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Characterizations of Lie n-derivations of Unital Algebras with Nontrivial Idempotents

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Abstract. Let $\mathcal{A}$ be a unital algebra with a nontrivial idempotent e, and $f = 1 - e$. Suppose that $\mathcal{A}$ satisfies that $exe \cdot e\mathcal{A}f = \{0\} = f\mathcal{A}e \cdot exe$ implies $exe = 0$ and $e\mathcal{A}f \cdot fxf = \{0\} = fxf \cdot f\mathcal{A}e$ implies $fxf = 0$ for each x in $\mathcal{A}$. For a Lie n-derivation φ on $\mathcal{A}$, we obtain the necessary and sufficient conditions for φ to be standard, i.e., $\varphi = d + \gamma$, where d is a derivation on $\mathcal{A}$, and γ is a linear mapping from $\mathcal{A}$ into the centre $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$−th commutators of $\mathcal{A}$. Furthermore, we also consider the sufficient conditions under which each Lie n-derivation on $\mathcal{A}$ can be standard.

1. Introduction

Let $\mathcal{A}$ be a unital algebra over a unital commutative ring $\mathcal{R}$. The algebra $\mathcal{A}$ is called to be *n-torsion free* if $nx = 0$ implies $x = 0$ for some positive integer n and each x in $\mathcal{A}$, and is called to be *torsion-free* if $nx = 0$ implies $x = 0$ for each positive integer n and each x in $\mathcal{A}$. A linear mapping δ on $\mathcal{A}$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ for each x, y in $\mathcal{A}$, is called a *Jordan derivation* if $\delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y)$ for each x, y in $\mathcal{A}$, is called a *Lie derivation* if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for each x, y in $\mathcal{A}$, and is called a *Lie triple derivation* if $\delta([[x, y], z]) = [[\delta(x), y], z] + [[x, \delta(y)], z] + [[x, y], \delta(z)]$ for each x, y, z in $\mathcal{A}$, where $x \circ y = xy + yx$ and $[x, y] = xy - yx$ for each x, y in $\mathcal{A}$. A derivation δ is called an *inner derivation* if there exists some a in $\mathcal{A}$ such that $\delta(x) = ax - xa$ for each x in $\mathcal{A}$. Now we define a sequence of polynomials as follows:

$$p_1(x_1) = x_1,$$
$$p_n(x_1, x_2, ..., x_n) = [p_{n-1}(x_1, x_2, ..., x_{n-1}), x_n]$$

for all $x_1, x_2, ..., x_n \in \mathcal{A}$ and each positive integer $n \geq 2$. Thus, $p_2(x_1, x_2) = [x_1, x_2]$ and $p_3(x_1, x_2, x_3) = [[x_1, x_2], x_3]$. For $n \geq 2$, $p_n(x_1, x_2, ..., x_n) = [...[[x_1, x_2], x_3], ..., x_n]$ is also called an $(n-1)$−th *commutator* of $x_1, x_2, ..., x_n \in \mathcal{A}$. A linear mapping δ on $\mathcal{A}$ is called a *Lie n-derivation* ($n \geq 2$) if

$$\delta(p_n(x_1, x_2, ..., x_n)) = \sum_{i=1}^{n} p_n(x_1, ..., x_{i-1}, \delta(x_i), x_{i+1}, ..., x_n)$$

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for all $x_1, x_2, ..., x_n \in \mathcal{A}$. Thus, δ is a Lie derivation when $n = 2$, and is a Lie triple derivation when $n = 3$. The notion of Lie n-derivations is firstly proposed by Abdullaev in [1], where the author describes the form of Lie n-derivations of a certain von Neumann algebra (or of its skew-adjoint part). A Lie n-derivation δ on $\mathcal{A}$ is called to be *standard* if $\delta = h + \tau$, where h is a derivation on $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into its centre $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)-$th commutators of $\mathcal{A}$.

Let e be a nontrivial idempotent in $\mathcal{A}$, and $f = 1 - e$. Then $\mathcal{A}$ can be represented in the so called Pierce decomposition form

$$\mathcal{A} = e\mathcal{A}e + e\mathcal{A}f + f\mathcal{A}e + f\mathcal{A}f \tag{1.1}$$

where $e\mathcal{A}e$ is a subalgebra with unit e, $f\mathcal{A}f$ is a subalgebra with unit f, $e\mathcal{A}f$ is an $(e\mathcal{A}e, f\mathcal{A}f)$-bimodule, and $f\mathcal{A}e$ is an $(f\mathcal{A}f, e\mathcal{A}e)$-bimodule. In this paper, we study the conditions under which a Lie n-derivation on $\mathcal{A}$ is standard. Benkovič and Širovnik [4] consider Jordan derivations on unital algebras with nontrivial idempotents, and introduce the notion of singular Jordan derivations which comes out to be very important in study of mappings on unital algebras with nontrivial idempotents. Benkovič [2] obtains several sufficient (and necessary) conditions for a Lie triple derivation on $\mathcal{A}$ to be expressed as the sum of a derivation, a singular Jordan derivation and a linear mapping from $\mathcal{A}$ into the centre $\mathcal{Z}(\mathcal{A})$ vanishing on all second commutators of $\mathcal{A}$. Wang [15] discusses the sufficient conditions for a Lie n-derivation on $\mathcal{A}$ to be expressed as the sum of a derivation, a singular Jordan derivation and a linear mapping from $\mathcal{A}$ into the centre $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)-$th commutators of $\mathcal{A}$. It is worth to mention that $\mathcal{A}$ is isomorphic to a generalized matrix algebra $\mathcal{G} = (A, M, N, B)$ (which is first introduced by Morita in [13]), where A and B are two unital algebras, and ${}_AM_B$ and ${}_BN_A$ are two bimodules. Many papers discuss mappings on generalized matrix algebras such as [7, 12, 16]. With a quite common assumption that the bimodule ${}_AM_B$ is faithful which means that $aM = 0$ implies $a = 0$ for each $a \in A$ and that $Mb = 0$ implies $b = 0$ for each $b \in B$, several authors [12, 16] obtain sufficient conditions for Lie derivations and Lie n-derivations on generalized matrix algebras to be standard. In this paper, we consider a milder assumption which arises from [2] that the Pierce decomposition (1.1) satisfies

$$\begin{array}{lll} exe \cdot e\mathcal{A}f = \{0\} = f\mathcal{A}e \cdot exe & \text{implies} & exe = 0 \quad \text{and} \\ e\mathcal{A}f \cdot fxf = \{0\} = fxf \cdot f\mathcal{A}e & \text{implies} & fxf = 0 \end{array} \tag{1.2}$$

for each x in $\mathcal{A}$. Important examples of unital algebras with nontrivial idempotents satisfying the property (1.2) include triangular algebras, matrix algebras, algebras of all bounded linear operators of Banach space and prime algebras with nontrivial idempotents.

This paper is organized as follows. In Section 2, we consider that $\mathcal{A}$ is a unital algebra with a nontrivial idempotent e satisfying the property (1.2). For a Lie n-derivation ($n \geq 2$) φ on $\mathcal{A}$, we discuss the necessary and sufficient conditions for φ to be standard, or to be described as the sum of a derivation, a singular Jordan derivation and a central mapping. And we discuss the sufficient conditions under which each Lie n-derivation ($n \geq 2$) on $\mathcal{A}$ can be standard, or can be described as the sum of a derivation, a singular Jordan derivation and a central mapping. These results improve the corresponding main results in [2, 15].

In Section 3, as applications of the results in Section 2, we characterize Lie n-derivations on matrix algebras, triangular algebras, unital prime algebras with nontrivial idempotents and von Neumann algebras.

2. Main Results

In this section, we assume that $\mathcal{A}$ is a unital algebra with a nontrivial idempotent e. By the Pierce decomposition (1.1), $\mathcal{A}$ can be represented as $\mathcal{A} = e\mathcal{A}e + e\mathcal{A}f + f\mathcal{A}e + f\mathcal{A}f$, where $f = 1 - e$. In [4], Benkovič and Širovnik introduce the term *singular Jordan derivations*, which turns out to play an important role in the study of mappings on unital algebras with nontrivial idempotents. Denote that $\mathcal{Z}(\mathcal{A})$ is the centre of $\mathcal{A}$.

Definition 2.1. *A Jordan derivation δ on $\mathcal{A}$ is a* singular Jordan derivation *if*

$$\delta(e\mathcal{A}e) = \{0\}, \quad \delta(f\mathcal{A}f) = \{0\}, \quad \delta(e\mathcal{A}f) \subseteq f\mathcal{A}e \quad \text{and} \quad \delta(f\mathcal{A}e) \subseteq e\mathcal{A}f. \tag{2.1}$$

It's obvious that singular Jordan derivations on $\mathcal{A}$ is zero when $\mathcal{A}$ is a triangular algebra, since $f\mathcal{A}e = \{0\}$.

The following Lemma is very important, and is repeatedly used in the remaining paper.

Lemma 2.2. *[2, Proposition 2.1 and Remark 2.2] If $\mathcal{A}$ satisfies the property (1.2), then*

(i) $\mathcal{Z}(\mathcal{A}) = \left\{ a + b \;\middle|\; \begin{array}{l} a \in e\mathcal{A}e,\ b \in f\mathcal{A}f, \\ am = mb,\ ta = bt \text{ for each } m \in e\mathcal{A}f \text{ and } t \in f\mathcal{A}e \end{array} \right\}.$

(ii) *There exists a unique algebra isomorphism τ from $e\mathcal{Z}(\mathcal{A})e$ to $f\mathcal{Z}(\mathcal{A})f$, such that for each $a \in e\mathcal{Z}(\mathcal{A})e$ we have that $am = m\tau(a)$ and $ta = \tau(a)t$ for each $m \in e\mathcal{A}f$ and $t \in f\mathcal{A}e$.*
(Thus, $a + \tau(a) \in \mathcal{Z}(\mathcal{A})$ for each $a \in e\mathcal{Z}(\mathcal{A})e$ and $\tau^{-1}(b) + b \in \mathcal{Z}(\mathcal{A})$ for each $b \in f\mathcal{Z}(\mathcal{A})f$.)

(iii) *For $x \in \mathcal{A}$, if $[x, e\mathcal{A}f] = \{0\}$ and $[x, f\mathcal{A}e] = \{0\}$, then $exe + fxf \in \mathcal{Z}(\mathcal{A})$.*

2.1. The necessary and sufficient conditions for a Lie n-derivation to be standard.

At first, we consider the necessary and sufficient conditions under which a Lie n-derivation $(n \geq 2)$ φ can be described as the sum of a derivation, a singular Jordan derivation and a central mapping.

Theorem 2.3. *Let φ be a Lie n-derivation on $\mathcal{A}$. Suppose that $\mathcal{A}$ is 2- and $(n-1)$-torsion free, and that $\mathcal{A}$ satisfies the property (1.2). Then φ is of the form*

$$\varphi = d + \delta + \gamma \tag{2.2}$$

where d is a derivation on $\mathcal{A}$, δ is a singular Jordan derivation on $\mathcal{A}$ and γ is a linear mapping from $\mathcal{A}$ into the centre $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$–th commutators of $\mathcal{A}$, if and only if both conditions (i) *and* (ii) *hold:*

(i) *$f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$,*

(ii) *$e\varphi(tm)e + f\varphi(mt)f \in \mathcal{Z}(\mathcal{A})$ for each $m \in e\mathcal{A}f$ and $t \in f\mathcal{A}e$.*

Before we prove Theorem 2.3, we need to recognize the following Remark 2.4.

Remark 2.4. *Let φ be a Lie n-derivation on $\mathcal{A}$. Similar to the proofs of [2, Lemma 3.1] and [15, Theorem 2.1], we can assume that φ satisfies $e\varphi(e)f = f\varphi(e)e = 0$. Actually, let $x_0 = e\varphi(e)f - f\varphi(e)e$ and d be an inner derivation on $\mathcal{A}$ that $d(x) = [x, x_0]$ for each x in $\mathcal{A}$. Clearly $\varphi' = \varphi - d$ is also a Lie n-derivation. Since*

$$\begin{aligned}
\varphi'(e) &= \varphi(e) - [e, e\varphi(e)f - f\varphi(e)e] \\
&= \varphi(e) - e\varphi(e)f - f\varphi(e)e \\
&= e\varphi(e)e + f\varphi(e)f,
\end{aligned}$$

we obtain $e\varphi'(e)f = f\varphi'(e)e = 0$. Thus, it suffices to consider the Lie n-derivation φ on $\mathcal{A}$ satisfying $e\varphi(e)f = f\varphi(e)e = 0$.

Proof. [Proof of Theorem 2.3] Suppose that φ is of the form (2.2) $\varphi = d + \delta + \gamma$. Let $\delta' = d + \delta$, then δ' is a Jordan derivation and

$$2\delta'(a) = \delta'(e \circ a) = \delta'(e)a + a\delta'(e) + e\delta'(a) + \delta'(a)e, \tag{2.3}$$
$$2\delta'(b) = \delta'(f \circ b) = \delta'(f)b + b\delta'(f) + f\delta'(b) + \delta'(b)f, \tag{2.4}$$

for each a in $e\mathcal{A}e$ and b in $f\mathcal{A}f$. Since $\mathcal{A}$ is 2-torsion free, left and right multiplication of (2.3) by f implies that $f\delta'(a)f = 0$ for each a in $e\mathcal{A}e$, and left and right multiplication of (2.4) by e implies that $e\delta'(b)e = 0$ for each b in $f\mathcal{A}f$. Since $\gamma(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$, we have that

$$f\varphi(a)f = f\delta'(a)f + f\gamma(a)f = f\gamma(a)f \in f\mathcal{Z}(\mathcal{A})f,$$
$$e\varphi(b)e = e\delta'(b)e + e\gamma(b)e = e\gamma(b)e \in e\mathcal{Z}(\mathcal{A})e,$$

for each a in $e\mathcal{A}e$ and b in $f\mathcal{A}f$. Hence, (i) holds. For each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$, we have

$$p_n(t, e, ..., e, m) = p_{n-1}(t, e, ..., e, m) = ... = [t, m] = tm - mt.$$

Since γ vanishes on all $(n-1)-$th commutators of $\mathcal{A}$, we have that $\gamma(tm - mt) = 0$. We may as well assume that $\gamma(mt) = \gamma(tm) = a_0 + b_0 \in \mathcal{Z}(\mathcal{A})$ where $a_0 \in e\mathcal{Z}(\mathcal{A})e$ and $b_0 \in \mathcal{Z}(\mathcal{A})f$. Since $mt \in e\mathcal{A}e$, $tm \in f\mathcal{A}f$, we have that

$$\varphi(mt) = d(mt) + \delta(mt) + \gamma(mt) = d(m)t + md(t) + a_0 + b_0, \tag{2.5}$$
$$\varphi(tm) = d(tm) + \delta(tm) + \gamma(tm) = d(t)m + td(m) + a_0 + b_0. \tag{2.6}$$

Left and right multiplication of (2.5) by f implies that $f\varphi(mt)f = b_0$, and left and right multiplication of (2.6) by e implies that $e\varphi(tm)e = a_0$. Thus, $e\varphi(tm)e + f\varphi(mt)f = a_0 + b_0 \in \mathcal{Z}(\mathcal{A})$, (ii) holds.

Suppose that (i) and (ii) hold. According to Remark 2.4, it suffices to consider a Lie n-derivation φ on $\mathcal{A}$ satisfying $e\varphi(e)f = f\varphi(e)e = 0$. Thus, $\varphi(e) = e\varphi(e)e + f\varphi(e)f$. We organize the following proof by a series of claims.

Claim 1. For each $x \in \mathcal{A}$, we have $p_n(x, e, ..., e) = (-1)^{n-1}exf + fxe$ and $p_n(x, f, ..., f) = exf + (-1)^{n-1}fxe$.

It's obvious that

$$p_n(x, e, ..., e) = p_{n-1}([x, e], e, ..., e) = p_{n-1}(-exf + fxe, e, ..., e) = ... = (-1)^{n-1}exf + fxe.$$

The case of $p_n(x, f, ..., f)$ could be similarly proved.

Claim 2.
$\varphi(a) = e\varphi(a)e + f\varphi(a)f$ for each $a \in e\mathcal{A}e$,
$\varphi(b) = e\varphi(b)e + f\varphi(b)f$ for each $b \in f\mathcal{A}f$,
$\varphi(m) = e\varphi(m)f + f\varphi(m)e$ for each $m \in e\mathcal{A}f$,
$\varphi(t) = e\varphi(t)f + f\varphi(t)e$ for each $t \in f\mathcal{A}e$.

For each a in $e\mathcal{A}e$, since $[a, e] = 0$ and $p_n(a, e, ..., e) = 0$, according to Claim 1, we have that

$$\begin{aligned}
0 =& \varphi(p_n(a, e, ..., e)) \\
=& p_n(\varphi(a), e, ..., e) + p_n(a, \varphi(e), ..., e) + \sum_{j=3}^{n} p_n(a, e, ..., \varphi(e), ..., e) \\
=& (-1)^{n-1}e\varphi(a)f + f\varphi(a)e + (-1)^{n-2}e[a, \varphi(e)]f + f[a, \varphi(e)]e \\
=& (-1)^{n-1}e\varphi(a)f + f\varphi(a)e.
\end{aligned}$$

Left and right multiplying by e and f respectively in the above equations, we obtain that

$$e\varphi(a)f = f\varphi(a)e = 0. \tag{2.7}$$

Thus, $\varphi(a) = e\varphi(a)e + f\varphi(a)f$. For each b in $f\mathcal{A}f$, since $[b, f] = 0$ and $p_n(b, f, ..., f) = 0$, we can similarly prove that

$$e\varphi(b)f = f\varphi(b)e = 0 \tag{2.8}$$

and $\varphi(b) = e\varphi(b)e + f\varphi(b)f$. For each m in $e\mathcal{A}f$, according to Claim 1, we have that $p_n(m, e, ..., e) = (-1)^{n-1}m$ and

$$\begin{aligned}(-1)^{n-1}\varphi(m) =&\varphi(p_n(m, e, ..., e))\\ =&p_n(\varphi(m), e, ..., e) + \sum_{j=2}^{n} p_n(m, e, ..., \varphi(e), ..., e)\\ =&(-1)^{n-1}e\varphi(m)f + f\varphi(m)e + \sum_{j=2}^{n}(-1)^{j-2}p_{n-j+2}(m, \varphi(e), e, ..., e)\\ =&(-1)^{n-1}e\varphi(m)f + f\varphi(m)e + \sum_{j=2}^{n}(-1)^{j-2}(-1)^{n-j}[m, \varphi(e)]\\ =&(-1)^{n-1}e\varphi(m)f + f\varphi(m)e + (-1)^{n-2}(n-1)[m, \varphi(e)].\end{aligned}$$

Left and right multiplying by e and f respectively in the above equations, under the assumption that $\mathcal{A}$ is $(n-1)$-torsion free, we obtain that

$$\begin{aligned}&e\varphi(m)e = f\varphi(m)f = 0,\\ &f\varphi(m)e = (-1)^{n-1}f\varphi(m)e, && (2.9)\\ &[m, \varphi(e)] = 0. && (2.10)\end{aligned}$$

Thus, $\varphi(m) = e\varphi(m)f + f\varphi(m)e$. For each t in $f\mathcal{A}e$, by Claim 1, we have $p_n(t, e, ..., e) = t$. We can similarly prove that

$$\varphi(t) = (-1)^{n-1}e\varphi(t)f + f\varphi(t)e + (n-1)[t, \varphi(e)]$$

and

$$\begin{aligned}&e\varphi(t)e = f\varphi(t)f = 0,\\ &e\varphi(t)f = (-1)^{n-1}e\varphi(t)f, && (2.11)\\ &[t, \varphi(e)] = 0. && (2.12)\end{aligned}$$

Thus, $\varphi(t) = e\varphi(t)f + f\varphi(t)e$.

According to Lemma 2.2, there exists a unique algebra isomorphism τ from $e\mathcal{Z}(\mathcal{A})e$ to $f\mathcal{Z}(\mathcal{A})f$, such that for each $a \in e\mathcal{Z}(\mathcal{A})e$, we have that $am = m\tau(a)$ and $ta = \tau(a)t$ for each $m \in e\mathcal{A}f$ and $t \in f\mathcal{A}e$. For each $a \in e\mathcal{A}e$, $m \in e\mathcal{A}f$, $t \in f\mathcal{A}e$ and $b \in f\mathcal{A}f$, we define a linear mapping d on $\mathcal{A}$ as follows:

$$d(a)=e\varphi(a)e-\tau^{-1}(f\varphi(a)f),\ \ d(b)=f\varphi(b)f-\tau(e\varphi(b)e),\ \ d(m)=e\varphi(m)f\ \text{ and }\ d(t)=f\varphi(t)e, \tag{2.13}$$

and a linear mapping δ on $\mathcal{A}$ as follows:

$$\delta(a)=0,\quad \delta(b)=0,\quad \delta(m)=f\varphi(m)e \quad \text{and} \quad \delta(t)=e\varphi(t)f. \tag{2.14}$$

Denote $\gamma = \varphi - d - \delta$. Then γ is a linear mapping satisfying

$$\gamma(a)=\tau^{-1}(f\varphi(a)f)+f\varphi(a)f,\ \ \gamma(b)=e\varphi(b)e+\tau(e\varphi(b)e),\ \ \gamma(m)=0\ \text{ and }\ \gamma(t)=0 \tag{2.15}$$

for each $a \in e\mathcal{A}e$, $m \in e\mathcal{A}f$, $t \in f\mathcal{A}e$ and $b \in f\mathcal{A}f$. By (i) and Lemma 2.2, γ maps $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$.

Claim 3

$$\begin{array}{ll}
\varphi(am) = [\varphi(a), m] + a \circ \varphi(m) & \textit{for each } a \in e\mathcal{A}e, m \in e\mathcal{A}f,\\
\varphi(mb) = [m, \varphi(b)] + \varphi(m) \circ b & \textit{for each } b \in f\mathcal{A}f, m \in e\mathcal{A}f,\\
\varphi(ta) = [t, \varphi(a)] + \varphi(t) \circ a & \textit{for each } a \in e\mathcal{A}e, t \in f\mathcal{A}e,\\
\varphi(bt) = [\varphi(b), t] + b \circ \varphi(t) & \textit{for each } b \in f\mathcal{A}f, t \in f\mathcal{A}e,\\
\varphi(mt) - \varphi(tm) = [\varphi(m), t] + [m, \varphi(t)] & \textit{for each } m \in e\mathcal{A}f, t \in f\mathcal{A}e,\\
(-1)^{n-2}[\varphi(m_1), m_2] + [m_1, \varphi(m_2)] = 0 & \textit{for each } m_1, m_2 \in e\mathcal{A}f,\\
(-1)^{n-2}[\varphi(t_1), t_2] + [t_1, \varphi(t_2)] = 0 & \textit{for each } t_1, t_2 \in f\mathcal{A}e.
\end{array}$$

For each a in $e\mathcal{A}e$ and m in $e\mathcal{A}f$, since $[a, m] = am \in e\mathcal{A}f$, according to Claim 1, we have that $p_n(a, m, e, ..., e) = (-1)^{n-2}am$ and

$$\begin{aligned}
(-1)^{n-2}\varphi(am) =&\varphi(p_n(a, m, e, ..., e))\\
=&p_n(\varphi(a), m, e, ..., e) + p_n(a, \varphi(m), e, ..., e) + \sum_{j=3}^{n} p_n(a, m, e, ..., \varphi(e), ..., e)\\
=&(-1)^{n-2}e[\varphi(a), m]f + (-1)^{n-2}e[a, \varphi(m)]f + f[a, \varphi(m)]e\\
&+ \sum_{j=3}^{n}(-1)^{j-3}p_{n-j+3}(a, m, \varphi(e), e, ..., e)\\
=&(-1)^{n-2}[\varphi(a), m] + (-1)^{n-2}a\varphi(m) - \varphi(m)a + (-1)^{n-3}(n-2)[am, \varphi(e)].
\end{aligned}$$

Left and right multiplying by e and f respectively in the above equations, we obtain that

$$\begin{aligned}
&f\varphi(am)e = (-1)^{n-1}\varphi(m)a,\\
&e\varphi(am)f = [\varphi(a), m] + a\varphi(m) - (n-2)[am, \varphi(e)].
\end{aligned}$$

Associating with (2.9) and (2.10), we have $[am, \varphi(e)] = 0$ and $f\varphi(am)e = (-1)^{n-1}f\varphi(m)e \cdot eae = f\varphi(m)e \cdot eae = \varphi(m)a$. Thus,

$$\varphi(am) = e\varphi(am)f + f\varphi(am)e = [\varphi(a), m] + a \circ \varphi(m)$$

for each a in $e\mathcal{A}e$ and m in $e\mathcal{A}f$. Let's make similar discussions on mb, ta and bt. For each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$, t in $f\mathcal{A}e$ and b in $f\mathcal{A}f$, since $[m, b] = mb \in e\mathcal{A}f$, $[t, a] = ta \in f\mathcal{A}e$ and $[b, t] = bt \in f\mathcal{A}e$, we have that

$$p_n(m, b, e, ..., e) = (-1)^{n-2}mb, \quad p_n(t, a, e, ..., e) = ta \quad \text{and} \quad p_n(b, t, e, ..., e) = bt.$$

It follows that

$$\begin{aligned}
&(-1)^{n-2}\varphi(mb) = (-1)^{n-2}\varphi(m)b - b\varphi(m) + (-1)^{n-2}[m, \varphi(b)] + (-1)^{n-3}(n-2)[mb, \varphi(e)],\\
&\varphi(ta) = -(-1)^{n-2}a\varphi(t) + \varphi(t)a + [t, \varphi(a)] + (n-2)[ta, \varphi(e)],\\
&\varphi(bt) = [\varphi(b), t] - (-1)^{n-2}\varphi(t)b + b\varphi(t) + (n-2)[bt, \varphi(e)].
\end{aligned}$$

Left and right multiplying by e and f respectively in the above equations, and associating with (2.9), (2.10), (2.11) and (2.12), we obtain that

$$\begin{aligned}
&[mb, \varphi(e)] = [ta, \varphi(e)] = [bt, \varphi(e)] = 0,\\
&f\varphi(mb)e = b\varphi(m),\ e\varphi(ta)f = a\varphi(t) \text{ and } e\varphi(bt)f = \varphi(t)b,\\
&\varphi(mb) = [m, \varphi(b)] + \varphi(m) \circ b,\ \varphi(ta) = [t, \varphi(a)] + \varphi(t) \circ a \text{ and } \varphi(bt) = [\varphi(b), t] + b \circ \varphi(t)
\end{aligned}$$

for each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$, t in $f\mathcal{A}e$ and b in $f\mathcal{A}f$.

When $n=2$, for each m, m_1, m_2 in $e\mathcal{A}f$ and t, t_1, t_2 in $f\mathcal{A}e$, it is obvious that

$$\begin{aligned}
&\varphi(mt)-\varphi(tm)=\varphi([m,t])=[\varphi(m),t]+[m,\varphi(t)],\\
&(-1)^{n-2}[\varphi(m_1),m_2]+[m_1,\varphi(m_2)]=[\varphi(m_1),m_2]+[m_1,\varphi(m_2)]=\varphi([m_1,m_2])=0,\\
&(-1)^{n-2}[\varphi(t_1),t_2]+[t_1,\varphi(t_2)]=[\varphi(t_1),t_2]+[t_1,\varphi(t_2)]=\varphi([t_1,t_2])=0.
\end{aligned}$$

If $n\geq 3$, for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$, since $p_n(m,e,...,e,t)=[p_{n-1}(m,e,...,e),t]=(-1)^{n-2}[m,t]$, then we have that

$$\begin{aligned}
(-1)^{n-2}(\varphi(mt)-\varphi(tm)) =&\varphi(p_n(m,e,...,e,t))\\
=&p_n(\varphi(m),e,...,e,t)+\sum_{j=2}^{n-1}p_n(m,e,...,\varphi(e),...,e,t)+p_n(m,e,...,e,\varphi(t))\\
=&[(-1)^{n-2}e\varphi(m)f+f\varphi(m)e,t]+(-1)^{n-3}(n-2)[[m,\varphi(e)],t]\\
&+(-1)^{n-2}[m,\varphi(t)]\\
=&(-1)^{n-2}[\varphi(m),t]+(-1)^{n-2}[m,\varphi(t)].
\end{aligned}$$

For each m_1, m_2 in $e\mathcal{A}f$, since $p_n(m_1,e,...,e,m_2)=[p_{n-1}(m_1,e,...,e),m_2]=0$, we have that

$$\begin{aligned}
0 =&\varphi(p_n(m_1,e,...,e,m_2))\\
=&p_n(\varphi(m_1),e,...,e,m_2)+\sum_{j=2}^{n-1}p_n(m_1,e,...,\varphi(e),...,e,m_2)+p_n(m_1,e,...,e,\varphi(m_2))\\
=&[(-1)^{n-2}e\varphi(m_1)f+f\varphi(m_1)e,m_2]+(-1)^{n-3}(n-2)[[m_1,\varphi(e)],m_2]+(-1)^{n-2}[m_1,\varphi(m_2)]\\
=&[\varphi(m_1),m_2]+(-1)^{n-2}[m_1,\varphi(m_2)].
\end{aligned}$$

For each t_1, t_2 in $f\mathcal{A}e$, since $p_n(t_1,e,...,e,t_2)=[p_{n-1}(t_1,e,...,e),t_2]=0$, we have that

$$\begin{aligned}
0 =&\varphi(p_n(t_1,e,...,e,t_2))\\
=&p_n(\varphi(t_1),e,...,e,t_2)+\sum_{j=2}^{n-1}p_n(t_1,e,...,\varphi(e),...,e,t_2)+p_n(t_1,e,...,e,\varphi(t_2))\\
=&[(-1)^{n-2}e\varphi(t_1)f+f\varphi(t_1)e,t_2]+(n-2)[[t_1,\varphi(e)],t_2]+[t_1,\varphi(t_2)]\\
=&(-1)^{n-2}[\varphi(t_1),t_2]+[t_1,\varphi(t_2)].
\end{aligned}$$

Thus, for each $n\geq 2$, we conclude that

$$\begin{aligned}
&\varphi(mt)-\varphi(tm)=[\varphi(m),t]+[m,\varphi(t)],\\
&(-1)^{n-2}[\varphi(m_1),m_2]+[m_1,\varphi(m_2)]=0,\\
&(-1)^{n-2}[\varphi(t_1),t_2]+[t_1,\varphi(t_2)]=0
\end{aligned}$$

for each m, m_1, m_2 in $e\mathcal{A}f$ and t, t_1, t_2 in $f\mathcal{A}e$.

Claim 4 d is a derivation.

According to the definition (2.13) of d, we have that

$$d(a)=ed(a)e,\quad d(m)=ed(m)f,\quad d(t)=fd(t)e\quad\text{and}\quad d(b)=fd(b)f \tag{2.16}$$

for each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$, t in $f\mathcal{A}e$ and b in $f\mathcal{A}f$. For each a in $e\mathcal{A}e$ and m in $e\mathcal{A}f$, by Claim 3, we have $\varphi(am) = [\varphi(a), m] + a \circ \varphi(m)$. Thus,

$$e\varphi(am)f = [\varphi(a), m] + a\varphi(m) = e\varphi(a)e \cdot emf - emf \cdot f\varphi(a)f + eae \cdot e\varphi(m)f.$$

By (i) and the definition of τ, we know that $emf \cdot f\varphi(a)f = \tau^{-1}(f\varphi(a)f) \cdot emf$. Thus,

$$d(am) = e\varphi(am)f = (e\varphi(a)e - \tau^{-1}(f\varphi(a)f)) \cdot emf + eae \cdot e\varphi(m)f = d(a)m + ad(m) \tag{2.17}$$

for each a in $e\mathcal{A}e$ and m in $e\mathcal{A}f$. Make similar discussions on mb, ta and bt, and we obtain that

$$d(mb) = md(b) + d(m)b, \quad d(ta) = td(a) + d(t)a \quad \text{and} \quad d(bt) = d(b)t + bd(t) \tag{2.18}$$

for each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$, t in $f\mathcal{A}e$ and b in $f\mathcal{A}f$. For each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$, by Claim 3, we have $\varphi(mt) - \varphi(tm) = [\varphi(m), t] + [m, \varphi(t)]$. Thus,

$$\begin{aligned} e\varphi(mt)e - e\varphi(tm)e =& \varphi(m)t + m\varphi(t) = e\varphi(m)f \cdot fte + emf \cdot f\varphi(t)e, \\ -f\varphi(mt)f + f\varphi(tm)f =& t\varphi(m) + \varphi(t)m = fte \cdot e\varphi(m)f + f\varphi(t)e \cdot emf. \end{aligned}$$

Since $mt \in e\mathcal{A}e$ and $tm \in f\mathcal{A}f$, we obtain that

$$\begin{aligned} d(m)t + md(t) =& e\varphi(m)f \cdot fte + emf \cdot f\varphi(t)e = e\varphi(mt)e - e\varphi(tm)e \\ =& d(mt) + \tau^{-1}(f\varphi(mt)f) - e\varphi(tm)e, \\ d(t)m + td(m) =& f\varphi(t)e \cdot emf + fte \cdot e\varphi(m)f = -f\varphi(mt)f - f\varphi(tm)f \\ =& d(tm) + \tau(e\varphi(tm)e) - f\varphi(mt)f, \end{aligned}$$

By (ii) and Lemma 2.2, $\tau(e\varphi(tm)e) = f\varphi(mt)f$ and $e\varphi(tm)e = \tau^{-1}(f\varphi(mt)f)$. Thus,

$$d(m)t + md(t) = d(mt) \quad \text{and} \quad d(t)m + td(m) = d(tm) \tag{2.19}$$

for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$. By (2.16), (2.17), (2.18), (2.19) and [4, Lemma 2.3], we obtain that d is a derivation.

Claim 5. δ is a singular Jordan derivation.

According to the definition (2.14) of δ, we only need to prove that δ is a Jordan derivation. For each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$, t in $f\mathcal{A}e$ and b in $f\mathcal{A}f$, by Claim 3, we know that

$$f\varphi(am)e = \varphi(m)a, \quad f\varphi(mb)e = b\varphi(m), \quad e\varphi(ta)f = a\varphi(t) \quad \text{and} \quad e\varphi(bt)f = \varphi(t)b.$$

In view of (2.14), we obtain that

$$\delta(am) = \delta(m)a, \quad \delta(mb) = b\delta(m), \quad \delta(ta) = a\delta(t) \quad \text{and} \quad \delta(bt) = \delta(t)b \tag{2.20}$$

for each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$, t in $f\mathcal{A}e$ and b in $f\mathcal{A}f$.

For each m in $e\mathcal{A}f$, if n is even, then $2f\varphi(m)e = 0$ by (2.9). Since $\mathcal{A}$ is 2-torsion free, $f\varphi(m)e = 0$, i.e. $\delta(m) = 0$. If n is odd, then by Claim 3, we have $2[m, \varphi(m)] = 0$. Since $\mathcal{A}$ is 2-torsion free, we have that $[m, \varphi(m)] = 0$. Left and right multiplication by e and f respectively implies that $m\varphi(m) = \varphi(m)m = 0$, i.e. $m\delta(m) = \delta(m)m = 0$. Thus,

$$m\delta(m) = \delta(m)m = 0 \tag{2.21}$$

for each $n \geq 2$.

For each t in $f\mathcal{A}e$, if n is even, then $2e\varphi(t)f = 0$ by (2.11). Since $\mathcal{A}$ is 2-torsion free, $e\varphi(t)f = 0$, i.e. $\delta(t) = 0$. If n is odd, then by Claim 3, we have $2[t, \varphi(t)] = 0$. Since $\mathcal{A}$ is 2-torsion free, we have that $[t, \varphi(t)] = 0$. Left and right multiplication by e and f respectively implies that $t\varphi(t) = \varphi(t)t = 0$, i.e. $t\delta(t) = \delta(t)t = 0$. Thus,

$$t\delta(t) = \delta(t)t = 0 \tag{2.22}$$

for each $n \geq 2$.

Let $x = a + m + t + b$ be an arbitrary element in $\mathcal{A}$ where a, m, t, b are elements in $e\mathcal{A}e, e\mathcal{A}f, f\mathcal{A}e, f\mathcal{A}f$, respectively. By (2.14), (2.20), (2.21) and (2.22), we obtain that

$$\begin{aligned}
\delta(x^2) =& \delta((a + m + t + b)^2)\\
=& \delta(m)a + b\delta(m) + a\delta(t) + \delta(t)b,\\
x\delta(x) + \delta(x)x =& (a + m + t + b)\delta(a + m + t + b) + \delta(a + m + t + b)(a + m + t + b)\\
=& b\delta(m) + a\delta(t) + \delta(m)a + \delta(t)b.
\end{aligned}$$

So $\delta(x^2) = x\delta(x) + \delta(x)x$ for each x in $\mathcal{A}$. According to the definition (2.14) of δ, we obtain that δ is a singular Jordan derivation.

Claim 6. γ vanishes on all $(n-1)-$th commutators of $\mathcal{A}$.

For each $x_1, x_2, ..., x_n$ in $\mathcal{A}$, we have that

$$\begin{aligned}
\gamma(p_n(x_1, x_2, ..., x_n)) =& \varphi(p_n(x_1, x_2, ..., x_n)) - d(p_n(x_1, x_2, ..., x_n)) - \delta(p_n(x_1, x_2, ..., x_n))\\
=& \sum_{i=1}^{n} p_n(x_1, ..., \varphi(x_i), ..., x_n) - d(p_n(x_1, x_2, ..., x_n)) - \delta(p_n(x_1, x_2, ..., x_n))\\
=& \sum_{i=1}^{n} p_n(x_1, ..., d(x_i) + \delta(x_i) + \gamma(x_i), ..., x_n) - d(p_n(x_1, x_2, ..., x_n))\\
& - \delta(p_n(x_1, x_2, ..., x_n)).
\end{aligned}$$

Since d is a derivation and $\gamma(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$, it follows that

$$\gamma(p_n(x_1, x_2, ..., x_n)) = \sum_{i=1}^{n} p_n(x_1, ..., x_{i-1}, \delta(x_i), x_{i+1}, ..., x_n) - \delta(p_n(x_1, x_2, ..., x_n)). \tag{2.23}$$

If n is even, then in view of (2.9), (2.11), (2.14) and that $\mathcal{A}$ is 2-torsion free, we obtain that $\delta(m) = f\varphi(m)e = 0$ and $\delta(t) = e\varphi(t)f = 0$ for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$. Thus, $\delta(\mathcal{A}) = \{0\}$. By (2.23), we have $\gamma(p_n(x_1, x_2, ..., x_n)) = 0$ for each $x_1, x_2, ..., x_n$ in $\mathcal{A}$.

If n is odd, then by Claim 3 and (2.14), we have that

$$\begin{aligned}
&\delta(am) = f\varphi(am)e = f\varphi(m)e \cdot eae = \delta(m)a,\\
&\delta(mb) = f\varphi(mb)e = fbf \cdot f\varphi(m)e = b\delta(m),\\
&\delta(ta) = e\varphi(ta)f = eae \cdot e\varphi(t)f = a\delta(t),\\
&\delta(bt) = e\varphi(bt)f = e\varphi(t)f \cdot fbf = \delta(t)b,\\
&\delta(m_1)m_2 + \delta(m_2)m_1 = f\varphi(m_1)e \cdot em_2f + f\varphi(m_2)e \cdot em_1f = 0,\\
&m_2\delta(m_1) + m_1\delta(m_2) = em_2f \cdot f\varphi(m_1)e + em_1f \cdot f\varphi(m_2)e = 0,\\
&\delta(t_1)t_2 + \delta(t_2)t_1 = e\varphi(t_1)f \cdot ft_2e + e\varphi(t_2)f \cdot ft_1e = 0,\\
&t_2\delta(t_1) + t_1\delta(t_2) = ft_2e \cdot e\varphi(t_1)f + ft_1e \cdot e\varphi(t_2)f = 0
\end{aligned}$$

for each a in $e\mathcal{A}e$, m, m_1, m_2 in $e\mathcal{A}f$, t, t_1, t_2 in $f\mathcal{A}e$ and b in $f\mathcal{A}f$. It follows that

$$\begin{aligned}
&\delta(a_1a_2m) = \delta(m)a_1a_2 = \delta(a_1m)a_2 = \delta(a_2a_1m) = \delta(m)a_2a_1,\\
&\delta(mb_1b_2) = b_1b_2\delta(m) = b_1\delta(mb_2) = \delta(mb_2b_1) = b_2b_1\delta(m),\\
&\delta(ta_1a_2) = a_1a_2\delta(t) = a_1\delta(ta_2) = \delta(ta_2a_1) = a_2a_1\delta(t),\\
&\delta(b_1b_2t) = \delta(t)b_1b_2 = \delta(b_1t)b_2 = \delta(b_2b_1t) = \delta(t)b_2b_1,\\
&\delta(amb) = \delta(mb)a = b\delta(m)a,\\
&\delta(bta) = \delta(ta)b = a\delta(t)b,\\
&\delta(t_1)t_2m + mt_2\delta(t_1) = \delta(t_2mt_1 + t_1mt_2) = mt_1\delta(t_2) + mt_2\delta(t_1) = 0,\\
&\delta(m_1)m_2t + tm_2\delta(m_1) = \delta(m_2tm_1 + m_1tm_2) = tm_1\delta(m_2) + tm_2\delta(m_1) = 0
\end{aligned}$$

for each a, a_1, a_2 in $e\mathcal{A}e$, m, m_1, m_2 in $e\mathcal{A}f$, t, t_1, t_2 in $f\mathcal{A}e$ and b, b_1, b_2 in $f\mathcal{A}f$. Thus, for each $x_1 = a_1 + m_1 + t_1 + b_1$, $x_2 = a_2 + m_2 + t_2 + b_2$ and $x_3 = a_3 + m_3 + t_3 + b_3$ in $\mathcal{A}$ where $a_1, a_2, a_3 \in e\mathcal{A}e$, $m_1, m_2, m_3 \in e\mathcal{A}f$, $t_1, t_2, t_3 \in f\mathcal{A}e$ and $b_1, b_2, b_3 \in f\mathcal{A}f$, we obtain that

$$\begin{aligned}
\delta([[x_1, x_2], x_3]) =&\delta([[a_1 + m_1 + t_1 + b_1, a_2 + m_2 + t_2 + b_2], a_3 + m_3 + t_3 + b_3])\\
=&\delta(t_1a_2a_3 + b_1t_2a_3 - t_2a_1a_3 - b_2t_1a_3 - b_3t_1a_2 - b_3b_1t_2 + b_3t_2a_1 + b_3b_2t_1 + a_1m_2b_3\\
&+ m_1b_2b_3 - a_2m_1b_3 - m_2b_1b_3 - a_3a_1m_2 - a_3m_1b_2 + a_3a_2m_1 + a_3m_2b_1),\\
[[\delta(x_1), x_2], x_3]+&[[x_1, \delta(x_2)], x_3] + [[x_1, x_2], \delta(x_3)]\\
=&[[\delta(a_1+m_1+t_1+b_1), a_2+m_2+t_2+b_2], a_3+m_3+t_3+b_3]\\
&+[[a_1+m_1+t_1+b_1, \delta(a_2+m_2+t_2+b_2)], a_3+m_3+t_3+b_3]\\
&+[[a_1+m_1+t_1+b_1, a_2+m_2+t_2+b_2], \delta(a_3+m_3+t_3+b_3)]\\
=&\delta(t_1)b_2b_3 - a_2\delta(t_1)b_3 - \delta(t_2)b_1b_3 + a_1\delta(t_2)b_3 - a_3\delta(t_1)b_2 + a_3a_2\delta(t_1) + a_3\delta(t_2)b_1\\
&- a_3a_1\delta(t_2) + \delta(m_1)a_2a_3 - b_2\delta(m_1)a_3 - \delta(m_2)a_1a_3 + b_1\delta(m_2)a_3 - b_3\delta(m_1)a_2\\
&+ b_3b_2\delta(m_1) + b_3\delta(m_2)a_1 - b_3b_1\delta(m_2).
\end{aligned}$$

It follows that $\delta([[x_1, x_2], x_3]) = [[\delta(x_1), x_2], x_3] + [[x_1, \delta(x_2)], x_3] + [[x_1, x_2], \delta(x_3)]$, i.e., δ is a Lie triple derivation. Since n is odd, we can deduce that

$$\begin{aligned}
\delta(p_n(x_1, x_2, ..., x_n) =&\delta([[p_n(x_1, x_2, ..., x_{n-2}), x_{n-1}], x_n])\\
=&[[\delta(p_{n-2}(x_1, x_2, ..., x_{n-2})), x_{n-1}], x_n] + [[p_{n-2}(x_1, x_2, ..., x_{n-2}), \delta(x_{n-1})], x_n]\\
&+ [[p_{n-2}(x_1, x_2, ..., x_{n-2}), x_{n-1}], \delta(x_n)]\\
=&p_{n-2}(\delta([[x_1, x_2], x_3]), x_4 ..., x_n) + \sum_{i=4}^{n} p_{n-2}([[x_1, x_2], x_3]), x_4, ..., \delta(x_i), ..., x_n)\\
=&\sum_{i=1}^{n} p_n(x_1, ..., x_{i-1}, \delta(x_i), x_{i+1}, ..., x_n),
\end{aligned}$$

i.e., δ is a Lie n-derivation. By (2.23), we have $\gamma(p_n(x_1, x_2, ..., x_n)) = 0$ for each $x_1, x_2, ..., x_n$ in $\mathcal{A}$. Thus, Claim 6 holds.

With the definitions (2.13), (2.14) and (2.15) of d, δ, γ and Claims 4, 5 and 6, the proof is finished. □

Lemma 2.5. *[4, Remark 3.2] Let δ be a singular Jordan derivation on $\mathcal{A}$.*

(i) *δ is an antiderivation if and only if δ satisfies*

$$\delta(e\mathcal{A}f) \cdot e\mathcal{A}f = e\mathcal{A}f \cdot \delta(e\mathcal{A}f) = \delta(f\mathcal{A}e) \cdot f\mathcal{A}e = f\mathcal{A}e \cdot \delta(f\mathcal{A}e) = \{0\}. \tag{2.24}$$

(ii) *If $\mathcal{A}$ satisfies*

$$e\mathcal{A}f \cdot f\mathcal{A}e = f\mathcal{A}e \cdot e\mathcal{A}f = \{0\}, \tag{2.25}$$

then δ is an antiderivation.

Corollary 2.6. *Let φ be a Lie n-derivation on $\mathcal{A}$. Suppose that $\mathcal{A}$ is 2- and $(n-1)$-torsion free, and that $\mathcal{A}$ satisfies the property (1.2) and (2.25). Then φ is of the form*

$$\varphi = d + \delta + \gamma \tag{2.26}$$

where d is a derivation on $\mathcal{A}$, δ is a singular Jordan derivation and antiderivation on $\mathcal{A}$, and γ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)-$th commutators of $\mathcal{A}$, if and only if $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$.

Proof. Since $\mathcal{A}$ satisfies (2.25), we have that $mt = tm = 0$ and $e\varphi(tm)e + f\varphi(mt)f = 0$ for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$. Thus, the condition (ii) in Theorem 2.3 holds. It follows that φ is of the form (2.26) $\varphi = d + \delta + \gamma$ where δ is a singular Jordan derivation if and only if $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$. By Lemma 2.5, δ is also an antiderivation. □

Associated with Theorem 2.3, we can get the necessary and sufficient conditions under which a Lie n-derivation φ can be standard.

Corollary 2.7. *Let φ be a Lie n-derivation on $\mathcal{A}$. Suppose that $\mathcal{A}$ is 2- and $(n-1)$-torsion free, and that $\mathcal{A}$ satisfies the property (1.2). Then φ is standard if and only if the following conditions hold:*

(i) *$f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$,*

(ii) *$e\varphi(tm)e + f\varphi(mt)f \in \mathcal{Z}(\mathcal{A})$ for each $m \in e\mathcal{A}f$ and $t \in f\mathcal{A}e$,*

(iii) *$f\varphi(e\mathcal{A}f)e = e\varphi(f\mathcal{A}e)f = \{0\}$.*

Proof. Suppose that φ is standard. That is, there exists a derivation d on $\mathcal{A}$ and a linear mapping γ from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)-$th commutators of $\mathcal{A}$, such that $\varphi = d + \gamma$. Let δ be a linear mapping on $\mathcal{A}$ and $\delta = 0$. Then δ is a singular Jordan derivation and $\varphi = d + \delta + \gamma$. According to Theorem 2.3, we only need to prove (iii). For each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$, we have $p_n(m, f, ..., f) = p_{n-1}(m, f, ..., f) = ... = [m, f] = m$ and $p_n(t, e, ..., e) = p_{n-1}(t, e, ..., e) = ... = [t, e] = t$. Since γ vanishes on all $(n-1)-$th commutators of $\mathcal{A}$, we have that $\gamma(m) = \gamma(t) = 0$. Thus,

$$\begin{aligned} \varphi(m) &= d(em) + \gamma(m) = d(e)m + ed(m), \\ \varphi(t) &= d(te) + \gamma(t) = d(t)e + td(e). \end{aligned}$$

Left and right multiplication by e and f respectively implies that $f\varphi(m)e = 0$ and $e\varphi(t)f = 0$. Hence, (iii) holds.

Suppose that (i), (ii) and (iii) hold. According to Theorem 2.3, φ is of the form (2.2) $\varphi = d+\delta+\gamma$. By (iii) and the definition (2.14) of δ in Theorem 2.3, we obtain that $\delta = 0$. □

Corollary 2.8. *Let φ be a Lie n-derivation on $\mathcal{A}$. Suppose that n is even, and that $\mathcal{A}$ is a 2- and $(n-1)$-torsion free algebra satisfying the property (1.2). Then φ is standard if and only if both conditions* (i) *and* (ii) *hold:*

(i) *$f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$,*

(ii) *$e\varphi(tm)e + f\varphi(mt)f \in \mathcal{Z}(\mathcal{A})$ for each $m \in e\mathcal{A}f$ and $t \in f\mathcal{A}e$.*

Proof. According to Corollary 2.7, we only need to prove that $f\varphi(e\mathcal{A}f)e = e\varphi(f\mathcal{A}e)f = \{0\}$ if (i) and (ii) hold.

Suppose that (i) and (ii) hold. For each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$, by (2.9) and (2.11), we have that

$$f\varphi(m)e = (-1)^{n-1} f\varphi(m)e \quad \text{and} \quad e\varphi(t)f = (-1)^{n-1} e\varphi(t)f.$$

Since n is even and $\mathcal{A}$ is 2-torsion free, we have $f\varphi(m)e = e\varphi(t)f = 0$. □

Remark 2.9. *Let $\mathcal{A}$ be a unital algebra with a nontrivial idempotent e satisfying the property (1.2). Theorem 2.3 generalizes [2, Theorem 3.4] which considers Lie triple derivations on $\mathcal{A}$. Corollaries 2.6 and 2.8 generalize [2, Corollaries 3.5, 3.6 and 3.7] which considers Lie derivations and Lie triple derivations on $\mathcal{A}$.*

2.2. The sufficient conditions for each Lie n-derivation to be standard.

Now, we want to enhance the sufficient conditions proposed in Theorem 2.3, and to give the sufficient conditions under which each Lie n-derivation on $\mathcal{A}$ can be described as the sum of a derivation, a singular Jordan derivation and a central mapping. For this purpose, we need to find some sufficient conditions independent of any Lie n-derivation φ.

Let $\tilde{\mathcal{A}}$ be an arbitrary algebra. $\mathcal{Z}(\tilde{\mathcal{A}})$ denotes the centre of $\tilde{\mathcal{A}}$. $\tilde{\mathcal{A}}$ is said to *have no nonzero central ideal* if there is no nonzero ideal of $\tilde{\mathcal{A}}$ in $\mathcal{Z}(\tilde{\mathcal{A}})$. $\mathcal{S}(\tilde{\mathcal{A}})$ denotes the subalgebra generated with all idempotents and commutators of $\tilde{\mathcal{A}}$. For each x in $\tilde{\mathcal{A}}$, we consider that

$$[x, \tilde{\mathcal{A}}] \subseteq \mathcal{Z}(\tilde{\mathcal{A}}) \text{ implies } x \in \mathcal{Z}(\tilde{\mathcal{A}}). \tag{2.27}$$

That is,

$$[[x, \tilde{\mathcal{A}}], \tilde{\mathcal{A}}] = \{0\} \text{ implies } [x, \tilde{\mathcal{A}}] = \{0\},$$

which is equivalent to the condition that there exists no nonzero central inner derivation of $\tilde{\mathcal{A}}$. Important examples of algebras satisfying (2.27) include commutative algebras, prime algebras, triangular algebras and matrix algebras.

Since $\mathcal{A}$ is a unital algebra with a nontrivial idempotent e, we would mention that

$$\mathcal{S}(\mathcal{A}) = (\mathcal{S}(e\mathcal{A}e) + e\mathcal{A}f \cdot f\mathcal{A}e) + e\mathcal{A}f + f\mathcal{A}e + (\mathcal{S}(f\mathcal{A}f) + f\mathcal{A}e \cdot e\mathcal{A}f).$$

If $e\mathcal{A}f \cdot f\mathcal{A}e = f\mathcal{A}e \cdot e\mathcal{A}f = \{0\}$, then $\mathcal{A} = \mathcal{S}(\mathcal{A})$ if and only if $e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ and $f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$. If $\mathcal{A}$ satisfies the property (1.2), then $\mathcal{A}$ satisfies (2.27) and has no nonzero central ideal, but we cannot confirm that $e\mathcal{A}e$ or $f\mathcal{A}f$ satisfies (2.27) or has no nonzero central ideal.

Theorem 2.10. *Let φ be a Lie n-derivation on $\mathcal{A}$. Suppose that $\mathcal{A}$ is 2- and $(n-1)$-torsion free, and that $\mathcal{A}$ satisfies the property (1.2). Suppose that one of following conditions* (i-1) - (i-4) *holds:*

(i-1) *$e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ and $f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$,*

(i-2) *$e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ and $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$,*

(i-3) *$f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$ and $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$,*

(i-4) *$e\mathcal{A}e$ or $f\mathcal{A}f$ satisfies (2.27) when $n \geq 3$, $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$ and $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$.*

And suppose that one of the following conditions (ii-1) - (ii-4) *also holds:*

(ii-1) *$e\mathcal{A}e$ or $f\mathcal{A}f$ has no nonzero central ideal,*

(ii-2) $\mathcal{Z}(\mathcal{A}) = \{\ a + b \mid a \in e\mathcal{Z}(\mathcal{A})e,\ b \in f\mathcal{Z}(\mathcal{A})f,\ am_0 = m_0 b\ \}$ *for some* $m_0 \in e\mathcal{A}f$,

(ii-3) $\mathcal{Z}(\mathcal{A}) = \{\ a + b \mid a \in e\mathcal{Z}(\mathcal{A})e,\ b \in f\mathcal{Z}(\mathcal{A})f,\ t_0 a = bt_0\ \}$ *for some* $t_0 \in f\mathcal{A}e$,

(ii-4) *$\mathcal{A}$ satisfies (2.25).*

Then φ is of the form $\varphi = d + \delta + \gamma$, where d is a derivation on $\mathcal{A}$, δ is a singular Jordan derivation on $\mathcal{A}$, and γ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$−th commutators of $\mathcal{A}$.

In addition, δ is also an antiderivation on $\mathcal{A}$ when (ii-1) *or* (ii-4) *holds.*

Remark 2.11. *In Theorem 2.10, if we make a further assumption that n is even, or that $f\varphi(e\mathcal{A}f)e = e\varphi(f\mathcal{A}e)f = \{0\}$, then φ is standard. Thus, we can obtain the sufficient conditions under which every Lie n-derivation can be standard.*

Corollary 2.12. *Suppose that n is even, and that $\mathcal{A}$ is a 2- and $(n-1)$-torsion free algebra satisfying the property (1.2). Suppose that one of the following conditions* (i-1) - (i-4) *holds:*

(i-1) *$e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ and $f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$,*

(i-2) *$e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ and $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$,*

(i-3) *$f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$ and $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$,*

(i-4) *$e\mathcal{A}e$ or $f\mathcal{A}f$ satisfies (2.27) when $n \geq 3$, $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$ and $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$.*

And suppose that one of the following conditions (ii-1) - (ii-4) *also holds:*

(ii-1) *$e\mathcal{A}e$ or $f\mathcal{A}f$ has no nonzero central ideal,*

(ii-2) *$\mathcal{Z}(\mathcal{A}) = \{\ a + b \mid a \in e\mathcal{Z}(\mathcal{A})e,\ b \in f\mathcal{Z}(\mathcal{A})f,\ am_0 = m_0 b\ \}$ for some $m_0 \in e\mathcal{A}f$,*

(ii-3) *$\mathcal{Z}(\mathcal{A}) = \{\ a + b \mid a \in e\mathcal{Z}(\mathcal{A})e,\ b \in f\mathcal{Z}(\mathcal{A})f,\ t_0 a = b t_0\ \}$ for some $t_0 \in f\mathcal{A}e$,*

(ii-4) *$\mathcal{A}$ satisfies (2.25).*

Then every Lie n-derivation on $\mathcal{A}$ is standard.

Corollary 2.13. *Suppose that $\mathcal{A}$ is 2- and $(n-1)$-torsion free, and that $\mathcal{A}$ satisfies the property (1.2). Suppose that one of following conditions* (i-1) - (i-4) *holds:*

(i-1) *$e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ and $f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$,*

(i-2) *$e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ and $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$,*

(i-3) *$f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$ and $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$,*

(i-4) *$e\mathcal{A}e$ or $f\mathcal{A}f$ satisfies (2.27) when $n \geq 3$, $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$ and $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$.*

And suppose that one of the following conditions (ii-1) - (ii-2) *also holds:*

(ii-1) *$e\mathcal{A}e$ or $f\mathcal{A}f$ has no nonzero central ideal,*

(ii-2) *$\mathcal{A}$ satisfies (2.25).*

If

(iii) *for each x in $\mathcal{A}$, we have $exf \cdot f\mathcal{A}e = \{0\} = f\mathcal{A}e \cdot exf$ implies $exf = 0$ and $e\mathcal{A}f \cdot fxe = \{0\} = fxe \cdot e\mathcal{A}f$ implies $fxe = 0$,*

then every Lie n-derivation on $\mathcal{A}$ is standard.

Remark 2.14. *Theorem 2.10 improves [15, Theorem 2.1] and [2, Theorem 5.1] which consider Lie triple derivations and Lie n-derivations on a unital algebra with a nontrivial idempotent e satisfying the property (1.2).*

To prove Theorem 2.10, we need to prove several lemmas.

Lemma 2.15. *Let φ be a Lie n-derivation on $\mathcal{A}$. Suppose that $\mathcal{A}$ is $(n-1)$-torsion free, and that $\mathcal{A}$ satisfies the property (1.2). Suppose that both the conditions* (i) *and* (ii) *hold:*

(i) $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ *and* $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$,

(ii) $e\mathcal{A}e$ *or* $f\mathcal{A}f$ *has no nonzero central ideal.*

Then φ satisfies $e\varphi(tm)e + f\varphi(mt)f \in \mathcal{Z}(\mathcal{A})$ for each $m \in e\mathcal{A}f$ and $t \in f\mathcal{A}e$.

Lemma 2.16. *Let φ be a Lie n-derivation on $\mathcal{A}$. Suppose that $\mathcal{A}$ is* 2- *and* $(n-1)$-*torsion free, and that $\mathcal{A}$ satisfies the property (1.2). Suppose that*

(i) $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ *and* $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$.

And suppose that one of the following conditions (ii-1) - (ii-2) *also holds:*

(ii-1) $\mathcal{Z}(\mathcal{A}) = \{\ a+b \mid a \in e\mathcal{Z}(\mathcal{A})e,\ b \in f\mathcal{Z}(\mathcal{A})f,\ am_0 = m_0 b\ \}$ *for some* $m_0 \in e\mathcal{A}f$,

(ii-2) $\mathcal{Z}(\mathcal{A}) = \{\ a+b \mid a \in e\mathcal{Z}(\mathcal{A})e,\ b \in f\mathcal{Z}(\mathcal{A})f,\ t_0 a = b t_0\ \}$ *for some* $t_0 \in f\mathcal{A}e$.

Then φ satisfies $e\varphi(tm)e + f\varphi(mt)f \in \mathcal{Z}(\mathcal{A})$ for each $m \in e\mathcal{A}f$ and $t \in f\mathcal{A}e$.

Lemma 2.17. *Let φ be a Lie n-derivation on $\mathcal{A}$. Suppose that $\mathcal{A}$ is* 2- *and* $(n-1)$-*torsion free, and that $\mathcal{A}$ satisfies the property (1.2). Suppose that one of following conditions* (i) - (iv) *holds:*

(i) $e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ *and* $f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$,

(ii) $e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ *and* $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$,

(iii) $f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$ *and* $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$,

(iv) $e\mathcal{A}e$ *or* $f\mathcal{A}f$ *satisfies (2.27) when* $n \geq 3$, $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$ *and* $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$.

Then φ satisfies $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$.

Proof. [Proof of Lemma 2.15] Without loss of generality, we suppose that $e\mathcal{A}e$ has no nonzero central ideal. Since φ is a Lie n-derivation on $\mathcal{A}$. Discussing similarly as Theorem 2.3, we obtain same results as Claims 1, 2 and 3 in Theorem 2.3. According to (i) and the definition (2.13) of d, we conclude that for each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$,

$$\begin{aligned} &d(am) = d(a)m + ad(m),\\ &d(ta) = td(a) + d(t)a,\\ &d(m)t + md(t) = d(mt) + \tau^{-1}(f\varphi(mt)f) - e\varphi(tm)e. \end{aligned} \tag{2.28}$$

For each a_1, a_2 in $e\mathcal{A}e$, m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$, it follows that

$$\begin{aligned} &d(a_1a_2m) = d(a_1a_2)m + a_1a_2d(m),\\ &d(a_1a_2m) = d(a_1)a_2m + a_1d(a_2m) = d(a_1)a_2m + a_1d(a_2)m + a_1a_2d(m),\\ &d(ta_1a_2) = td(a_1a_2) + d(t)a_1a_2,\\ &d(ta_1a_2) = ta_1d(a_2) + d(ta_1)a_2 = ta_1d(a_2) + td(a_1)a_2 + d(t)a_1a_2. \end{aligned}$$

Thus,

$$\begin{aligned} &(d(a_1a_2) - d(a_1)a_2 - a_1d(a_2))m = 0,\\ &t(d(a_1a_2) - a_1d(a_2) - d(a_1)a_2) = 0. \end{aligned}$$

Since $\mathcal{A}$ satisfies the property (1.2), we obtain that

$$d(a_1a_2) = d(a_1)a_2 + a_1d(a_2).$$

Denote that $\epsilon(m,t) = e\varphi(tm)e - \tau^{-1}(f\varphi(mt)f)$ for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$. Then $\epsilon(m,t) = d(mt) - d(m)t - md(t)$ by (2.28). According to (i), we have that $\epsilon(m,t) \in e\mathcal{Z}(\mathcal{A})e \subseteq \mathcal{Z}(e\mathcal{A}e)$. Since

$$\begin{aligned}\epsilon(am,t) =& d(amt) - d(am)t - amd(t)\\ =& d(a)mt + ad(mt) - d(a)mt - ad(m)t - amd(t)\\ =& a(d(mt) - d(m)t - md(t))\\ =& a\epsilon(m,t)\end{aligned}$$

for each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$, we obtain that $\epsilon(m,t)$ is a central ideal of $e\mathcal{A}e$. According to the assumption that $e\mathcal{A}e$ has no nonzero central ideal, we have $\epsilon(m,t) = 0$, i.e., $e\varphi(tm)e - \tau^{-1}(f\varphi(mt)f) = 0$. Thus, $e\varphi(tm)e + f\varphi(mt)f \in \mathcal{Z}(\mathcal{A})$.

The proof in case that $f\mathcal{A}f$ has no nonzero central ideal goes in a similar way. □

Proof. [Proof of Lemma 2.16] Without loss of generality, we suppose that (i) and (ii-1) hold. Discussing similarly as Theorem 2.3, we obtain same results as Claims 1, 2 and 3 in Theorem 2.3. According to (i) and the definition (2.13) of d, we conclude that for each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$, t in $f\mathcal{A}e$ and b in $f\mathcal{A}f$,

$$d(am) = d(a)m + ad(m), \tag{2.29}$$

$$d(mb) = md(b) + d(m)b, \tag{2.30}$$

$$d(m)t + md(t) = d(mt) + \tau^{-1}(f\varphi(mt)f) - e\varphi(tm)e, \tag{2.31}$$

$$d(t)m + td(m) = d(tm) + \tau(e\varphi(tm)e) - f\varphi(mt)f. \tag{2.32}$$

For each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$, it follows from (2.29) and (2.30) that

$$d(mtm) = d((mt)m) = d(mt)m + mtd(m) \quad \text{and} \quad d(mtm) = d(m(tm)) = md(tm) + d(m)tm.$$

Thus,

$$(d(mt) - d(m)t)m = m(d(tm) - td(m)). \tag{2.33}$$

Denote that $\epsilon(m,t) = e\varphi(tm)e - \tau^{-1}(f\varphi(mt)f)$ for m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$. By (2.31) and (2.32), we have that

$$\begin{aligned}d(mt) - d(m)t &= md(t) + \epsilon(m,t),\\ d(tm) - td(m) &= d(t)m - \tau(\epsilon(m,t)).\end{aligned}$$

In view of (2.33), it follows that

$$md(t)m + \epsilon(m,t)m = md(t)m - m\tau(\epsilon(m,t)).$$

Thus, $2\epsilon(m,t)m = 0$. Since $\mathcal{A}$ is 2-torsion free, we have

$$\epsilon(m,t)m = 0 \tag{2.34}$$

for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$. Let m_0 be as in (ii-1), then $\epsilon(m_0,t)m_0 = 0 = m_0 0$. It follows from (ii-1) that $\epsilon(m_0,t) + 0 \in \mathcal{Z}(\mathcal{A})$. Thus, $\epsilon(m_0,t) = 0$. Since (2.34) and

$$0 = \epsilon(m+m_0,t)(m+m_0) = \epsilon(m,t)m + \epsilon(m,t)m_0 + \epsilon(m_0,t)(m+m_0),$$

we conclude that $\epsilon(m,t)m_0 = 0 = m_0 0$, which follows from (ii-1) that $\epsilon(m,t) + 0 \in \mathcal{Z}(\mathcal{A})$. Thus, $\epsilon(m,t) = 0$ and $e\varphi(tm)e + f\varphi(mt)f \in \mathcal{Z}(\mathcal{A})$ for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$.

The proof in case that (i) and (ii-2) hold goes in a similar way. □

Proof. [Proof of Lemma 2.17] Since φ is a Lie n-derivation on $\mathcal{A}$, discussing similarly as Theorem 2.3, we obtain same results as Claims 1, 2 and 3 in Theorem 2.3. Thus for each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$, t in $f\mathcal{A}e$ and b in $f\mathcal{A}f$, we have that

$$e\varphi(am)f = e\varphi(a)e \cdot m + a \cdot e\varphi(m)f - m \cdot f\varphi(a)f, \tag{2.35}$$
$$e\varphi(mb)f = m \cdot f\varphi(b)f + e\varphi(m)f \cdot b - e\varphi(b)e \cdot m, \tag{2.36}$$
$$f\varphi(ta)e = t \cdot e\varphi(a)e + f\varphi(t)e \cdot a - f\varphi(a)f \cdot t, \tag{2.37}$$
$$f\varphi(bt)e = f\varphi(b)f \cdot t + b \cdot f\varphi(t)e - t \cdot e\varphi(b)e. \tag{2.38}$$

Then the remaining proof could be organized by the following claims.

Claim 1. $e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ implies $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$.

Let $A_0 = \{\ a \in e\mathcal{A}e \mid f\varphi(a)f \in f\mathcal{Z}(\mathcal{A})f\ \}$. We only need to prove that $A_0 = e\mathcal{A}e$.

According to the linearity of φ, we have that A_0 is a $\mathcal{R}$-submodule of $e\mathcal{A}e$. Take arbitrary elements a, a' in A_0. We have $f\varphi(a)f, f\varphi(a')f \in f\mathcal{Z}(\mathcal{A})f$. By Lemma 2.2, we have that for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$,

$$\begin{aligned}
&m \cdot f\varphi(a)f = \tau^{-1}(f\varphi(a)f) \cdot m,\\
&m \cdot f\varphi(a')f = \tau^{-1}(f\varphi(a')f) \cdot m,\\
&f\varphi(a)f \cdot t = t \cdot \tau^{-1}(f\varphi(a)f),\\
&f\varphi(a')f \cdot t = t \cdot \tau^{-1}(f\varphi(a')f).
\end{aligned}$$

It follows from (2.35) and (2.37) that

$$\begin{aligned}
e\varphi((aa')m)f =& e\varphi(aa')e \cdot m + aa' \cdot e\varphi(m)f - m \cdot f\varphi(aa')f,\\
e\varphi(a(a'm))f =& e\varphi(a)e \cdot a'm + a \cdot e\varphi(a'm)f - a'm \cdot f\varphi(a)f\\
=& e\varphi(a)e \cdot a'm + a \cdot e\varphi(a')e \cdot m + aa' \cdot e\varphi(m)f - am \cdot f\varphi(a')f - a'm \cdot f\varphi(a)f\\
=& (e\varphi(a)e \cdot a' + a \cdot e\varphi(a')e - a\tau^{-1}(f\varphi(a')f) - a'\tau^{-1}(f\varphi(a)f)) \cdot m + aa' \cdot e\varphi(m)f,\\
f\varphi(t(aa'))e =& t \cdot e\varphi(aa')e + f\varphi(t)e \cdot aa' - f\varphi(aa')f \cdot t,\\
f\varphi((ta)a')e =& ta \cdot e\varphi(a')e + f\varphi(ta)e \cdot a' - f\varphi(a')f \cdot ta\\
=& ta \cdot e\varphi(a')e + t \cdot e\varphi(a)e \cdot a' + f\varphi(t)e \cdot aa' - f\varphi(a)f \cdot ta' - f\varphi(a')f \cdot ta\\
=& t \cdot (a \cdot e\varphi(a')e + e\varphi(a)e \cdot a' - a'\tau^{-1}(f\varphi(a)f) - a\tau^{-1}(f\varphi(a')f)) + f\varphi(t)e \cdot aa'.
\end{aligned}$$

Then

$$\begin{aligned}
&(e\varphi(aa')e - e\varphi(a)e \cdot a' - a \cdot e\varphi(a')e + a\tau^{-1}(f\varphi(a')f) + a'\tau^{-1}(f\varphi(a)f)) \cdot m = m \cdot f\varphi(aa')f,\\
&f\varphi(aa')f \cdot t = t \cdot (e\varphi(aa')e - e\varphi(a)e \cdot a' - a \cdot e\varphi(a')e + a\tau^{-1}(f\varphi(a')f) + a'\tau^{-1}(f\varphi(a)f)).
\end{aligned}$$

According to Lemma 2.2, $f\varphi(aa')f \in f\mathcal{Z}(\mathcal{A})f$. That is, $aa' \in A_0$. Thus, A_0 is a subalgebra of $e\mathcal{A}e$.

Take an arbitrary element a in $e\mathcal{A}e$ satisfying $a = a^2$. By (2.35) and (2.37), we have that for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$,

$$\begin{aligned}
e\varphi(am)f =& e\varphi(a(am))f = e\varphi(a)e \cdot am + a \cdot e\varphi(am)f - am \cdot f\varphi(a)f\\
=& e\varphi(a)e \cdot am + a \cdot e\varphi(a)e \cdot m + a \cdot e\varphi(m)f - 2am \cdot f\varphi(a)f,\\
f\varphi(ta)e =& f\varphi((ta)a)e = ta \cdot e\varphi(a)e + f\varphi(ta)e \cdot a - f\varphi(a)f \cdot ta\\
=& ta \cdot e\varphi(a)e + t \cdot e\varphi(a)e \cdot a + f\varphi(t)e \cdot a - 2f\varphi(a)f \cdot ta.
\end{aligned}$$

Then combining with (2.35) and (2.37), we obtain that

$$(e\varphi(a)e - e\varphi(a)e \cdot a - a \cdot e\varphi(a)e) \cdot m + 2am \cdot f\varphi(a)f = m \cdot f\varphi(a)f, \tag{2.39}$$
$$t \cdot (e\varphi(a)e - a \cdot e\varphi(a)e - e\varphi(a)e \cdot a) + 2f\varphi(a)f \cdot ta = f\varphi(a)f \cdot t. \tag{2.40}$$

Left and right multiplication by a respectively implies that

$$am \cdot f\varphi(a)f = a \cdot e\varphi(a)e \cdot am, \tag{2.41}$$
$$f\varphi(a)f \cdot ta = ta \cdot e\varphi(a)e \cdot a. \tag{2.42}$$

In view of (2.41) and (2.42), equations (2.39) and (2.40) can be reformed to

$$(e\varphi(a)e - e\varphi(a)e \cdot a - a \cdot e\varphi(a)e + 2a \cdot e\varphi(a)e \cdot a) \cdot m = m \cdot f\varphi(a)f,$$
$$t \cdot (e\varphi(a)e - e\varphi(a)e \cdot a - a \cdot e\varphi(a)e + 2a \cdot e\varphi(a)e \cdot a) = f\varphi(a)f \cdot t.$$

According to Lemma 2.2, $f\varphi(a)f \in f\mathcal{Z}(\mathcal{A})f$. That is, $a = a^2 \in A_0$. Thus, A_0 contains all idempotents in $e\mathcal{A}e$.

Take arbitrary elements a, a' in $e\mathcal{A}e$. By (2.35) and (2.37), we have that for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$,

$$\begin{aligned} e\varphi([a,a']m)f =& e\varphi([a,a'])e \cdot m + [a,a'] \cdot e\varphi(m)f - m \cdot f\varphi([a,a'])f, \\ e\varphi([a,a']m)f =& e\varphi(aa'm)f - e\varphi(a'am)f \\ =& [e\varphi(a)e, a']m + [a, e\varphi(a')e] \cdot m + [a,a'] \cdot e\varphi(m)f, \\ f\varphi(t[a,a'])e =& t \cdot e\varphi([a,a'])e + f\varphi(t)e \cdot [a,a'] - f\varphi([a,a'])f \cdot t, \\ f\varphi(t[a,a'])e =& f\varphi(taa')e - f\varphi(ta'a)e \\ =& t \cdot [a, e\varphi(a')e] + t \cdot [e\varphi(a)e, a'] + f\varphi(t)e \cdot [a,a']. \end{aligned}$$

Then

$$(e\varphi([a,a'])e - [e\varphi(a)e, a'] - [a, e\varphi(a')e]) \cdot m = m \cdot f\varphi([a,a'])f,$$
$$f\varphi([a,a'])f \cdot t = t \cdot (e\varphi([a,a'])e - [a, e\varphi(a')e] - [e\varphi(a)e, a']).$$

According to Lemma 2.2, $f\varphi([a,a'])f \in f\mathcal{Z}(\mathcal{A})f$. That is, $[a,a'] \in A_0$. Thus, A_0 contains all commutators in $e\mathcal{A}e$.

Since A_0 is a subalgebra of $e\mathcal{A}e$, and A_0 contains all idempotent and commutators in $e\mathcal{A}e$, we have that $\mathcal{S}(e\mathcal{A}e) \subseteq A_0$. Since $e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$, we conclude that $e\mathcal{A}e = A_0$.

Claim 2. $f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$ implies $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$.

The proof of Claim 2 is similar to the proof of Claim 1.

Claim 3. (i) implies that $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$.

It's obvious according to Claim 1 and 2.

Claim 4. (ii) implies that $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$.

By Claim 1, we only need to prove $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$. If $n \geq 3$, for each a in $e\mathcal{A}e$, m in $e\mathcal{A}f$, t in $f\mathcal{A}e$ and b in $f\mathcal{A}f$, since $[a,b] = 0$ and $p_n(a,b,m,f,...,f) = p_n(a,b,t,e,...,e) = 0$, we obtain that

$$\begin{aligned} 0 =& \varphi(p_n(a,b,m,f,...,f)) = p_n(\varphi(a),b,m,f,...,f) + p_n(a,\varphi(b),m,f,...,f) \\ =& [[\varphi(a),b] + [a,\varphi(b)], m], \\ 0 =& \varphi(p_n(a,b,t,e,...,e)) = p_n(\varphi(a),b,t,e,...,e) + p_n(a,\varphi(b),t,e,...,e) \\ =& [[\varphi(a),b] + [a,\varphi(b)], t]. \end{aligned}$$

According to Lemma 2.2, we have

$$[f\varphi(a)f, b] + [a, e\varphi(b)e] \in \mathcal{Z}(\mathcal{A}).$$

By Claim 1, we have $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f \subseteq \mathcal{Z}(f\mathcal{A}f)$, and $[f\varphi(a)f, b] = 0$. If $n \geq 3$, it follows that $[a, e\varphi(b)e] = 0$ for each a in $e\mathcal{A}e$ and b in $f\mathcal{A}f$. If $n = 2$, for each a in $e\mathcal{A}e$ and b in $f\mathcal{A}f$, we have that

$$0 = \varphi([a, b]) = [f\varphi(a)f, b] + [a, e\varphi(b)e]$$

which follows that $[a, e\varphi(b)e] = 0$. Thus, we have $e\varphi(f\mathcal{A}f)e \subseteq \mathcal{Z}(e\mathcal{A}e)$ for each $n \geq 2$. Since $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$, we conclude that $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$.

Claim 5. (iii) implies that $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$.
The proof of Claim 5 is similar to the proof of Claim 4.

Claim 6. (iv) implies that $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$.
If $n \geq 3$, suppose that $e\mathcal{A}e$ satisfies (2.27). Similar to the proof of Claim 4, we have that

$$[f\varphi(a)f, b] + [a, e\varphi(b)e] \in \mathcal{Z}(\mathcal{A}) \tag{2.43}$$

for each a in $e\mathcal{A}e$ and b in $f\mathcal{A}f$. Then $[e\mathcal{A}e, e\varphi(f\mathcal{A}f)e] \subseteq e\mathcal{Z}(\mathcal{A})e \subseteq \mathcal{Z}(e\mathcal{A}e)$. Since $e\mathcal{A}e$ satisfies (2.27), it follows that

$$e\varphi(f\mathcal{A}f)e \subseteq \mathcal{Z}(e\mathcal{A}e). \tag{2.44}$$

Since $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$, we have that $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$. On the other hand, by (2.44), we have $[a, e\varphi(b)e] = 0$ for each a in $e\mathcal{A}e$ and b in $f\mathcal{A}f$. In view of (2.43), we obtain $[f\varphi(a)f, b] = 0$ for each a in $e\mathcal{A}e$ and b in $f\mathcal{A}f$. That is, $f\varphi(e\mathcal{A}e)f \subseteq \mathcal{Z}(f\mathcal{A}f)$. Since $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$, we obtain that $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$. The proof in case that $f\mathcal{A}f$ satisfies (2.27) goes in a similar way.

If $n = 2$, for each a in $e\mathcal{A}e$ and b in $f\mathcal{A}f$, we have that

$$0 = \varphi([a, b]) = [f\varphi(a)f, b] + [a, e\varphi(b)e].$$

Then, $[f\varphi(a)f, b] = [a, e\varphi(b)e] = 0$. That is, $f\varphi(e\mathcal{A}e)f \subseteq \mathcal{Z}(f\mathcal{A}f)$ and $e\varphi(f\mathcal{A}f)e \subseteq \mathcal{Z}(e\mathcal{A}e)$. Since $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$ and $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$, we conclude that $f\varphi(e\mathcal{A}e)f \subseteq f\mathcal{Z}(\mathcal{A})f$ and $e\varphi(f\mathcal{A}f)e \subseteq e\mathcal{Z}(\mathcal{A})e$. □

Proof. [Proof of Theorem 2.10] According to Lemmas 2.5, 2.15, 2.16, 2.17 and Theorem 2.3, we only need to prove that δ is an antiderivation when (ii-1) holds.

Without loss of generality, we suppose that $e\mathcal{A}e$ has no nonzero central ideal. The following discussion is partially similar as Claim 6 in Theorem 2.3, and we will omit several complicated procedures. If n is even, then in view of (2.9), (2.11), (2.14) and that $\mathcal{A}$ is 2-torsion free, we obtain that $\delta(m) = 0$ and $\delta(t) = 0$ for each m in $e\mathcal{A}f$ and t in $f\mathcal{A}e$. Thus, $\delta = 0$. If n is odd, then in view of (2.14) and Claim 3 in Theorem 2.3, we have that for each m_1, m_2 in $e\mathcal{A}f$ and t_1, t_2 in $f\mathcal{A}e$,

$$[\delta(m_1), m_2] + [\delta(m_2), m_1] = 0 \quad \text{and} \quad [\delta(t_1), t_2] + [\delta(t_2), t_1] = 0. \tag{2.45}$$

Take arbitrary elements m, m_1, m_2 in $e\mathcal{A}f$. If $n \geq 3$, since $p_n(m_1, m_2, m, f, ..., f) = 0$, we have that

$$\begin{aligned}
0 =& \varphi(p_n(m_1, m_2, m, f, ..., f)) \\
=& [[\varphi(m_1), m_2], m] + [[m_1, \varphi(m_2)], m] \\
=& [[\varphi(m_1), m_2] + [m_1, \varphi(m_2)], m] \\
=& [[\delta(m_1), m_2] + [m_1, \delta(m_2)], m].
\end{aligned}$$

Or if $n = 2$, then

$$\begin{aligned}
0 =& \varphi([[m_1, m_2], m]) \\
=& [\varphi([m_1, m_2]), m] \\
=& [[\varphi(m_1), m_2] + [m_1, \varphi(m_2)], m] \\
=& [[\delta(m_1), m_2] + [m_1, \delta(m_2)], m].
\end{aligned}$$

Thus, for each $n \geq 2$ and for each m, m_1, m_2 in $e\mathcal{A}f$, we have that

$$[[\delta(m_1), m_2] + [m_1, \delta(m_2)], m] = 0. \tag{2.46}$$

Similarly, since $p_n(m_1, m_2, t, e, ..., e) = p_n(t_1, t_2, m, f, ..., f) = p_n(t_1, t_2, t, e, ..., e) = 0$ when $n \geq 3$, we can conclude that

$$[[\delta(m_1), m_2] + [m_1, \delta(m_2)], t] = 0, \tag{2.47}$$
$$[[\delta(t_1), t_2] + [t_1, \delta(t_2)], m] = 0, \tag{2.48}$$
$$[[\delta(t_1), t_2] + [t_1, \delta(t_2)], t] = 0 \tag{2.49}$$

for each $n \geq 2$ and for each m, m_1, m_2 in $e\mathcal{A}f$ and t, t_1, t_2 in $f\mathcal{A}e$. Considering (2.46), (2.47), (2.48), (2.49) and Lemma 2.4, it follows that

$$[\delta(m_1), m_2] + [m_1, \delta(m_2)] \in \mathcal{Z}(\mathcal{A}) \quad \text{and} \quad [\delta(t_1), t_2] + [t_1, \delta(t_2)] \in \mathcal{Z}(\mathcal{A}). \tag{2.50}$$

The subtraction of (2.45) and (2.50) leads to $2[\delta(m_1), m_2] \in \mathcal{Z}(\mathcal{A})$ and $2[\delta(t_1), t_2] \in \mathcal{Z}(\mathcal{A})$. Since $\mathcal{A}$ is 2-torsion free, we have that

$$[\delta(m_1), m_2] = \delta(m_1)m_2 - m_2\delta(m_1) \in \mathcal{Z}(\mathcal{A}) \quad \text{and} \quad [\delta(t_1), t_2] = \delta(t_1)t_2 - t_2\delta(t_1) \in \mathcal{Z}(\mathcal{A}). \tag{2.51}$$

Hence,

$$m_2\delta(m_1) \in \mathcal{Z}(e\mathcal{A}e) \quad \text{and} \quad \delta(t_1)t_2 \in \mathcal{Z}(e\mathcal{A}e)$$

for each m_1, m_2 in $e\mathcal{A}f$ and t_1, t_2 in $f\mathcal{A}e$. It follows obviously that $e\mathcal{A}f \cdot \delta(e\mathcal{A}f)$ and $\delta(f\mathcal{A}e) \cdot f\mathcal{A}e$ are central ideals of $e\mathcal{A}e$. Since $e\mathcal{A}e$ has no nonzero central ideal, we confirm that $e\mathcal{A}f \cdot \delta(e\mathcal{A}f) = \delta(f\mathcal{A}e) \cdot f\mathcal{A}e = \{0\}$. According to (2.51), we confirm that $\delta(e\mathcal{A}f) \cdot e\mathcal{A}f = f\mathcal{A}e \cdot \delta(f\mathcal{A}e) = \{0\}$. By lemma 2.5, we obtain that δ is an antiderivation.

The proof in case that $f\mathcal{A}f$ has no nonzero central ideal goes in a similar way. □

Proof. [Proof of Corollary 2.13] According to Theorem 2.10, if $\mathcal{A}$ satisfies one of (i-1), (i-2), (i-3) and (i-4), and if $\mathcal{A}$ satisfies (ii-1) or (ii-2), then arbitrary Lie n-derivation φ on $\mathcal{A}$ is of the form $\varphi = d + \delta + \gamma$, where d is a derivation on $\mathcal{A}$, δ is a singular Jordan derivation and antiderivation on $\mathcal{A}$, and γ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ vanishing on all $(n-1)$–th commutators of $\mathcal{A}$. By Lemma 2.5, we know that δ satisfies (2.24):

$$\delta(e\mathcal{A}f) \cdot e\mathcal{A}f = e\mathcal{A}f \cdot \delta(e\mathcal{A}f) = \delta(f\mathcal{A}e) \cdot f\mathcal{A}e = f\mathcal{A}e \cdot \delta(f\mathcal{A}e) = \{0\}.$$

Since (iii), we conclude that $\delta(e\mathcal{A}f) = \delta(f\mathcal{A}e) = \{0\}$. That is, $\delta = 0$. □

In Corollary 2.13, we would mention that $e\mathcal{A}f = f\mathcal{A}e = \{0\}$ if and only if both conditions (ii-2) and (iii) hold. And if condition (ii-2) holds, then $\mathcal{A} = \mathcal{S}(\mathcal{A})$ if and only if condition (i-1) holds. Thus, we can obtain the following corollary.

Corollary 2.18. *Suppose that $\mathcal{A}$ is a 2- and $(n-1)$-torsion free algebra satisfying the property (1.2), and that $\mathcal{A} = \mathcal{S}(\mathcal{A})$. Suppose that one of conditions* (i) *and* (ii) *holds:*

(i) *n is even, and $\mathcal{A}$ satisfies (2.25),*

(ii) *$e\mathcal{A}f = f\mathcal{A}e = \{0\}$.*

Then every Lie n-derivation on $\mathcal{A}$ is standard.

1

3. Applications

Let $\mathcal{G} = (A, M, N, B)$ be a generalized matrix algebra, where A and B are two unital algebras, and ${}_AM_B$ and ${}_BN_A$ are two bimodules. M is said to be *faithful*, if M satisfies that $aM = 0$ implies $a = 0$ and that $Mb = 0$ implies $b = 0$ for each $a \in A$ and $b \in B$. $\mathcal{G}$ is said to be *trivial* if $MN = NM = \{0\}$.

Corollary 3.1. *Let $\mathcal{G} = (A, M, N, B)$ be a 2- and $(n-1)$-torsion free generalized matrix algebra, where A and B are two unital algebras, M is a faithful (A, B)-bimodule, and N is a (B, A)-bimodule. Suppose that n is even, and that one of following conditions* (i-1) - (i-4) *holds:*

(i-1) *$A = \mathcal{S}(A)$ and $B = \mathcal{S}(B)$,*

(i-2) *$A = \mathcal{S}(A)$ and $\mathcal{Z}(A) = e\mathcal{Z}(\mathcal{G})e$,*

(i-3) *$B = \mathcal{S}(B)$ and $\mathcal{Z}(B) = f\mathcal{Z}(\mathcal{G})f$,*

(i-4) *A or B satisfies (2.27) when $n \geq 3$, $\mathcal{Z}(A) = e\mathcal{Z}(\mathcal{G})e$ and $\mathcal{Z}(B) = f\mathcal{Z}(\mathcal{G})f$.*

And suppose that one of the following conditions (ii-1) - (ii-4) *also holds:*

(ii-1) *A or B has no nonzero central ideal,*

(ii-2) *$\mathcal{Z}(\mathcal{G}) = \{\ a + b \mid a \in e\mathcal{Z}(\mathcal{G})e,\ b \in f\mathcal{Z}(\mathcal{G})f,\ am_0 = m_0b\ \}$ for some $m_0 \in M$,*

(ii-3) *$\mathcal{Z}(\mathcal{G}) = \{\ a + b \mid a \in e\mathcal{Z}(\mathcal{G})e,\ b \in f\mathcal{Z}(\mathcal{G})f,\ t_0a = bt_0\ \}$ for some $t_0 \in N$,*

(ii-4) *$\mathcal{G}$ satisfies (2.25).*

Then every Lie n-derivation on $\mathcal{G}$ is standard.

Corollary 3.2. *Let $\mathcal{G} = (A, M, N, B)$ be a 2- and $(n-1)$-torsion free generalized matrix algebra, where A and B are two unital algebras, M is a faithful (A, B)-bimodule, and N is a (B, A)-bimodule. If one of the following statements holds:*

(i-1) *$A = \mathcal{S}(A)$ and $B = \mathcal{S}(B)$,*

(i-2) *$A = \mathcal{S}(A)$ and $\mathcal{Z}(A) = e\mathcal{Z}(\mathcal{G})e$,*

(i-3) *$B = \mathcal{S}(B)$ and $\mathcal{Z}(B) = f\mathcal{Z}(\mathcal{G})f$,*

(i-4) *A or B satisfies (2.27) when $n \geq 3$, $\mathcal{Z}(A) = e\mathcal{Z}(\mathcal{G})e$ and $\mathcal{Z}(B) = f\mathcal{Z}(\mathcal{G})f$.*

And suppose that one of conditions (ii-1) *and* (ii-2) *also holds:*

(ii-1) *A or B has no nonzero central ideal,*

(ii-2) *$\mathcal{G}$ satisfies (2.25).*

If

(iii) *for each m in M and t in N, it follows that $m \cdot N = \{0\} = N \cdot m$ implies $m = 0$, and that $M \cdot t = \{0\} = t \cdot M$ implies $t = 0$,*

then every Lie n-derivation on $\mathcal{G}$ is standard.

Proof. [proof of Corollaries 3.1 and 3.2] Obviously, $\mathcal{G}$ satisfies (1.2). According to Corollaries 2.12 and 2.13, we obtain corollaries 3.1 and 3.2. □

Corollary 3.3. *Let $\mathcal{G} = (A, M, N, B)$ be a 2- and $(n-1)$-torsion free trivial generalized matrix algebra, where A and B are two unital algebras, M is a faithful (A, B)-bimodule, and N is a (B, A)-bimodule. Suppose that one of following conditions* (i) - (v) *holds:*

(i) *$A = \mathcal{S}(A)$ and $B = \mathcal{S}(B)$,*

(ii) *$\mathcal{G} = \mathcal{S}(\mathcal{G})$,*

(iii) *$A = \mathcal{S}(A)$ and $\mathcal{Z}(A) = e\mathcal{Z}(\mathcal{G})e$,*

(iv) *$B = \mathcal{S}(B)$ and $\mathcal{Z}(B) = f\mathcal{Z}(\mathcal{G})f$,*

(v) *A or B satisfies (2.27) when $n \geq 3$, $\mathcal{Z}(A) = e\mathcal{Z}(\mathcal{G})e$ and $\mathcal{Z}(B) = f\mathcal{Z}(\mathcal{G})f$.*

If n is even or $M = N = \{0\}$, then every Lie n-derivation on $\mathcal{G}$ is standard.

Proof. Since $\mathcal{G}$ is trivial, $\mathcal{G}$ satisfies (2.25). In fact, a trivial generalized matrix algebra satisfying (iii) is a generalized matrix algebra satisfying $M = N = \{0\}$. □

Corollary 3.4. *Let $\mathcal{A} = M_s(A)$ be a 2- and $(n-1)$-torsion free full matrix algebra, where A is a unital algebra with center $\mathcal{Z}(A)$ and $s \geq 3$. Then every Lie n-derivation on $M_s(A)$ is standard.*

Proof. $M_s(A)$ can be represented as of the form $\begin{pmatrix} A & M_{1\times(s-1)}(A) \\ M_{(s-1)\times 1}(A) & M_{s-1}(A) \end{pmatrix}$, which is a generalized matrix algebra and $M_{1\times(s-1)}(A)$ is faithful. In view of [3, Example 5.6] and [7, Lemma 1], conditions (i-4) and (ii-1) in Theorem 2.10 hold. According to [4, Corollary 4.4], every Jordan derivation of $M_s(A)$ is a derivation. Thus, the proof is finished. □

Corollary 3.5. *Let $\mathcal{A} = M_2(A)$ be a 2- and $(n-1)$-torsion free full matrix algebra, where A is a unital algebra with center $\mathcal{Z}(A)$. Suppose that A is a commutative algebra or a noncommutative prime algebra. Then every Lie n-derivation on $M_2(A)$ is standard.*

Proof. If A is commutative, we can assert that (i-4) and (ii-2) in Theorem 2.10 hold. Actually, take arbitrary elements a in $e\mathcal{Z}(M_2(A))e$ and b in $f\mathcal{Z}(M_2(A))f$, which follows that $am = m\tau(a)$ and $bt = t\tau^{-1}(b)$ for each m in $e\mathcal{Z}(M_2(A))f$ and t in $f\mathcal{Z}(M_2(A))e$. Assume that there exists nonzero element m_0 in $eM_2(A)f$ satisfying $am_0 = m_0b$. We can obtain that $b = \tau(a)$. Thus, $am = mb$ and $bt = ta$ for each m in $e\mathcal{Z}(M_2(A))f$ and t in $f\mathcal{Z}(M_2(A))e$ if $am_0 = m_0b$, i.e., (ii-2) in Theorem 2.10 holds. If A is noncommutative prime, it's not difficult to prove that (i-4) and (ii-1) in Theorem 2.10 hold. According to [4, Corollary 4.4], every Jordan derivation of $M_2(A)$ is a derivation. Thus, the proof is finished. □

Corollary 3.6. *Let $\mathcal{T} = (A, M, B)$ be a 2- and $(n-1)$-torsion free triangular algebra, where A and B are two unital algebras, and M is a faithful (A, B)-bimodule. Suppose that one of following conditions* (i) - (v) *holds:*

(i) *$A = \mathcal{S}(A)$ and $B = \mathcal{S}(B)$,*

(ii) *$\mathcal{T} = \mathcal{S}(\mathcal{T})$,*

(iii) *$A = \mathcal{S}(A)$ and $\mathcal{Z}(A) = e\mathcal{Z}(\mathcal{T})e$,*

(iv) *$B = \mathcal{S}(B)$ and $\mathcal{Z}(B) = f\mathcal{Z}(\mathcal{T})f$,*

(v) *A or B satisfies (2.27) when $n \geq 3$, $\mathcal{Z}(A) = e\mathcal{Z}(\mathcal{T})e$ and $\mathcal{Z}(B) = f\mathcal{Z}(\mathcal{T})f$.*

Then every Lie n-derivation on $\mathcal{T}$ is standard.

Proof. Since $f\mathcal{T}e = \{0\}$, it's obvious that $\mathcal{T}$ satisfies (1.2), (2.25), (2.27) and $f\varphi(e\mathcal{A}f)e = e\varphi(f\mathcal{A}e)f = \{0\}$. According to Theorem 2.10, the proof is finished. □

Corollary 3.7. *Let $\mathcal{A} = T_s(A)$ be a 2- and $(n-1)$-torsion free upper triangular matrix algebra, where A is a unital algebra with center $\mathcal{Z}(A)$ and $s \geq 3$. Then every Lie n-derivation on $T_s(A)$ is standard.*

Corollary 3.8. *Let $\mathcal{A} = T_2(A)$ be a 2- and $(n-1)$-torsion free upper triangular matrix algebra, where A is a unital algebra with center $\mathcal{Z}(A)$. Suppose that A is a commutative algebra or a noncommutative prime algebra. Then every Lie n-derivation on $T_2(A)$ is standard.*

Remark 3.9. *The proof of Corollaries 3.7 and 3.8 is more or less similar to the proof of Corollaries 3.4 and 3.5. We would mention that Corollary 3.7 is not true in case that $s = 2$. In [3, Section 7], the authors construct an example. Let A be a $\mathbb{Z}_2-$graded algebra over $\mathcal{R}$, i.e., an algebra of the form $A = A_0 + A_1$, where $A_0, A_1 \subseteq A$ and multiplication in A is such that $A_0A_0 \subseteq A_0$, $A_1A_1 \subseteq A_1$, $A_0A_1 \subseteq A_1$ and $A_1A_0 \subseteq A_1$. Suppose that $\mathcal{Z}(A) = A_0$ and φ is a map on $\mathcal{A} = T_2(A)$. Define that*

$$\varphi\begin{pmatrix} a_0 + a_1 & m_0 + m_1 \\ & b_0 + b_1 \end{pmatrix} = \begin{pmatrix} b_1 & -m_1 \\ & a_1 \end{pmatrix}$$

for each $a_0 + a_1$, $m_0 + m_1$ and $b_0 + b_1$ in A. Then φ is a Lie n-derivation and is not standard.

Corollary 3.10. *Let $\mathcal{A} = \mathrm{alg}\mathcal{N}$ be a nest algebra, where $\mathcal{N}$ is a non-trivial nest in a Hilbert space $\mathcal{H}$ and* $\dim\mathcal{H} \geq 2$. *Then every Lie n-derivation on* $\mathrm{alg}\mathcal{N}$ *is standard.*

Proof. By [6], $\mathrm{alg}\mathcal{N}$ can be viewed as a triangular algebra. Let $\mathcal{A}_0 = E(\mathrm{alg}\mathcal{N})E$, where E is the orthogonal projection on $\mathcal{N}$. Assume that d is a central derivation of $\mathcal{A}_0$. Since $\mathcal{A}_0$ is also a nest, we can find an orthonormal projection e onto $\mathcal{A}_0$, and view $\mathcal{A}_0$ as a triangular algebra. Since $d(e) \in \mathcal{Z}(\mathcal{A}_0)$ and $d(e) = d(ee) = ed(e) + d(e)e$, we have $d(e) = ed(e) = d(e)e = ed(e)e = 0$. Similarly, we have $d(E-e) = 0$. For each $m \in e\mathcal{A}_0(E-e)$, since $d(m) \in \mathcal{Z}(\mathcal{A}_0)$ and $d(m) = d(em(E-e)) = ed(m)(E-e)$, we have $d(m) = 0$. For each $a \in e\mathcal{A}_0e$ and $m \in e\mathcal{A}_0(E-e)$, since $0 = d(am) = d(a)m$, we have $d(a) = 0$. Similarly, we have $d(b) = 0$ for each $b \in (E-e)\mathcal{A}_0(E-e)$. Then $d = 0$. That is, there exists no nonzero central derivation on $\mathcal{A}_0$. Since $\mathcal{Z}(\mathrm{alg}\mathcal{N}) = \mathbb{C}I$, it obviously follows that (v) in Corollary 3.6 holds. Thus, the proof is finished. □

Corollary 3.11. *Let $\mathcal{A}$ be a unital 2- and $(n-1)$-torsion free prime algebra with a nontrivial idempotent e. Suppose that one of following conditions* (i-1) - (i-4) *holds:*

(i-1) *$e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ and $f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$,*

(i-2) *$e\mathcal{A}e = \mathcal{S}(e\mathcal{A}e)$ and $\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$,*

(i-3) *$f\mathcal{A}f = \mathcal{S}(f\mathcal{A}f)$ and $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$,*

(i-4) *$\mathcal{Z}(e\mathcal{A}e) = e\mathcal{Z}(\mathcal{A})e$ and $\mathcal{Z}(f\mathcal{A}f) = f\mathcal{Z}(\mathcal{A})f$.*

And suppose that one of the following conditions (ii-1) - (ii-5) *also holds:*

(ii-1) *$e\mathcal{A}e$ or $f\mathcal{A}f$ is noncommutative,*

(ii-2) *$e\mathcal{A}e$ or $f\mathcal{A}f$ has no nonzero central ideal,*

(ii-3) *$\mathcal{Z}(\mathcal{A}) = \{\ a + b \mid a \in e\mathcal{Z}(\mathcal{A})e,\ b \in f\mathcal{Z}(\mathcal{A})f,\ am_0 = m_0b\ \}$ for some $m_0 \in e\mathcal{A}f$,*

(ii-4) *$\mathcal{Z}(\mathcal{A}) = \{\ a + b \mid a \in e\mathcal{Z}(\mathcal{A})e,\ b \in f\mathcal{Z}(\mathcal{A})f,\ t_0a = bt_0\ \}$ for some $t_0 \in f\mathcal{A}e$,*

(ii-5) *$\mathcal{A}$ satisfies (2.25).*

Then every Lie n-derivation on $\mathcal{A}$ is standard.

Proof. If $exe \cdot e\mathcal{A}f = \{0\}$, i.e., $(exe)\mathcal{A}f = \{0\}$, then $exe = 0$. And if $e\mathcal{A}f \cdot fxf = \{0\}$, i.e., $e\mathcal{A}(fxf) = \{0\}$, then $fxf = 0$. Thus, $\mathcal{A}$ satisfies (1.2). According to [4, Corollary 4.5], every Jordan derivation of $\mathcal{A}$ is a derivation. If $e\mathcal{A}e$ is commutative, then $e\mathcal{A}e$ obviously satisfies (2.27). Or if $e\mathcal{A}e$ is noncommutative, then $e\mathcal{A}e$ satisfies (2.27) by [14, Theorem 2], and it's not difficult to prove that $e\mathcal{A}e$ has no nonzero central ideal. According to Theorem 2.10, the proof is finished. □

Corollary 3.12. *Let $\mathcal{A} = B(X)$ be an algebra of all bounded linear operators, where X is a Banach space over the complex field $\mathbb{C}$ and $\dim X \geq 2$. Then every Lie n-derivation on $B(X)$ is standard.*

Proof. Obviously, $B(X)$ is a unital prime algebra with a nontrivial idempotent e. Since the center $\mathcal{Z}(B(X)) = \mathbb{C}I$, we have that $\mathcal{Z}(eB(X)e) = e\mathcal{Z}(B(X))e$ and $\mathcal{Z}(fB(X)f) = f\mathcal{Z}(B(X))f$. If $eB(X)e$ is commutative and $eB(X)f = \{0\}$, then $B(X)$ satisfies (2.25). Or if $eB(X)e$ is commutative and $eB(X)f \neq \{0\}$, then we can choose an arbitrary nonzero element m_0 in $eB(X)f$. For arbitrary elements $\lambda \cdot eIe$ in $\mathcal{Z}(eB(X)e)$ and $\mu \cdot fIf$ in $\mathcal{Z}(fB(X)f)$ satisfying $(\lambda \cdot eIe)m_0 = m_0(\mu \cdot fIf)$, since $(\lambda \cdot eIe)m_0 = \lambda m_0$ and $m_0(\mu \cdot fIf) = \mu m_0$, we have that $\lambda = \mu$ and $\lambda \cdot eIe + \mu \cdot fIf = \lambda I \in \mathcal{Z}(B(X))$, which follows that (ii-3) in Corollary 3.11 holds. According to Corollary 3.11, the proof is finished. □

Corollary 3.13. *Let $\mathcal{A}$ be a factor von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with $\dim(\mathcal{A}) \geq 2$. Then every Lie n-derivation on $\mathcal{A}$ is standard.*

Proof. Obviously, $\mathcal{A}$ is a unital prime algebra with nontrivial idempotents p_1, p_2. Let $\mathcal{A}_{ij} = p_i\mathcal{A}p_j$ where $1 \leq i, j \leq 2$. Since the center $\mathcal{Z}(\mathcal{A}) = \mathbb{C}I$, we have that $\mathcal{Z}(\mathcal{A}_{11}) = p_1\mathcal{Z}(\mathcal{A})p_1$ and $\mathcal{Z}(\mathcal{A}_{22}) = p_2\mathcal{Z}(\mathcal{A})p_2$. If $\mathcal{A}_{11}$ is commutative and $\mathcal{A}_{12} = \{0\}$, then $\mathcal{A}$ satisfies (2.25). Or if $\mathcal{A}_{11}$ is commutative and $\mathcal{A}_{12} \neq \{0\}$, then we can choose an arbitrary nonzero element m_0 in $\mathcal{A}_{12}$. For arbitrary elements $\lambda \cdot p_1Ip_1$ in $\mathcal{Z}(\mathcal{A}_{11})$ and $\mu \cdot p_2Ip_2$ in $\mathcal{Z}(\mathcal{A}_{22})$ satisfying $(\lambda \cdot p_1Ip_1)m_0 = m_0(\mu \cdot p_2Ip_2)$, since $(\lambda \cdot p_1Ip_1)m_0 = \lambda m_0$ and $m_0(\mu \cdot p_2Ip_2) = \mu m_0$, we have that $\lambda = \mu$ and $\lambda \cdot p_1Ip_1 + \mu \cdot p_2Ip_2 = \lambda I \in \mathcal{Z}(\mathcal{A})$, which follows that (ii-3) in Corollary 3.11 holds. According to Corollary 3.11, the proof is finished. □

Corollary 3.14. *Let $\mathcal{A}$ be a von Neumann algebra with no central summand of type I_1. Then every Lie n-derivation on $\mathcal{A}$ is standard.*

Proof. According to [10, Lemmas 4] and [11, Lemma 1], $\mathcal{A}$ is a unital algebra with a nontrivial idempotent p satisfying (1.2) and (iii) in Corollary 2.13. Denote that $q = I - p$. Let $\mathcal{A}_{11} = p\mathcal{A}p$, $\mathcal{A}_{12} = p\mathcal{A}q$, $\mathcal{A}_{21} = q\mathcal{A}p$ and $\mathcal{A}_{22} = q\mathcal{A}q$. By [10, Lemma 5], we have that $\mathcal{Z}(\mathcal{A}_{11}) = p\mathcal{Z}(\mathcal{A})p$ and $\mathcal{Z}(\mathcal{A}_{22}) = q\mathcal{Z}(\mathcal{A})q$. Let d_0 be an arbitrary central inner derivation of $\mathcal{A}_{11}$, i.e., there exists an element a'_{11} in $\mathcal{A}_{11}$ such that $d_0(a_{11}) = [a'_{11}, a_{11}] \in \mathcal{Z}(\mathcal{A}_{11})$ for each a_{11} in $\mathcal{A}_{11}$. By the Kleinecke-Shirokov theorem [8, Lemma 2.2], we have $d_0(a_{11})^2 = 0$ for each a_{11} in $\mathcal{A}_{11}$. It follows form [11, Lemma 1] that $d_0 = 0$. Thus, (i-4) in Corollary 2.13 holds. Let $\mathcal{I}$ be the central ideal of $\mathcal{A}_{11}$. For each a_{11} in $\mathcal{I}$, since $a_{11}\mathcal{A}_{11} \subseteq \mathcal{I} \subseteq \mathcal{Z}(\mathcal{A}_{11})$ and [5, Lemma 5], we have $a_{11} = 0$. Thus, (ii-1) in Corollary 2.13 holds. According to Corollary 2.13, the proof is finished. □

Remark 3.15. *In this section, we give several applications of the results in Section 2. Some results in this section can be seen in other papers. Corollaries 3.1 and 3.2 are partially proved by [16, Theorem 1]. Corollaries 3.4 and 3.5 are partially proved by [17, Theorem 2.1], [2, Corollaries 5.5 and 5.6] and [16, Corollaries 2 and 3]. Corollary 3.6 is partially proved by [3, Theorem 5.9]. Corollary 3.10 can be seen in [3, Corollary 6.4]. Corollary 3.12 improves [9, Theorem 1.1]. Corollary 3.14 can be seen in [8, Theorem 2.3].*

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