{\gamma}(s)\big)k(s)\boldsymbol{t}(s) + \tau(s)\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) &= \tau(s)\boldsymbol{n}(s), \end{aligned}$$

where $\tau(s) = \delta\big(\boldsymbol{\gamma}(s)\big)\langle \boldsymbol{b}'(s), \boldsymbol{n}(s) \rangle$ is the torsion of $\boldsymbol{\gamma}$ at any $s \in I$ (Cf. [14], [19] and [20].). Here, it can be easily verified that $\tau(s) = -\delta\big(\boldsymbol{\gamma}(s)\big)\det\big(\boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s)\big)\big/k^2(s)$.

3. Slant ruled surfaces with the director curve $\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}$

In this section, for any fixed $\phi \in [0, \pi/2]$, we define a *slant ruled surface* by

$$\begin{aligned} F_{\left(\boldsymbol{\gamma}, \mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)} : I \times J &\longrightarrow \mathbb{R}^3_1 \\ (s, u) &\longmapsto \boldsymbol{\gamma}(s) + u\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}(s) \end{aligned}$$

such that $\boldsymbol{\gamma}: I \to \mathbb{R}^3_1$ is a spacelike base curve parametrized by arc length s, $\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm} = \cos\phi\, \boldsymbol{n} \pm \boldsymbol{b}$ is a director curve and the straight lines $u \longmapsto \boldsymbol{\gamma}(s) + u\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}(s)$ are rulings, where I and J are open intervals in $\mathbb{R}$ or unit circles S^1. Here, we remark that

$$\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}(s) \in \begin{cases} S^2_1(\sin^2\phi) & \text{if } \boldsymbol{n}(s) \text{ is timelike,} \\ H^2(-\sin^2\phi) & \text{if } \boldsymbol{n}(s) \text{ is spacelike} \end{cases}$$

for any fixed $\phi \in [0, \pi/2]$ and we say that $F_{\left(\gamma, \mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$ is

$$\begin{cases} \textit{a } \phi\textit{-de Sitter normal surface of } \boldsymbol{\gamma} & \text{if } \boldsymbol{n}(s) \text{ is timelike,} \\ \textit{a } \phi\textit{-hyperbolic normal surface of } \boldsymbol{\gamma} & \text{if } \boldsymbol{n}(s) \text{ is spacelike.} \end{cases}$$

We briefly say that $F_{\left(\gamma, \mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$ is *a slant normal surface of* $\boldsymbol{\gamma}$ if it is either a ϕ-de Sitter normal surface or a ϕ-hyperbolic normal surface of $\boldsymbol{\gamma}$. Especially, we say that $F_{\left(\gamma, \mathbb{N}[\pi/2]^{\boldsymbol{nb}}_{\pm}\right)}$ is

$$\begin{cases} \textit{a de Sitter binormal surface of } \boldsymbol{\gamma} & \text{if } \boldsymbol{n}(s) \text{ is timelike,} \\ \textit{a hyperbolic binormal surface of } \boldsymbol{\gamma} & \text{if } \boldsymbol{n}(s) \text{ is spacelike} \end{cases}$$

(See [17] in Euclidean sense.). Moreover, $F_{\left(\gamma, \mathbb{N}[0]^{\boldsymbol{nb}}_{\pm}\right)}$ is said to be *the lightcone normal surface of* $\boldsymbol{\gamma}$, where $\mathbb{N}[0]^{\boldsymbol{nb}}_{\pm}(s) \in LC^*$. Here, we point out that this case was studied in [14].

For the normal vector of a slant ruled surface $F_{\left(\gamma, \mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$, we get

$$\frac{\partial F_{\left(\gamma, \mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}}{\partial s}(s,u) \times \frac{\partial F_{\left(\gamma, \mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}}{\partial u}(s,u) = \boldsymbol{\gamma}'(s) \times \mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}(s) + u(\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm})'(s) \times \mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}(s)$$

at any $(s,u) \in I \times J$. If we denote this normal vector by $N^{\phi,\boldsymbol{nb}}_{\pm}(s,u)$, then we obtain

$$\begin{aligned} N^{\phi,\boldsymbol{nb}}_{\pm}(s,u) &= -u\sin^2\phi\, \delta\big(\boldsymbol{\gamma}(s)\big)\tau(s)\boldsymbol{t}(s) \pm \Big(1 - u\cos\phi\, \delta\big(\boldsymbol{\gamma}(s)\big)k(s)\Big)\boldsymbol{n}(s) \\ &\quad + \cos\phi\Big(1 - u\cos\phi\, \delta\big(\boldsymbol{\gamma}(s)\big)k(s)\Big)\boldsymbol{b}(s). \end{aligned}$$

As a result, we have the following propositions and remark:

Proposition 3.1. *Let* $\phi \in (0, \pi/2)$. (s_0, u_0) *is a singular point of* $F_{\left(\gamma, \mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$ *if and only if* $\tau(s_0) = 0$ *and* $u_0 = \dfrac{1}{\cos\phi\, \delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}$.

Proposition 3.2. $\left(s_0, \dfrac{1}{\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}\right)$ *is a singular point of* $F_{\left(\gamma, \mathbb{N}[0]^{\boldsymbol{nb}}_{\pm}\right)}$.

We emphasize that $\phi = 0$ case was investigated in [14].

Remark 3.3. $F_{\left(\gamma, \mathbb{N}[\pi/2]^{\boldsymbol{nb}}_{\pm}\right)}$ *is always regular.*

Now, we consider the following cases:

(1) $\phi = 0$ and $u\delta\big(\boldsymbol{\gamma}(s)\big)k(s) \neq 1$.

(2) $\phi \in (0, \pi/2]$, $\boldsymbol{n}(s)$ is spacelike and at least one of the following conditions holds:

(i) $\tau(s) \neq 0$,

(ii) $u\cos\phi\, k(s) \neq 1$.

(3) $\phi \in (0, \pi/2]$, $\boldsymbol{n}(s)$ is timelike and $u^2\sin^2\phi\,\tau^2(s) > \Big(1 + u\cos\phi\, k(s)\Big)^2$, where one of the following conditions holds:

(i) $\tau(s) \neq 0$ and $u\cos\phi\, k(s) = -1$,

(ii) $\tau(s) \neq 0$ and $u\cos\phi\, k(s) \neq -1$.

(4) $\phi \in (0, \pi/2]$, $\boldsymbol{n}(s)$ is timelike and $u^2\sin^2\phi\,\tau^2(s) = \Big(1 + u\cos\phi\, k(s)\Big)^2$, where $\tau(s) \neq 0$ and $u\cos\phi\, k(s) \neq -1$.

(5) $\phi \in (0, \pi/2]$, $\boldsymbol{n}(s)$ is timelike and $u^2\sin^2\phi\,\tau^2(s) < \Big(1 + u\cos\phi\, k(s)\Big)^2$, where one of the following conditions holds:

(i) $\tau(s) = 0$ and $u\cos\phi\, k(s) \neq -1$,

(ii) $\tau(s) \neq 0$ and $u\cos\phi\, k(s) \neq -1$.

By means of the above cases, we classify the normal vector $N^{\phi,\boldsymbol{nb}}_{\pm}(s,u)$ of $F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$ at any regular $(s,u) \in I \times J$ as follows:

$$N^{\phi,\boldsymbol{nb}}_{\pm}(s,u) \text{ is } \begin{cases} \text{spacelike if either } \textbf{(2)} \text{ or } \textbf{(3)} \text{ is satisfied,} \\ \text{lightlike if either } \textbf{(1)} \text{ or } \textbf{(4)} \text{ is satisfied,} \\ \text{timelike if } \textbf{(5)} \text{ is satisfied.} \end{cases}$$

Example 3.4. *Let $\boldsymbol{\gamma}(s) = (0, \cos s, \sin s)$, where $0 \le s < 2\pi$. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}(s,u) = \Big(\mp u, (1 - u\cos\phi)\cos s, (1 - u\cos\phi)\sin s\Big),$$

where the points $\left(s, \frac{1}{\cos\phi}\right)$ are its singular points for $\phi \in [0, \pi/2)$. Moreover, $F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$ is timelike (respectively, lightlike) for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$), where $u\cos\phi \neq 1$ (respectively, $u \neq 1$).

Example 3.5. *Let $\boldsymbol{\gamma}(s) = (\cosh s, \sinh s, 0)$. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}(s,u) = \Big((1 + u\cos\phi)\cosh s, (1 + u\cos\phi)\sinh s, \mp u\Big),$$

where the points $\left(s, -\frac{1}{\cos\phi}\right)$ are its singular points for $\phi \in [0, \pi/2)$. Moreover, $F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$ is spacelike (respectively, lightlike) for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$), where $u\cos\phi \neq -1$ (respectively, $u \neq -1$).

Example 3.6. *Let $\boldsymbol{\gamma}(s) = \left(\cosh s, \dfrac{\sinh s}{\sqrt{2}}, \dfrac{\sinh s}{\sqrt{2}}\right)$. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}(s,u) = \left((1 + u\cos\phi)\cosh s, (1 + u\cos\phi)\frac{\sinh s}{\sqrt{2}} \pm \frac{u}{\sqrt{2}}, (1 + u\cos\phi)\frac{\sinh s}{\sqrt{2}} \mp \frac{u}{\sqrt{2}}\right),$$

where the points $\left(s, -\frac{1}{\cos\phi}\right)$ are its singular points for $\phi \in [0, \pi/2)$. Moreover, $F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$ is spacelike (respectively, lightlike) for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$), where $u\cos\phi \neq -1$ (respectively, $u \neq -1$).

Example 3.7. *Let $\boldsymbol{\gamma}(s) = \left(\sin s, \sqrt{2}\sin s, \cos s\right)$, where $0 \le s < 2\pi$. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}(s,u) = \Big((1 - u\cos\phi)\sin s \pm \sqrt{2}u, \sqrt{2}(1 - u\cos\phi)\sin s \pm u, (1 - u\cos\phi)\cos s\Big),$$

where the points $\left(s, \frac{1}{\cos\phi}\right)$ are its singular points for $\phi \in [0, \pi/2)$. Moreover, $F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$ is timelike (respectively, lightlike) for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$), where $u\cos\phi \neq 1$ (respectively, $u \neq 1$).

We can define the unit non-lightlike normal vector denoted by $\mathfrak{n}^{\phi,\boldsymbol{nb}}_{\pm}(s,u)$ of $F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$ at any regular $(s,u)\in I\times J$ as follows:

$$\mathfrak{n}^{\phi,\boldsymbol{nb}}_{\pm}(s,u)=\begin{cases}\dfrac{-u\sin^2\phi\,\tau(s)\boldsymbol{t}(s)\pm\Big(1-u\cos\phi\,k(s)\Big)\boldsymbol{n}(s)+\cos\phi\Big(1-u\cos\phi\,k(s)\Big)\boldsymbol{b}(s)}{\sin\phi\,\sqrt{u^2\sin^2\phi\ \tau^2(s)+\Big(1-u\cos\phi\,k(s)\Big)^2}} & \text{if } \textbf{(2)} \text{ is satisfied,}\\ \dfrac{u\sin^2\phi\,\tau(s)\boldsymbol{t}(s)\pm\Big(1+u\cos\phi\,k(s)\Big)\boldsymbol{n}(s)+\cos\phi\Big(1+u\cos\phi\,k(s)\Big)\boldsymbol{b}(s)}{\sin\phi\,\sqrt{u^2\sin^2\phi\ \tau^2(s)-\Big(1+u\cos\phi\,k(s)\Big)^2}} & \text{if } \textbf{(3)} \text{ is satisfied,}\\ \dfrac{u\sin^2\phi\,\tau(s)\boldsymbol{t}(s)\pm\Big(1+u\cos\phi\,k(s)\Big)\boldsymbol{n}(s)+\cos\phi\Big(1+u\cos\phi\,k(s)\Big)\boldsymbol{b}(s)}{\sin\phi\,\sqrt{-\Big(u^2\sin^2\phi\ \tau^2(s)-\Big(1+u\cos\phi\,k(s)\Big)^2\Big)}} & \text{if } \textbf{(5)} \text{ is satisfied.}\end{cases}$$

In terms of the Frenet-Serret type equations, we obtain

$$\begin{aligned}\frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}}{\partial s}(s,u)&=\Big(1-u\cos\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)k(s)\Big)\boldsymbol{t}(s)\pm u\tau(s)\boldsymbol{n}(s)+u\cos\phi\,\tau(s)\boldsymbol{b}(s),\\ \frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}}{\partial u}(s,u)&=\cos\phi\,\boldsymbol{n}(s)\pm\boldsymbol{b}(s),\\ \frac{\partial^2 F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}}{\partial^2 s}(s,u)&=\Big(-u\cos\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)k'(s)\mp u\delta\big(\boldsymbol{\gamma}(s)\big)k(s)\tau(s)\Big)\boldsymbol{t}(s)\\ &\quad+\Big(k(s)-u\cos\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)k^2(s)\pm u\tau'(s)+u\cos\phi\,\tau^2(s)\Big)\boldsymbol{n}(s)\\ &\quad+\Big(\pm u\tau^2(s)+u\cos\phi\,\tau'(s)\Big)\boldsymbol{b}(s),\\ \frac{\partial^2 F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}}{\partial u\partial s}(s,u)&=-\cos\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)k(s)\boldsymbol{t}(s)\pm\tau(s)\boldsymbol{n}(s)+\cos\phi\,\tau(s)\boldsymbol{b}(s),\\ \frac{\partial^2 F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}}{\partial^2 u}(s,u)&=\boldsymbol{0}.\end{aligned}$$

Therefore, for the Gauss curvature denoted by $K^{\phi,\boldsymbol{nb}}_{\pm}$ of a non-lightlike (either timelike or spacelike) slant ruled surface $F_{\left(\gamma,\mathbb{N}[\phi]^{\boldsymbol{nb}}_{\pm}\right)}$, we have the following classifications:

$$K^{\phi,\boldsymbol{nb}}_{\pm}(s,u)=\begin{cases}\dfrac{\tau^2(s)}{\Big(u^2\sin^2\phi\,\tau^2(s)+\Big(1-u\cos\phi\,k(s)\Big)^2\Big)^2}\geq 0 & \text{if } \textbf{(2)} \text{ is satisfied,}\\ \dfrac{\tau^2(s)}{\Big(u^2\sin^2\phi\,\tau^2(s)-\Big(1+u\cos\phi\,k(s)\Big)^2\Big)^2}>0 & \text{if } \textbf{(3)} \text{ is satisfied,}\\ -\dfrac{\tau^2(s)}{\Big(u^2\sin^2\phi\,\tau^2(s)-\Big(1+u\cos\phi\,k(s)\Big)^2\Big)^2}\leq 0 & \text{if } \textbf{(5)} \text{ is satisfied}\end{cases}$$

by the formula

$$K^{\phi,\boldsymbol{nb}}_{\pm}(s,u)=\varepsilon\frac{ln-m^2}{EG-F^2},$$

where $\varepsilon = sign(\mathfrak{n}_{\pm}^{\phi,\boldsymbol{nb}}(s,u))$ and

$$\begin{aligned}
l &= \left\langle \mathfrak{n}_{\pm}^{\phi,\boldsymbol{nb}}(s,u), \frac{\partial^2 F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}}{\partial s^2}(s,u) \right\rangle,\\
m &= \left\langle \mathfrak{n}_{\pm}^{\phi,\boldsymbol{nb}}(s,u), \frac{\partial^2 F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}}{\partial s \partial u}(s,u) \right\rangle,\\
n &= \left\langle \mathfrak{n}_{\pm}^{\phi,\boldsymbol{nb}}(s,u), \frac{\partial^2 F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}}{\partial u^2}(s,u) \right\rangle,\\
E &= \left\langle \frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}}{\partial s}(s,u), \frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}}{\partial s}(s,u) \right\rangle,\\
F &= \left\langle \frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}}{\partial s}(s,u), \frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}}{\partial u}(s,u) \right\rangle,\\
G &= \left\langle \frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}}{\partial u}(s,u), \frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}}{\partial u}(s,u) \right\rangle
\end{aligned}$$

(See [27] and [29].). Thus, for a non-lightlike slant ruled surface $F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}$, we can conclude that

$$K_{\pm}^{\phi,\boldsymbol{nb}}(s,u) = 0 \Longleftrightarrow \tau(s) = 0.$$

So, taking into account [1], [4] and the proposition which was given in [17] for the Euclidean case, we have the following proposition:

Proposition 3.8. *Singular points of a non-lightlike slant ruled surface $F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}$ are located on the line of striction on which the Gauss curvature $K_{\pm}^{\phi,\boldsymbol{nb}}$ is zero. At regular points of a timelike (respectively, spacelike) slant ruled surface $F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}$, $K_{\pm}^{\phi,\boldsymbol{nb}}$ satisfies $K_{\pm}^{\phi,\boldsymbol{nb}} \geq 0$ (respectively, $K_{\pm}^{\phi,\boldsymbol{nb}} \leq 0$) and $K_{\pm}^{\phi,\boldsymbol{nb}}$ is zero only along the rulings which meet the line of striction at a singular point.*

4. Singularities of non-lightlike and non-cylindrical slant developable surfaces with the director curve $\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}$

For any fixed $\phi \in [0, \pi/2]$, we say that a non-lightlike slant ruled surface $F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}$ is a *non-lightlike slant developable surface* if the Gauss curvature $K_{\pm}^{\phi,\boldsymbol{nb}}$ of the regular part of $F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}$ is identically zero. Moreover, we say that a slant developable surface $F_{\left(\gamma,\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\right)}(s,u)$ is a *ϕ-de Sitter* (respectively, *ϕ-hyperbolic*) *normal developable surface of $\boldsymbol{\gamma}(s)$* if $\boldsymbol{n}(s)$ is timelike (respectively, spacelike). Furthermore, $F_{\left(\gamma,\mathbb{N}[0]_{\pm}^{\boldsymbol{nb}}\right)}(s,u) = \boldsymbol{\gamma}(s) + u\Big(\boldsymbol{n}(s) \pm \boldsymbol{b}(s)\Big)$ is said to be *the lightcone developable surface of $\boldsymbol{\gamma}(s)$*, where $\mathbb{N}[0]_{\pm}^{\boldsymbol{nb}} \in LC^*$. Here, we remark that this case was studied in [14].

It can be easily seen that

$$\det\Big(\boldsymbol{\gamma}'(s), \mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}(s), \Big(\mathbb{N}[\phi]_{\pm}^{\boldsymbol{nb}}\Big)'(s)\Big) = \sin^2\phi\, \delta\Big(\boldsymbol{\gamma}(s)\Big)\tau(s).$$

Hence, taking into account [17], [36]-[38], [40] and [41], we have the following proposition:

Proposition 4.1. *Let* $\phi \in (0, \pi/2]$. *Then, a non-lightlike slant ruled surface* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *is a non-lightlike slant developable surface if and only if*

$$\det\left(\gamma'(s), \mathbb{N}[\phi]^{nb}_{\pm}(s), \left(\mathbb{N}[\phi]^{nb}_{\pm}\right)'(s)\right) = 0.$$

On the other hand, since

$$\mathbb{N}[\phi]^{nb}_{\pm}(s) \times \left(\mathbb{N}[\phi]^{nb}_{\pm}\right)'(s) = \sin^2\phi\,\delta\big(\gamma(s)\big)\tau(s)\boldsymbol{t}(s) \pm \cos\phi\,\delta\big(\gamma(s)\big)k(s)\boldsymbol{n}(s) + \cos^2\phi\,\delta\big(\gamma(s)\big)k(s)\boldsymbol{b}(s),$$

following [17] in Euclidean sense, we have the following proposition:

Proposition 4.2. *A slant ruled surface*

$$F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)} \text{ is } \begin{cases} \text{non-cylindrical} & \text{if } \phi \in [0, \pi/2), \\ \text{non-cylindrical} & \text{if } \phi = \pi/2 \text{ and } \tau(s) \neq 0, \\ \text{cylindrical} & \text{if } \phi = \pi/2 \text{ and } \tau(s) = 0. \end{cases}$$

As a result, the space of non-lightlike and non-cylindrical slant developable surfaces $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$ is given by

$$Dev[\phi]^{nb}_{\pm}(I, \mathbb{R}^3_1) = \{\, \gamma : I \to \mathbb{R}^3_1 \text{ is a spacelike curve which is parametrized by arc length } s \mid k(s) \neq 0 \text{ and } \tau(s) = 0 \text{ for any } s \in I\},$$

where $\phi \in (0, \pi/2)$ (See [17] for the Euclidean case.).

Example 4.3. *In Example 3.4,* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *is a* ϕ*-hyperbolic normal developable surface of* γ. *Moreover, it is non-cylindrical (respectively, cylindrical) for* $\phi \in [0, \pi/2)$ *(respectively,* $\phi = \pi/2$*).*

Example 4.4. *In Example 3.5,* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *is a* ϕ*-de Sitter normal developable surface of* γ. *Moreover, it is non-cylindrical (respectively, cylindrical) for* $\phi \in [0, \pi/2)$ *(respectively,* $\phi = \pi/2$*).*

Example 4.5. *In Example 3.6,* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *is a* ϕ*-de Sitter normal developable surface of* γ. *Moreover, it is non-cylindrical (respectively, cylindrical) for* $\phi \in [0, \pi/2)$ *(respectively,* $\phi = \pi/2$*).*

Example 4.6. *In Example 3.7,* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *is a* ϕ*-hyperbolic normal developable surface of* γ. *Moreover, it is non-cylindrical (respectively, cylindrical) for* $\phi \in [0, \pi/2)$ *(respectively,* $\phi = \pi/2$*).*

Now, we investigate the singularities of non-lightlike and non-cylindrical slant developable surfaces $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$, where $\phi \in (0, \pi/2)$. Taking into account [17] in Euclidean sense, we have the following lemma and corollary:

Lemma 4.7. *Let* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *be a non-lightlike and non-cylindrical slant ruled surface, where* $\phi \in (0, \pi/2)$. *Then,* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *is a non-lightlike slant developable surface if and only if*

$$\gamma'(s) = -\frac{1}{\cos\phi\,\delta\big(\gamma(s)\big)k(s)}\left(\mathbb{N}[\phi]^{nb}_{\pm}\right)'(s).$$

Corollary 4.8. *Let $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ be a non-lightlike and non-cylindrical slant developable surface, where $\phi \in (0, \pi/2)$. In this case, the set of the singular points of $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ is a curve parametrized by*

$$\boldsymbol{\sigma}[\phi]^{nb}_{\pm}(s) = \boldsymbol{\gamma}(s) + \frac{1}{\cos\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)k(s)}\mathbb{N}[\phi]^{nb}_{\pm}(s).$$

If $\boldsymbol{\sigma}[\phi]^{nb}_{\pm}$ is non-singular, then $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ is the tangent developable surface of $\boldsymbol{\sigma}[\phi]^{nb}_{\pm}$.

Proof. Since $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ is a non-lightlike and non-cylindrical slant developable surface for $\phi \in (0, \pi/2)$, from Lemma 4.7, we have $\boldsymbol{\gamma}'(s) = -\frac{1}{\cos\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)k(s)}\left(\mathbb{N}[\phi]^{nb}_{\pm}\right)'(s)$. It is obvious that $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ is singular at a point $(s_0, u_0) \in I \times J$ if and only if

$$\boldsymbol{\gamma}'(s_0) \times \mathbb{N}[\phi]^{nb}_{\pm}(s_0) + u_0\left(\mathbb{N}[\phi]^{nb}_{\pm}\right)'(s_0) \times \mathbb{N}[\phi]^{nb}_{\pm}(s_0) = \mathbf{0}.$$

If we use $\boldsymbol{\gamma}'(s_0) = -\frac{1}{\cos\phi\,\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}\left(\mathbb{N}[\phi]^{nb}_{\pm}\right)'(s_0)$ in the above equation, we obtain $u_0 = \frac{1}{\cos\phi\,\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}$. Consequently, for the singular locus on $F_{(\gamma,\mathbb{N}[\phi]^{nb}_{\pm})}$, we get

$$\sum\left(F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}\right) = \boldsymbol{\sigma}[\phi]^{nb}_{\pm}(s) = \left\{\gamma(s) + \frac{1}{\cos\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)k(s)}\mathbb{N}[\phi]^{nb}_{\pm}(s)\,\middle|\, s \in I\right\}.$$

It can be easily seen that the singular locus $\boldsymbol{\sigma}[\phi]^{nb}_{\pm}(s)$ is the line of striction of $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$. Moreover, since

$$\left(\sigma[\phi]^{nb}_{\pm}\right)'(s) = -\frac{1}{\cos\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)}\frac{k'(s)}{k^2(s)}\mathbb{N}[\phi]^{nb}_{\pm}(s),$$

a non-lightlike and non-cylindrical slant developable surface $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ can be considered as the tangent developable surface of the singular locus $\sigma[\phi]^{nb}_{\pm}$ if $k'(s) \neq 0$ at any $s \in I$ (that is, if $\boldsymbol{\sigma}[\phi]^{nb}_{\pm}$ is non-singular). □

Now, taking into account [17], in terms of

$$\begin{aligned}\det\left(\mathbb{N}[\phi]^{nb}_{\pm}(s), \left(\mathbb{N}[\phi]^{nb}_{\pm}\right)'(s), \left(\mathbb{N}[\phi]^{nb}_{\pm}\right)''(s)\right) &= -\cos\phi\,\sin^2\phi\,\tau(s)k'(s) \mp \sin^2\phi\,k(s)\tau^2(s)\\ &\quad \mp \cos^2\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)k^3(s) + \cos\phi\,\sin^2\phi\,k(s)\tau'(s),\end{aligned}$$

we have the following theorem for the singularities of a non-lightlike and non-cylindrical slant developable surface $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$, where $\phi \in (0, \pi/2)$:

Theorem 4.9. *Let $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ be a non-lightlike and non-cylindrical slant developable surface, where $\phi \in (0, \pi/2)$. Moreover, let $(s_0, u_0) \in I \times J$ be a singular point of $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ and $x_0 = F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}(s_0, u_0) = \boldsymbol{\gamma}(s_0) + u_0\big(\cos\phi\,\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0)\big)$. In this case, we have the following:*

***(1)** The germ of $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}(I \times J)$ at x_0 is diffeomorphic to the cuspidal edge if $u_0 = \frac{1}{\cos\phi\,\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}$ and $k'(s_0) \neq 0$.*

***(2)** The germ of $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}(I \times J)$ at x_0 is diffeomorphic to the swallowtail if $u_0 = \frac{1}{\cos\phi\,\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}$, $k'(s_0) = 0$ and $k''(s_0) \neq 0$.*

***(3)** The cuspidal cross cap never appears as a singularity of $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$.*

Proof. A non-lightlike and non-cylindrical slant developable surface $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ can be taken into account as the tangent developable surface of the singular locus $\boldsymbol{\sigma}[\phi]^{nb}_{\pm}(s)$ of $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ around $\boldsymbol{\sigma}[\phi]^{nb}_{\pm}(s_0)$ under the condition $\left(\boldsymbol{\sigma}[\phi]^{nb}_{\pm}\right)''(s_0) \neq \mathbf{0}$ even if $\boldsymbol{\sigma}[\phi]^{nb}_{\pm}(s)$ has a singularity at s_0. Hence, the classifications of the singularities of $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ can be reduced to the classifications of the singularities of the tangent developable surface of a (not necessarily regular) space curve in $\mathbb{R}^3_1$ (Cf. [7], [10], [11], [17], [28] and [35] in Euclidean sense.). □

Example 4.10. *Let* $\gamma(s) = \left(\sinh(\sqrt{2s}) - \sqrt{2s}\cosh(\sqrt{2s}), -\cosh(\sqrt{2s}) + \sqrt{2s}\sinh(\sqrt{2s}), 0\right)$*,* $s > 0$*. In this case, we have the following slant ruled surface parametrized by*

$$\begin{aligned} F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}(s,u) &= \Big(\sinh(\sqrt{2s}) - \sqrt{2s}\cosh(\sqrt{2s}) - \cos\phi\,\cosh(\sqrt{2s})\,u, \\ &\qquad -\cosh(\sqrt{2s}) + \sqrt{2s}\sinh(\sqrt{2s}) + \cos\phi\,\sinh(\sqrt{2s})\,u, \pm u\Big), \end{aligned}$$

where $\left(s, -\frac{\sqrt{2s}}{\cos\phi}\right)$ *are its singular points for* $\phi \in [0, \pi/2)$*. Moreover, it is spacelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u\frac{\cos\phi}{\sqrt{2s}} \neq -1$ *(respectively,* $\frac{u}{\sqrt{2s}} \neq -1$*). Furthermore, it is non-cylindrical (respectively, cylindrical) for* $\phi \in [0, \pi/2)$ *(respectively,* $\phi = \pi/2$*). Since* $k(s) = \frac{1}{\sqrt{2s}}$ *and* $k'(s) = -\frac{1}{\sqrt{(2s)^3}}$*, the germ of the slant developable surface* $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}\left(s, -\frac{\sqrt{2s}}{\cos\phi}\right)$ *is diffeomorphic to the cuspidal edge for each* s*, where* $\phi \in (0, \pi/2)$*.*

Example 4.11. *Let* $\gamma(s) = \left(\operatorname{arccosh} s, \sqrt{s^2-1}, 0\right)$*,* $s > 1$*. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}(s,u) = \left(\operatorname{arccosh} s - \cos\phi\,\frac{s\,u}{\sqrt{s^2-1}}, \sqrt{s^2-1} - \cos\phi\,\frac{u}{\sqrt{s^2-1}}, \pm u\right),$$

where $\left(s, \frac{1-s^2}{\cos\phi}\right)$ *are its singular points for* $\phi \in [0, \pi/2)$*. Moreover, it is spacelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u\frac{\cos\phi}{s^2-1} \neq -1$ *(respectively,* $\frac{u}{s^2-1} \neq -1$*). Furthermore, it is non-cylindrical (respectively, cylindrical) for* $\phi \in [0, \pi/2)$ *(respectively,* $\phi = \pi/2$*). Since* $k(s) = \frac{1}{s^2-1}$ *and* $k'(s) = -\frac{2s}{(s^2-1)^2}$*, the germ of the slant developable surface* $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}\left(s, \frac{1-s^2}{\cos\phi}\right)$ *is diffeomorphic to the cuspidal edge for each* s*, where* $\phi \in (0, \pi/2)$*.*

Example 4.12. *Let* $\gamma(s) = \frac{1}{2}\left(s^2, s\sqrt{s^2+1} + \operatorname{arcsinh} s, 0\right)$*. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}(s,u) = \left(\frac{s^2}{2} + \cos\phi\,\sqrt{s^2+1}\,u, \frac{s\sqrt{s^2+1} + \operatorname{arcsinh} s}{2} + \cos\phi\, s\,u, \mp u\right),$$

where $\left(s, -\frac{\sqrt{s^2+1}}{\cos\phi}\right)$ *are its singular points for* $\phi \in [0, \pi/2)$*. Moreover, it is spacelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u\frac{\cos\phi}{\sqrt{s^2+1}} \neq -1$ *(respectively,* $\frac{u}{\sqrt{s^2+1}} \neq -1$*). Furthermore, it is non-cylindrical (respectively, cylindrical) for* $\phi \in [0, \pi/2)$ *(respectively,* $\phi = \pi/2$*). Since* $k(s) = \frac{1}{\sqrt{s^2+1}}$*,* $k'(s) = -\frac{s}{\sqrt{(s^2+1)^3}}$*,* $k''(s) = \frac{2s^2-1}{\sqrt{(s^2+1)^5}}$*,* $k'(0) = 0$ *and* $k''(0) = -1$*, the germ of the slant developable surface* $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}\left(s, -\frac{\sqrt{s^2+1}}{\cos\phi}\right)$ *is diffeomorphic to the cuspidal edge (respectively, the swallowtail) when* $s \neq 0$ *(respectively,* $s = 0$*), where* $\phi \in (0, \pi/2)$*.*

Example 4.13. *Let* $\gamma(s) = \Big(0, 2\arctan(e^s), \ln(2\cosh s)\Big)$*. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}(s,u) \;=\; \Big(\mp u, 2\arctan(e^s) - \cos\phi\, u\, \tanh s, \ln(2\cosh s) + \cos\phi\, \frac{u}{\cosh s}\Big),$$

where $\left(s, \frac{\cosh s}{\cos\phi}\right)$ *are its singular points for* $\phi \in [0, \pi/2)$*. Moreover, it is timelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u\frac{\cos\phi}{\cosh s} \neq 1$ *(respectively,* $\frac{u}{\cosh s} \neq 1$*). Furthermore, it is non-cylindrical (respectively, cylindrical) for* $\phi \in [0, \pi/2)$ *(respectively,* $\phi = \pi/2$*). Since* $k(s) = \frac{1}{\cosh s}$*,* $k'(s) = -\frac{\sinh s}{\cosh^2 s}$*,* $k''(s) = \frac{\sinh^2 s - 1}{\cosh^3 s}$*,* $k'(0) = 0$ *and* $k''(0) = -1$*, the germ of the slant developable surface* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}\left(s, \frac{\cosh s}{\cos\phi}\right)$ *is diffeomorphic to the cuspidal edge (respectively, the swallowtail) when* $s \neq 0$ *(respectively,* $s = 0$*), where* $\phi \in (0, \pi/2)$*.*

Example 4.14. *Let* $\gamma(s) = \frac{1}{2}\Big(0, 1 - s^2, \arccos s - s\sqrt{1 - s^2}\Big)$*,* $1 > s^2$*. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}(s,u) \;=\; \left(\pm u, \frac{1 - s^2}{2} - \cos\phi\, \sqrt{1 - s^2}\, u, \frac{\arccos s - s\sqrt{1 - s^2}}{2} + \cos\phi\, s\, u\right),$$

where $\left(s, \frac{\sqrt{1-s^2}}{\cos\phi}\right)$ *are its singular points for* $\phi \in [0, \pi/2)$*. Moreover, it is timelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u\frac{\cos\phi}{\sqrt{1-s^2}} \neq 1$ *(respectively,* $\frac{u}{\sqrt{1-s^2}} \neq 1$*). Furthermore, it is non-cylindrical (respectively, cylindrical) for* $\phi \in [0, \pi/2)$ *(respectively,* $\phi = \pi/2$*). Since* $k(s) = \frac{1}{\sqrt{1-s^2}}$*,* $k'(s) = \frac{s}{\sqrt{(1-s^2)^3}}$*,* $k''(s) = \frac{1+2s^2}{\sqrt{(1-s^2)^5}}$*,* $k'(0) = 0$ *and* $k''(0) = 1$*, the germ of the slant developable surface* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}\left(s, \frac{\sqrt{1-s^2}}{\cos\phi}\right)$ *is diffeomorphic to the cuspidal edge (respectively, the swallowtail) when* $s \neq 0$ *(respectively,* $s = 0$*), where* $\phi \in (0, \pi/2)$*.*

Example 4.15. *Let* $\gamma(s) = \Big(\ln(\sec s), \ln(\sec s + \tan s), 0\Big)$*,* $0 \leq s < \frac{\pi}{2}$*. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}(s,u) = \Big(\ln(\sec s) + \cos\phi\, u\, \sec s, \ln(\sec s + \tan s) + \cos\phi\, u\, \tan s, \mp u\Big),$$

where $\left(s, -\frac{\cos s}{\cos\phi}\right)$ *are its singular points for* $\phi \in [0, \pi/2)$*. Moreover, it is spacelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u\frac{\cos\phi}{\cos s} \neq -1$ *(respectively,* $\frac{u}{\cos s} \neq -1$*). Furthermore, it is non-cylindrical (respectively, cylindrical) for* $\phi \in [0, \pi/2)$ *(respectively,* $\phi = \pi/2$*). Since* $k(s) = \frac{1}{\cos s}$*,* $k'(s) = \frac{\sin s}{\cos^2 s}$*,* $k''(s) = \frac{1+\sin^2 s}{\cos^3 s}$*,* $k'(0) = 0$ *and* $k''(0) = 1$*, the germ of the slant developable surface* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}\left(s, -\frac{\cos s}{\cos\phi}\right)$ *is diffeomorphic to the cuspidal edge (respectively, the swallowtail) when* $s \neq 0$ *(respectively,* $s = 0$*), where* $\phi \in (0, \pi/2)$*.*

5. Singularities of slant ruled surfaces with the director curve $\mathbb{N}[\phi]^{nb}_{\pm}$

In this section, taking into account [17] for the principal normal surface of a unit speed curve with non-zero curvature in Euclidean 3-space, we investigate the singularities of slant ruled surfaces $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$, where $\phi \in (0, \pi/2)$.

Theorem 5.1. *Let* $\phi \in (0, \pi/2)$*. For a spacelike curve* $\gamma : I \to \mathbb{R}^3_1$ *which is parametrized by arc length s such that* $k(s) \neq 0$*, the slant normal surface* $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *of* γ *is the cross cap at* $(s_0, u_0) \in I \times J$ *if and only if*

$$u_0 = \frac{1}{\cos\phi\, \delta\big(\gamma(s_0)\big) k(s_0)}, \quad \tau(s_0) = 0 \;\; and \;\; \tau'(s_0) \neq 0.$$

Proof. For $\phi \in (0, \pi/2)$, we showed in Section 3 that (s_0, u_0) is a singular point of $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ if and only if $\tau(s_0) = 0$ and $u_0 = \dfrac{1}{\cos\phi\,\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}$. If we use these equations in the derivative equations of $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ at (s_0, u_0), we find

$$\frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}}{\partial s}(s_0,u_0) = \mathbf{0},$$
$$\frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}}{\partial u}(s_0,u_0) = \cos\phi\,\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0),$$
$$\frac{\partial^2 F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}}{\partial u\partial s}(s_0,u_0) = -\cos\phi\,\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)\boldsymbol{t}(s_0),$$
$$\frac{\partial^2 F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}}{\partial s^2}(s_0,u_0) = -\frac{k'(s_0)}{k(s_0)}\boldsymbol{t}(s_0) \pm \frac{\tau'(s_0)}{\cos\phi\,\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}\boldsymbol{n}(s_0) + \frac{\tau'(s_0)}{\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}\boldsymbol{b}(s_0).$$

By means of these relations, we deduce

$$\det\left(\frac{\partial F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}}{\partial u}(s_0,u_0), \frac{\partial^2 F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}}{\partial u\partial s}(s_0,u_0), \frac{\partial^2 F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}}{\partial s^2}(s_0,u_0)\right) = \sin^2\phi\,\delta\big(\boldsymbol{\gamma}(s_0)\big)\tau'(s_0).$$

In terms of the characterization of the cross cap which was given in [2], [5] and [17] for the Euclidean case, we find from the last equation that $\tau'(s_0) \neq 0$. Thus, the proof is completed. □

Example 5.2. *Let* $\boldsymbol{\gamma}(s) = \left(-\frac{(a^2-1)}{2}\left(\frac{\cosh((a+1)s)}{(a+1)^2} + \frac{\cosh((a-1)s)}{(a-1)^2}\right), -\frac{(a^2-1)}{2}\left(\frac{\sinh((a+1)s)}{(a+1)^2} - \frac{\sinh((a-1)s)}{(a-1)^2}\right), -\frac{\sqrt{a^2-1}}{a}\cosh(as)\right)$*, where* $a^2 > 1$*. It follows that* $k(s) = \sqrt{a^2-1}\cosh(as)$*,* $\tau(s) = -\sqrt{a^2-1}\sinh(as)$*,* $\tau(0) = 0$ *and so* $\left(0, \frac{1}{\cos\phi\,\sqrt{a^2-1}}\right)$ *(respectively,* $\left(s, \frac{1}{\sqrt{a^2-1}\cosh(as)}\right)$*) is the singular point (respectively, are the singular points) of* $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *for* $\phi \in (0, \pi/2)$ *(respectively,* $\phi = 0$*). Moreover, since* $\tau'(s) = -a\sqrt{a^2-1}\cosh(as)$ *and* $\tau'(0) = -a\sqrt{a^2-1}$*,* $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *is the cross cap at* $\left(0, \frac{1}{\cos\phi\,\sqrt{a^2-1}}\right)$ *for* $\phi \in (0, \pi/2)$*.*

Example 5.3. *Let* $\boldsymbol{\gamma}(s) = \left(\frac{s^2}{2}, s\cos s, s\sin s\right)$*. It follows that* $k(s) = \sqrt{s^2+3}$*,* $\tau(s) = -\frac{s(s^2+4)}{s^2+3}$*,* $\tau(0) = 0$ *and so* $\left(0, \frac{1}{\cos\phi\,\sqrt{3}}\right)$ *(respectively,*$\left(s, \frac{1}{\sqrt{s^2+3}}\right)$*) is the singular point (respectively, are the singular points) of* $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *for* $\phi \in (0, \pi/2)$ *(respectively,* $\phi = 0$*). Moreover, since* $\tau'(s) = -\frac{(s^4+5s^2+12)}{(s^2+3)^2}$ *and* $\tau'(0) = -\frac{4}{3}$*,* $F_{\left(\gamma,\mathbb{N}[\phi]^{nb}_{\pm}\right)}$ *is the cross cap at* $\left(0, \frac{1}{\cos\phi\,\sqrt{3}}\right)$ *for* $\phi \in (0, \pi/2)$*.*

Now, we take into account the following generic conditions on a space curve $\gamma : S^1 \to \mathbb{R}^3_1$ which is spacelike and parametrized by arc length s (See [5] and [17] for the Euclidean case.):

(1) There are no points on S^1 with $\tau(s) = \tau'(s) = 0$.
(2) The number of the points $s_0 \in S^1$ such that $\tau(s_0) = 0$ and $\tau'(s_0) \neq 0$ is finite.
(3) $k(s) \neq 0$ at any point $s \in S^1$.

Thus, taking into account [17], we have the following corollary:

Corollary 5.4. *For a "generic" spacelike curve $\boldsymbol{\gamma} : S^1 \to \mathbb{R}^3_1$, the number of the singular points of $F_{\left(\gamma, \mathbb{N}[\phi]^{nb}_{\pm}\right)}$ is finite and each singular point is the cross cap.*

6. Slant ruled surfaces with the director curve $\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}$

In this section, in a similar way to the one in Section 3, for any fixed $\phi \in [0, \pi/2]$, we define a *slant ruled surface* by

$$\begin{array}{rcccl} F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)} & : & I \times J & \longrightarrow & \mathbb{R}^3_1 \\ & & (s,u) & \longmapsto & \boldsymbol{\gamma}(s) + u\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s) \end{array}$$

such that $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ is a spacelike base curve parametrized by arc length s, $\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm} = \boldsymbol{n} \pm \cos\phi\, \boldsymbol{b}$ is a director curve and the straight lines $u \longmapsto \boldsymbol{\gamma}(s) + u\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s)$ are rulings, where I and J are open intervals in $\mathbb{R}$ or unit circles S^1. Here, we remark that

$$\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s) \in \begin{cases} S^2_1(\sin^2\phi) & \text{if } \boldsymbol{n}(s) \text{ is spacelike,} \\ H^2(-\sin^2\phi) & \text{if } \boldsymbol{n}(s) \text{ is timelike} \end{cases}$$

for any fixed $\phi \in [0, \pi/2]$ and we say that $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is

$$\begin{cases} \textit{a } \phi\textit{-de Sitter normal surface of } \boldsymbol{\gamma} & \text{if } \boldsymbol{n}(s) \text{ is spacelike,} \\ \textit{a } \phi\textit{-hyperbolic normal surface of } \boldsymbol{\gamma} & \text{if } \boldsymbol{n}(s) \text{ is timelike.} \end{cases}$$

We briefly say that $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is a *slant normal surface* of $\boldsymbol{\gamma}$ if it is either a ϕ-de Sitter normal surface or a ϕ-hyperbolic normal surface of $\boldsymbol{\gamma}$. Especially, we say that $F_{\left(\gamma, \widetilde{\mathbb{N}}[\pi/2]^{nb}_{\pm}\right)}$ is

$$\begin{cases} \textit{a de Sitter principal normal surface of } \boldsymbol{\gamma} & \text{if } \boldsymbol{n}(s) \text{ is spacelike,} \\ \textit{a hyperbolic principal normal surface of } \boldsymbol{\gamma} & \text{if } \boldsymbol{n}(s) \text{ is timelike} \end{cases}$$

(Cf. [17] in Euclidean sense.). Moreover, $F_{\left(\gamma, \widetilde{\mathbb{N}}[0]^{nb}_{\pm}\right)}$ is said to be *the lightcone normal surface of* $\boldsymbol{\gamma}$, where $\widetilde{\mathbb{N}}[0]^{nb}_{\pm}(s) \in LC^*$. Here, we remark that this case was investigated in [14].

For the normal vector of a slant ruled surface $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$, we obtain

$$\frac{\partial F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}}{\partial s}(s,u) \times \frac{\partial F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}}{\partial u}(s,u) = \boldsymbol{\gamma}'(s) \times \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s) + u\left(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)'(s) \times \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s)$$

at any $(s,u) \in I \times J$. If we denote this normal vector by $\widetilde{N}^{\phi,nb}_{\pm}(s,u)$, then we get

$$\widetilde{N}^{\phi,nb}_{\pm}(s,u) = u\sin^2\phi\, \delta\big(\boldsymbol{\gamma}(s)\big)\tau(s)\boldsymbol{t}(s) \pm \cos\phi\Big(1 - u\delta\big(\boldsymbol{\gamma}(s)\big)k(s)\Big)\boldsymbol{n}(s) + \Big(1 - u\delta\big(\boldsymbol{\gamma}(s)\big)k(s)\Big)\boldsymbol{b}(s).$$

Consequently, we have the following propositions and remark:

Proposition 6.1. *Let $\phi \in (0, \pi/2]$. (s_0, u_0) is a singular point of $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ if and only if $\tau(s_0) = 0$ and $u_0 = \dfrac{1}{\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}$.*

Proposition 6.2. $\left(s_0, \dfrac{1}{\delta\big(\gamma(s_0)\big)k(s_0)}\right)$ *is a singular point of* $F_{\left(\gamma, \widetilde{\mathbb{N}}[0]^{nb}_{\pm}\right)}$.

We point out that $\phi = 0$ case was studied in [14].

Remark 6.3. *The singular points of* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ *don't depend on* ϕ.

Now, we take into account the following cases:

(1) $\phi = 0$ and $u\delta\big(\gamma(s)\big)k(s) \neq 1$.

(2) $\phi \in (0, \pi/2]$, $n(s)$ is spacelike and $u^2 \sin^2\phi\, \tau^2(s) > \big(1 - uk(s)\big)^2$, where one of the following conditions holds:
(i) $\tau(s) \neq 0$ and $uk(s) = 1$,
(ii) $\tau(s) \neq 0$ and $uk(s) \neq 1$.

(3) $\phi \in (0, \pi/2]$, $n(s)$ is spacelike and $u^2 \sin^2\phi\, \tau^2(s) = \big(1 - uk(s)\big)^2$, where $\tau(s) \neq 0$ and $uk(s) \neq 1$.

(4) $\phi \in (0, \pi/2]$, $n(s)$ is spacelike and $u^2 \sin^2\phi\, \tau^2(s) < \big(1 - uk(s)\big)^2$, where one of the following conditions holds:
(i) $\tau(s) = 0$ and $uk(s) \neq 1$,
(ii) $\tau(s) \neq 0$ and $uk(s) \neq 1$.
(5) $\phi \in (0, \pi/2]$, $n(s)$ is timelike and at least one of the following conditions holds:
(i) $\tau(s) \neq 0$,
(ii) $uk(s) \neq -1$.

In terms of the above cases, we classify the normal vector $\widetilde{N}^{\phi,nb}_{\pm}(s,u)$ of $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ at any regular $(s,u) \in I \times J$ as follows:

$$\widetilde{N}^{\phi,nb}_{\pm}(s,u) \text{ is } \begin{cases} \text{spacelike} & \text{if either } \textbf{(2)} \text{ or } \textbf{(5)} \text{ is satisfied,} \\ \text{lightlike} & \text{if either } \textbf{(1)} \text{ or } \textbf{(3)} \text{ is satisfied,} \\ \text{timelike} & \text{if } \textbf{(4)} \text{ is satisfied.} \end{cases}$$

Example 6.4. *Let* $\gamma(s) = (0, \cos s, \sin s)$*, where* $0 \leq s < 2\pi$*. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s,u) = \big(\mp \cos\phi\, u, (1-u)\cos s, (1-u)\sin s\big),$$

where the points $(s, 1)$ *are its singular points for* $\phi \in [0, \pi/2]$*. Moreover,* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ *is spacelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u \neq 1$.

Example 6.5. *Let* $\gamma(s) = (\cosh s, \sinh s, 0)$*. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s,u) = \big((1+u)\cosh s, (1+u)\sinh s, \mp \cos\phi\, u\big),$$

where the points $(s, -1)$ *are its singular points for* $\phi \in [0, \pi/2]$*. Moreover,* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ *is timelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u \neq -1$.

Example 6.6. *Let* $\gamma(s) = \left(\cosh s, \dfrac{\sinh s}{\sqrt{2}}, \dfrac{\sinh s}{\sqrt{2}}\right)$*. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s,u) = \left((1+u)\cosh s, \frac{1}{\sqrt{2}}(1+u)\sinh s \pm \frac{\cos\phi}{\sqrt{2}}\, u, \frac{1}{\sqrt{2}}(1+u)\sinh s \mp \frac{\cos\phi}{\sqrt{2}}\, u\right),$$

where the points $(s,-1)$ are its singular points for $\phi \in [0,\pi/2]$. Moreover, $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is timelike (respectively, lightlike) for $\phi \in (0,\pi/2]$ (respectively, $\phi = 0$), where $u \neq -1$.

Example 6.7. *Let $\gamma(s) = \left(\sin s, \sqrt{2}\sin s, \cos s\right)$, where $0 \leq s < 2\pi$. In this case, we have the following slant ruled surface parametrized by*

$$F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s,u) = \left((1-u)\sin s \pm \sqrt{2}\cos\phi\, u,\ \sqrt{2}(1-u)\sin s \pm \cos\phi\, u, (1-u)\cos s\right),$$

where the points $(s,1)$ are its singular points for $\phi \in [0,\pi/2]$. Moreover, $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is spacelike (respectively, lightlike) for $\phi \in (0,\pi/2]$ (respectively, $\phi = 0$), where $u \neq 1$.

We can define the unit non-lightlike normal vector denoted by $\tilde{\mathfrak{n}}^{\phi,nb}_{\pm}(s,u)$ of $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ at any regular $(s,u) \in I \times J$ as follows:

$$\tilde{\mathfrak{n}}^{\phi,nb}_{\pm}(s,u) = \begin{cases} \dfrac{u\sin^2\phi\,\tau(s)\boldsymbol{t}(s) \pm \cos\phi\big(1-uk(s)\big)\boldsymbol{n}(s) + \big(1-uk(s)\big)\boldsymbol{b}(s)}{\sin\phi\,\sqrt{u^2\sin^2\phi\,\tau^2(s) - \big(1-uk(s)\big)^2}} & \text{if } \textbf{(2)} \text{ is satisfied,} \\ \dfrac{-u\sin^2\phi\,\tau(s)\boldsymbol{t}(s) \pm \cos\phi\big(1+uk(s)\big)\boldsymbol{n}(s) + \big(1+uk(s)\big)\boldsymbol{b}(s)}{\sin\phi\,\sqrt{u^2\sin^2\phi\,\tau^2(s) + \big(1+uk(s)\big)^2}} & \text{if } \textbf{(5)} \text{ is satisfied,} \\ \dfrac{u\sin^2\phi\,\tau(s)\boldsymbol{t}(s) \pm \cos\phi\big(1-uk(s)\big)\boldsymbol{n}(s) + \big(1-uk(s)\big)\boldsymbol{b}(s)}{\sin\phi\,\sqrt{-\left(u^2\sin^2\phi\,\tau^2(s) - \big(1-uk(s)\big)^2\right)}} & \text{if } \textbf{(4)} \text{ is satisfied.} \end{cases}$$

By the considerations similar to the ones in Section 3, for the Gauss curvature denoted by $\widetilde{K}^{\phi,nb}_{\pm}$ of a non-lightlike (either timelike or spacelike) slant ruled surface $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$, we obtain the following classifications:

$$\widetilde{K}^{\phi,nb}_{\pm}(s,u) = \begin{cases} \dfrac{\tau^2(s)}{\left(u^2\sin^2\phi\,\tau^2(s) - \big(1-uk(s)\big)^2\right)^2} > 0 & \text{if } \textbf{(2)} \text{ is satisfied,} \\ \dfrac{\tau^2(s)}{\left(u^2\sin^2\phi\,\tau^2(s) + \big(1+uk(s)\big)^2\right)^2} \geq 0 & \text{if } \textbf{(5)} \text{ is satisfied,} \\ -\dfrac{\tau^2(s)}{\left(u^2\sin^2\phi\,\tau^2(s) - \big(1-uk(s)\big)^2\right)^2} \leq 0 & \text{if } \textbf{(4)} \text{ is satisfied.} \end{cases}$$

As a result, for any non-lightlike slant ruled surface $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$, we can deduce that

$$\widetilde{K}^{\phi,nb}_{\pm}(s,u) = 0 \Longleftrightarrow \tau(s) = 0.$$

Thus, we have the following proposition which is similar to Proposition 3.8.

Proposition 6.8. *Singular points of a non-lightlike slant ruled surface $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ are located on the line of striction on which the Gauss curvature $\widetilde{K}^{\phi,nb}_{\pm}$ is zero. At regular points of a timelike (respectively, spacelike) slant ruled surface $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$, $\widetilde{K}^{\phi,nb}_{\pm}$ satisfies $\widetilde{K}^{\phi,nb}_{\pm} \geq 0$ (respectively, $\widetilde{K}^{\phi,nb}_{\pm} \leq 0$) and $\widetilde{K}^{\phi,nb}_{\pm}$ is zero only along the rulings which meet the line of striction at a singular point.*

7. Singularities of non-lightlike and non-cylindrical slant developable surfaces with the director curve $\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}$

Following Section 4, for any fixed $\phi \in [0, \pi/2]$, we say that a non-lightlike slant ruled surface $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is a *non-lightlike slant developable surface* if the Gauss curvature $\widetilde{K}^{\phi,nb}_{\pm}$ of the regular part of $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is identically zero. Moreover, we say that a slant developable surface $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s, u)$ is a ϕ-*de Sitter* (respectively, ϕ-*hyperbolic*) *normal developable surface of* $\boldsymbol{\gamma}(s)$ if $\boldsymbol{n}(s)$ is spacelike (respectively, timelike). Furthermore, $F_{\left(\gamma, \widetilde{\mathbb{N}}[0]^{nb}_{\pm}\right)}(s, u) = \boldsymbol{\gamma}(s) + u(\boldsymbol{n}(s) \pm \boldsymbol{b}(s))$ is said to be *the lightcone developable surface of* $\boldsymbol{\gamma}(s)$, where $\widetilde{\mathbb{N}}[0]^{nb}_{\pm} \in LC^{*}$. Here, we note that this case was investigated in [14]. We also remark that since the proofs of our results in this section are similar to the ones in Section 4, we omit them.

It can be easily verified that

$$\det\Big(\boldsymbol{\gamma}'(s), \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s), \big(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\big)'(s)\Big) = -\sin^2\phi\, \delta\big(\boldsymbol{\gamma}(s)\big)\tau(s).$$

So, taking into account [17], [36]-[38], [40] and [41], we have the following proposition which is similar to Proposition 4.1:

Proposition 7.1. *Let $\phi \in (0, \pi/2]$. Then, a non-lightlike slant ruled surface $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is a non-lightlike slant developable surface if and only if*

$$\det\Big(\boldsymbol{\gamma}'(s), \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s), \big(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\big)'(s)\Big) = 0.$$

On the other hand, since

$$\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s) \times \big(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\big)'(s) = -\sin^2\phi\, \delta\big(\boldsymbol{\gamma}(s)\big)\tau(s)\boldsymbol{t}(s) \pm \cos\phi\, \delta\big(\boldsymbol{\gamma}(s)\big)k(s)\boldsymbol{n}(s) + \delta\big(\boldsymbol{\gamma}(s)\big)k(s)\boldsymbol{b}(s),$$

we have the following proposition:

Proposition 7.2. *A slant ruled surface $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is always non-cylindrical for $\phi \in [0, \pi/2]$.*

As a result, the space of non-lightlike and non-cylindrical slant developable surfaces $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is given by

$$\widetilde{Dev}[\phi]^{nb}_{\pm}(I, \mathbb{R}^3_1) = \{\, \boldsymbol{\gamma} : I \to \mathbb{R}^3_1 \text{ is a spacelike curve which is parametrized by arc length } s \mid k(s) \neq 0 \text{ and } \tau(s) = 0 \text{ for any } s \in I\},$$

where $\phi \in (0, \pi/2]$ (See [17] for the Euclidean case.).

Example 7.3. *In Example 6.4, $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is a non-cylindrical and ϕ-de Sitter normal developable surface of $\boldsymbol{\gamma}$.*

Example 7.4. *In Example 6.5, $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is a non-cylindrical and ϕ-hyperbolic normal developable surface of $\boldsymbol{\gamma}$.*

Example 7.5. *In Example 6.6, $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is a non-cylindrical and ϕ-hyperbolic normal developable surface of $\boldsymbol{\gamma}$.*

Example 7.6. *In Example 6.7, $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is a non-cylindrical and ϕ-de Sitter normal developable surface of $\boldsymbol{\gamma}$.*

Now, taking into account Section 4, we have the following lemma and corollary:

Lemma 7.7. *Let $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ be a non-lightlike and non-cylindrical slant ruled surface, where $\phi \in (0,\pi/2]$. Then, $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is a non-lightlike slant developable surface if and only if*

$$\boldsymbol{\gamma}'(s) = -\frac{1}{\delta\big(\boldsymbol{\gamma}(s)\big)k(s)}\Big(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\Big)'(s).$$

Corollary 7.8. *Let $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ be a non-lightlike and non-cylindrical slant developable surface, where $\phi \in (0,\pi/2]$. In this case, the set of the singular points of $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is a curve parametrized by*

$$\tilde{\boldsymbol{\sigma}}[\phi]^{nb}_{\pm}(s) = \boldsymbol{\gamma}(s) + \frac{1}{\delta\big(\boldsymbol{\gamma}(s)\big)k(s)}\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s).$$

If $\tilde{\boldsymbol{\sigma}}[\phi]^{nb}_{\pm}$ is non-singular, then $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is the tangent developable surface of $\tilde{\boldsymbol{\sigma}}[\phi]^{nb}_{\pm}$.

Now, in a similar way to the one in Section 4, in terms of

$$\begin{aligned}\det\Big(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s),\big(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\big)'(s),\big(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\big)''(s)\Big) &= \sin^2\phi\,\tau(s)k'(s) \pm \cos\phi\,\sin^2\phi\,k(s)\tau^2(s)\\ &\quad \mp \cos\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)k^3(s) - \sin^2\phi\,k(s)\tau'(s)\end{aligned}$$

and

$$\begin{aligned}\det\Big(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}(s),\big(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\big)'(s),\big(\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\big)'''(s)\Big) &= \sin^2\phi\,\tau(s)k''(s) \pm 2\cos\phi\,\sin^2\phi\,k(s)\tau(s)\tau'(s)\\ &\quad \pm \cos\phi\,\sin^2\phi\,\tau^2(s)k'(s) \mp 3\cos\phi\,\delta\big(\boldsymbol{\gamma}(s)\big)k^2(s)k'(s)\\ &\quad - \sin^2\phi\,k(s)\tau''(s),\end{aligned}$$

we have the following theorem for the singularities of a non-lightlike and non-cylindrical slant developable surface $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$, where $\phi \in (0,\pi/2]$:

Theorem 7.9. *Let $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ be a non-lightlike and non-cylindrical slant developable surface, where $\phi \in (0,\pi/2]$. Moreover, let $(s_0,u_0) \in I \times J$ be a singular point of $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ and $x_0 = F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s_0,u_0) = \boldsymbol{\gamma}(s_0) + u_0\big(\boldsymbol{n}(s_0) \pm \cos\phi\,\boldsymbol{b}(s_0)\big)$. Then, we have the following:*
***(1)** Let $\phi \in (0,\pi/2)$. In this case,*
***(i)** the germ of $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(I\times J)$ at x_0 is diffeomorphic to the cuspidal edge if $u_0 = \dfrac{1}{\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}$ and $k'(s_0) \neq 0$.*
***(ii)** the germ of $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(I\times J)$ at x_0 is diffeomorphic to the swallowtail if $u_0 = \dfrac{1}{\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}$, $k'(s_0) = 0$ and $k''(s_0) \neq 0$.*
***(2)** The cuspidal cross cap never appears as a singularity of $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$.*

Example 7.10. *Consider the curve given in Example 4.10. In this case, we have the following non-cylindrical slant ruled surface parametrized by*

$$\begin{aligned}F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s,u) &= \Big(\sinh(\sqrt{2}s) - \sqrt{2}s\cosh(\sqrt{2}s) - u\,\cosh(\sqrt{2}s),\\ &\qquad -\cosh(\sqrt{2}s) + \sqrt{2}s\sinh(\sqrt{2}s) + u\,\sinh(\sqrt{2}s), \pm\cos\phi\,u\Big),\end{aligned}$$

where $\left(s, -\sqrt{2s}\right)$ *are its singular points and* $\phi \in [0, \pi/2]$*. Moreover, it is timelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u \neq -\sqrt{2s}$*. Furthermore, the germ of the slant developable surface* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}\left(s, -\sqrt{2s}\right)$ *is diffeomorphic to the cuspidal edge for each* s*, where* $\phi \in (0, \pi/2)$.

Example 7.11. *Consider the curve given in Example 4.11. In this case, we have the following non-cylindrical slant ruled surface parametrized by*

$$F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s, u) = \left(\operatorname{arccosh} s - \frac{s\,u}{\sqrt{s^2 - 1}}, \sqrt{s^2 - 1} - \frac{u}{\sqrt{s^2 - 1}}, \pm \cos\phi\, u\right),$$

where $(s, 1 - s^2)$ *are its singular points and* $\phi \in [0, \pi/2]$*. Moreover, it is timelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u \neq 1 - s^2$*. Furthermore, the germ of the slant developable surface* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s, 1 - s^2)$ *is diffeomorphic to the cuspidal edge for each* s*, where* $\phi \in (0, \pi/2)$.

Example 7.12. *Consider the curve given in Example 4.12. In this case, we have the following non-cylindrical slant ruled surface parametrized by*

$$F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s, u) = \left(\frac{s^2}{2} + \sqrt{s^2 + 1}\, u, \frac{s\sqrt{s^2 + 1} + \operatorname{arcsinh} s}{2} + s\, u, \mp \cos\phi\, u\right),$$

where $\left(s, -\sqrt{s^2 + 1}\right)$ *are its singular points and* $\phi \in [0, \pi/2]$*. Moreover, it is timelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u \neq -\sqrt{s^2 + 1}$*. Furthermore, the germ of the slant developable surface* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}\left(s, -\sqrt{s^2 + 1}\right)$ *is diffeomorphic to the cuspidal edge (respectively, the swallowtail) when* $s \neq 0$ *(respectively,* $s = 0$*), where* $\phi \in (0, \pi/2)$.

Example 7.13. *Consider the curve given in Example 4.13. In this case, we have the following non-cylindrical slant ruled surface parametrized by*

$$F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s, u) = \left(\mp \cos\phi\, u, 2\arctan(e^s) - u\, \tanh s, \ln(2\cosh s) + \frac{u}{\cosh s}\right),$$

where $(s, \cosh s)$ *are its singular points and* $\phi \in [0, \pi/2]$*. Moreover, it is spacelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u \neq \cosh s$*. Furthermore, the germ of the slant developable surface* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s, \cosh s)$ *is diffeomorphic to the cuspidal edge (respectively, the swallowtail) when* $s \neq 0$ *(respectively,* $s = 0$*), where* $\phi \in (0, \pi/2)$.

Example 7.14. *Consider the curve given in Example 4.14. In this case, we have the following non-cylindrical slant ruled surface parametrized by*

$$F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s, u) = \left(\pm \cos\phi\, u, \frac{1 - s^2}{2} - \sqrt{1 - s^2}\, u, \frac{\arccos s - s\sqrt{1 - s^2}}{2} + s\, u\right),$$

where $\left(s, \sqrt{1 - s^2}\right)$ *are its singular points and* $\phi \in [0, \pi/2]$*. Moreover, it is spacelike (respectively, lightlike) for* $\phi \in (0, \pi/2]$ *(respectively,* $\phi = 0$*), where* $u \neq \sqrt{1 - s^2}$*. Furthermore, the germ of the slant developable surface* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(I \times J)$ *at* $F_{\left(\gamma, \widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}\left(s, \sqrt{1 - s^2}\right)$ *is diffeomorphic to the cuspidal edge (respectively, the swallowtail) when* $s \neq 0$ *(respectively,* $s = 0$*), where* $\phi \in (0, \pi/2)$.

Example 7.15. *Consider the curve given in Example 4.15. In this case, we have the following non-cylindrical slant ruled surface parametrized by*

$$F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s,u) = \Big(\ln(\sec s) + u\ \sec s, \ln(\sec s + \tan s) + u\ \tan s, \mp \cos\phi\, u\Big),$$

where $(s, -\cos s)$ are its singular points and $\phi \in [0, \pi/2]$. Moreover, it is timelike (respectively, lightlike) for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$), where $u \neq -\cos s$. Furthermore, the germ of the slant developable surface $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(I \times J)$ at $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}(s, -\cos s)$ is diffeomorphic to the cuspidal edge (respectively, the swallowtail) when $s \neq 0$ (respectively, $s = 0$), where $\phi \in (0, \pi/2)$.

Now, taking into account [33] and [39] (See also [24] and [34].), we have the following theorems when $\phi = \pi/2$, where $k(s) \neq 0$ and $\tau(s) = 0$ for each $s \in I$:

Theorem 7.16. $F_{\left(\gamma,\widetilde{\mathbb{N}}[\pi/2]^{nb}_{\pm}\right)}$ *at* $\left(s_0, \dfrac{1}{\delta\big(\gamma(s_0)\big)k(s_0)}\right)$ *is $\mathscr{A}$-equivalent to*
(1) *the fold if and only if $k'(s_0) \neq 0$.*
(2) *the cusp if and only if $k'(s_0) = 0$ and $k''(s_0) \neq 0$.*
(3) *the swallowtail if and only if $k'(s_0) = k''(s_0) = 0$ and $k'''(s_0) \neq 0$.*

Proof. The proof is clear from the criteria given in the Fact 2 (See also [39]) and Theorem 3 in [33]. □

Theorem 7.17. *The lips and the beaks never appear as singularities of* $F_{\left(\gamma,\widetilde{\mathbb{N}}[\pi/2]^{nb}_{\pm}\right)}$.

Proof. The proof is clear from the criteria given in Theorem 3 in [33]. □

Example 7.18. *In Example 7.10,* $F_{\left(\gamma,\widetilde{\mathbb{N}}[\pi/2]^{nb}_{\pm}\right)}$ *at $(s, -\sqrt{2s})$ is $\mathscr{A}$-equivalent to the fold for each s.*

Example 7.19. *In Example 7.11,* $F_{\left(\gamma,\widetilde{\mathbb{N}}[\pi/2]^{nb}_{\pm}\right)}$ *at $(s, 1 - s^2)$ is $\mathscr{A}$-equivalent to the fold for each s.*

Example 7.20. *In Example 7.12,* $F_{\left(\gamma,\widetilde{\mathbb{N}}[\pi/2]^{nb}_{\pm}\right)}$ *at $(s, -\sqrt{s^2+1})$ is $\mathscr{A}$-equivalent to the fold (respectively, cusp) when $s \neq 0$ (respectively, $s = 0$).*

Example 7.21. *In Example 7.13,* $F_{\left(\gamma,\widetilde{\mathbb{N}}[\pi/2]^{nb}_{\pm}\right)}$ *at $(s, \cosh s)$ is $\mathscr{A}$-equivalent to the fold (respectively, cusp) when $s \neq 0$ (respectively, $s = 0$).*

Example 7.22. *In Example 7.14,* $F_{\left(\gamma,\widetilde{\mathbb{N}}[\pi/2]^{nb}_{\pm}\right)}$ *at $(s, \sqrt{1-s^2})$ is $\mathscr{A}$-equivalent to the fold (respectively, cusp) when $s \neq 0$ (respectively, $s = 0$).*

Example 7.23. *In Example 7.15,* $F_{\left(\gamma,\widetilde{\mathbb{N}}[\pi/2]^{nb}_{\pm}\right)}$ *at $(s, -\cos s)$ is $\mathscr{A}$-equivalent to the fold (respectively, cusp) when $s \neq 0$ (respectively, $s = 0$).*

8. Singularities of slant ruled surfaces with the director curve $\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}$

In this section, following Section 5, we investigate the singularities of slant ruled surfaces $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$, where $\phi \in (0, \pi/2]$. Since the proof of the following theorem is similar to the proof of Theorem 5.1, we omit it.

Theorem 8.1. *Let $\phi \in (0, \pi/2]$. For a spacelike curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ which is parametrized by arc length s such that $k(s) \neq 0$, the slant normal surface $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ of $\boldsymbol{\gamma}$ is the cross cap at $(s_0, u_0) \in I \times J$ if and only if*

$$u_0 = \frac{1}{\delta\big(\boldsymbol{\gamma}(s_0)\big)k(s_0)}, \quad \tau(s_0) = 0 \ \text{ and } \ \tau'(s_0) \neq 0.$$

Example 8.2. *Consider the curve given in Example 5.2. It is clear that $\left(0, \frac{1}{\sqrt{a^2-1}}\right)$ (respectively, $\left(s, \frac{1}{\sqrt{a^2-1}\cosh(as)}\right)$) is the singular point (are the singular points) of $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$). Moreover, $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is the cross cap at $\left(0, \frac{1}{\sqrt{a^2-1}}\right)$ for $\phi \in (0, \pi/2]$.*

Example 8.3. *Consider the curve given in Example 5.3. It is clear that $\left(0, \frac{1}{\sqrt{3}}\right)$ (respectively, $\left(s, \frac{1}{\sqrt{s^2+3}}\right)$) is the singular point (respectively, are the singular points) of $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$). Moreover, $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is the cross cap at $\left(0, \frac{1}{\sqrt{3}}\right)$ for $\phi \in (0, \pi/2]$.*

Now, considering the generic conditions expressed in Section 5 for a space curve $\boldsymbol{\gamma} : S^1 \to \mathbb{R}^3_1$ which is spacelike and parametrized by arc length s, we have the following corollary:

Corollary 8.4. *For a "generic" spacelike curve $\boldsymbol{\gamma} : S^1 \to \mathbb{R}^3_1$, the number of the singular points of $F_{\left(\gamma,\widetilde{\mathbb{N}}[\phi]^{nb}_{\pm}\right)}$ is finite and each singular point is the cross cap.*

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