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Existence of Solutions for a New Version of Generalized Operator Equilibrium Problems

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Abstract. In this paper, a system of generalized operator equilibrium problems(for short, SGOEP) in the setting of topological vector spaces is introduced. Applying some properties of the nonlinear scalarization mapping and the maximal element lemma an existence theorem for SGOEP is proved. Moreover, using Ky Fan's lemma an existence result for the generalized operator equilibrium problem(for short, GOEP) is established. The results of the paper can be viewed as a generalization and improvement of the corresponding results given in [1, 2, 5, 8].

1. Introduction and Preliminaries

Throughout the paper, unless otherwise specified, we use the following notations.
Let I be an index set, for each $i \in I$, let X_i and Y_i stand for topological vector spaces(for short, t.v.s.) and $L(X_i, Y_i)$, the space of all continuous linear operators from X_i into Y_i. Consider a family of nonempty convex subset $\{K_i\}_{i\in I}$ with K_i in $L(X_i, Y_i)$. The symbol $\Pi_{j\in I}K_j$ denotes the cartesian product of K_j. So for each $f \in \Pi_{j\in I}K_j$, we have $f = (f_j)_{j\in I}$, where $f_j \in K_j$.
For each $i \in I$, let $C_i : \Pi_{j\in I}K_j \to 2^{Y_i}$ be a set-valued mapping such that, for each $f \in \Pi_{j\in I}K_j$, $C_i(f)$ is closed, pointed convex cone such that $e_i \in intC_i(f)$ (we recall that a subset $C_i(f)$ of Y_i is convex cone and pointed whenever $\lambda C_i(f) + (1 - \lambda)C_i(f) \subseteq C_i(f)$, for all $0 < \lambda < 1$, $2C_i(f) \subseteq C_i(f)$ and $C_i(f) \cap -C_i(f) = \{0_{Y_i}\}$, resp.), for more details see [7].
Also for each $i \in I$, let $F_i : \Pi_{j\in I}K_j \times K_i \to 2^{Y_i}$ be a set-valued mapping. We consider the following problem which we call system of generalized operator equilibrium problem(for short, SGOEP):
Find $f^* = (f_j^*)_{j\in I} \in \Pi_{j\in I}K_j$ such that for each $i \in I$,

$$F_i(f^*, g_i) \nsubseteq -C_i(f^*), \ \forall g_i \in K_i. \tag{1}$$

We remark that, for suitable choices of I, F_i, K_i, X_i, Y_i and C_i, SGOEP (1) reduces to the preoblems presented in [1, 8] and the references therein.
When I is singelton, that is $F_i = F$, $X_i = X$, $Y_i = Y$, $K_i = K \subseteq L(X, Y)$, $C_i = C : K \to 2^Y$, then (1) reduces to the following problem which is called a generalized operator equilibrium problem(for short, GOEP) and

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studied in [8]:
Find $f^* \in K$ such that

$$F(f^*, g) \nsubseteq -C(f^*), \quad \forall g \in K. \tag{2}$$

Now, we recall some concepts and results which are used in the sequel.

Definition 1.1. *[2] Let X and Y be two topological spaces. A set valued mapping $G : X \longrightarrow 2^Y$ is called*

(i) upper semicontinuous(u.s.c.) at $x \in X$ if for each open set V containing $G(x)$, there is an open set U containing x such that for each $t \in U, G(t) \subseteq V$; G is said to be u.s.c. on X if it is u.s.c. at all $x \in X$;

(ii) lower semicontinuous(l.s.c.) at $x \in X$ if for each open set V with $G(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U, G(t) \cap V \neq \emptyset$; G is said to be l.s.c. on X if it is l.s.c. at all $x \in X$;

(iii) closed if the graph of G, that is, the set $\{(x, y) : x \in X, y \in G(x)\}$, is a closed set in $X \times Y$;

(iv) compact if the closure of range G, that is, $clG(X)$, is compact, where $G(X) = \bigcup_{x \in X} G(x)$.

Remark 1.2. *One can see that if $G(x)$ is compact and G is u.s.c., then for any net $\{x_\alpha\} \subseteq X$ such that $x_\alpha \longrightarrow x$ and for every $y_\alpha \in G(x_\alpha)$ there exist $y \in G(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \longrightarrow y$.*

The nonlinear scalarization mapping that has a crucial role in the paper, was first introduced in [6] in order to apply to study the vector optimization theory and vector equilibrium problems.

Definition 1.3. *[6, 10] Let X be a topological vector space with the convex and pointed cone C. The formula*

$$\xi_e(x) := inf\{r \in \mathbb{R} : re - x \in C\},$$

where $x \in X$ and $e \in intC$, defines a mapping from X into $\mathbb{R}$(The real line) and is called the nonlinear scalarization mapping on X(with respect to C and e).

The following lemma characterizes some of the important properties of the nonlinear scalarization mapping which are used in the sequel.

Lemma 1.4. *[3, 9] Let X be a t.v.s. and C be a closed, pointed convex cone of X with $e \in intC$. Then for each $r \in \mathbb{R}$ and $x \in X$ the following statements are satisfied:*

(i) $\xi_e(x) = min\{r \in \mathbb{R} : re - x \in C\}$.

(ii) $\xi_e(x) \leq r \Longleftrightarrow re - x \in C$.

(iii) $\xi_e(x) < r \Longleftrightarrow re - x \in intC$.

(iv) $\xi_e(x) = r \Longleftrightarrow x \in re - \partial C$, where ∂C is the topological boundary of C.

(v) $y_2 - y_1 \in C \Longrightarrow \xi_e(y_1) \leq \xi_e(y_2)$.

(vi) The mapping ξ_e is continuous, positively homogeneous and subadditive(that is sublinear) on X.

For proving an existence result of an eqeuilibrium problem, Ky Fan's lemma plays a key role. We are going now to state it. Before stating it we need the following definition.

Definition 1.5. *[4] Let K be a nonempty subset of topological vector space X. A set-valued mapping $T : K \to 2^X$ is called a KKM-mapping if, for every finite subset $\{x_1, x_2, ..., x_n\}$ of K, $conv\{x_1, x_2, ..., x_n\}$ is contained in $\bigcup_{i=1}^{n} T(x_i)$, where conv denotes the convex hull.*

Ky Fan in 1984 obtained the following result, which is known as Ky Fan's lemma.

Lemma 1.6. *(Ky Fan-1984) [4] Let K be a nonempty subset of topological vector space X and $T : K \to 2^X$ be a KKM-mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x\in B} T(x)$ is compact. Then*

$$\bigcap_{x\in K} T(x) \neq \emptyset.$$

Definition 1.7. *A set-valued mapping $F : K \longrightarrow 2^Y$ is called $C(.)$-natural quasi convex if for any $f, g \in K$ and $\lambda \in [0, 1]$, there exist $h \in K \subseteq L(X, Y)$ and $\mu \in [0, 1]$ such that*

$$F(\lambda f + (1-\lambda)g) \subseteq \mu F(f) + (1-\mu)F(g) - C(h).$$

In the spacial case if we take $Y = \mathbb{R}$ and $C(h) = \mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$, then the definition of $C(.)$-natural quasi convexity converse to $\mathbb{R}_+$-natural quasi convexity.

Definition 1.8. *Let $C : K \subseteq L(X, Y) \longrightarrow 2^Y$ be a set-valued mapping and $C(f)$ be a convex cone, for each $f \in K$. Then the set-valued mapping $F : K \times K \longrightarrow 2^Y$ is said to be*

(i) $C(.)$-pseudomonotone, if for any f and $g \in K$,
$F(f, g) \nsubseteq -C(f) \Longrightarrow F(g, f) \subseteq -C(g)$.

(ii) strongly $C(.)$-pseudomonotone, if for any f and $g \in K$,
$F(f, g) \nsubseteq -intC(f) \Longrightarrow F(g, f) \subseteq -C(g)$.

Note that every strongly $C(.)$-pseudomonotone map is $C(.)$-pseudomonotone map.

2. Main results

The following maximal element theorem which proved by Ky Fan's lemma will be used in establishing some existence results in this paper.

Theorem 2.1. *For each $i \in I$, let K_i be a nonempty convex subset of $L(X_i, Y_i)$ and let $\Gamma_i : \Pi_{j\in I}K_j \to 2^{K_i}$ be a set-valued mapping satisfying the following conditions:*

(i) $\forall i \in I$ and $\forall f = (f_j)_{j\in I} \in \Pi_{j\in I}K_j$; $f_i \notin conv\Gamma_i(f)$, where f_i is the ith projection of f;

(ii) $\forall i \in I$ and $\forall g_i \in K_i$; $\Gamma_i^{-1}(g_i)$ is open in $\Pi_{j\in I}K_j$;

(iii) There exist a nonempty compact subset D of $\Pi_{j\in I}K_j$ and a nonempty compact convex subset $E_j \subseteq K_j$, $\forall j \in I$ such that $\forall f \in \Pi_{j\in I}K_j \setminus D$ there exists $j \in I$ such that $\Gamma_j(f) \cap E_j \neq \emptyset$.

Then there exists $f^* \in \Pi_{j\in I}K_j$ such that $\Gamma_j(f^*) = \emptyset$, for each $j \in I$.

Proof. Let a mapping $\Gamma : \Pi_{j\in I}K_j \longrightarrow 2^{\Pi_{j\in I}K_j}$ be defined by

$$\Gamma(f) = \Pi_{j\in I}K_j \setminus \bigcup_{j\in I} \Gamma_j^{-1}(f_j), \quad \forall f = (f_j)_{j\in I} \in \Pi_{j\in I}K_j.$$

Applying (ii), $\Gamma(f)$ is closed in $\Pi_{j\in I}K_j$, for each $f \in \Pi_{j\in I}K_j$. We claim that Γ is a KKM-mapping.
To verify this, let $B = \{(z_j^1)_{j\in I}, (z_j^2)_{j\in I}, ...(z_j^n)_{j\in I}\} \subseteq \Pi_{j\in I}K_j$ and $z^* = (z_j^*)_{j\in I} \in convB$.
If, on the contrary, we asuume that $z^* \notin \bigcup_{m=1}^n \Gamma((z_j^m)_{j\in I})$, then for each $m = 1, 2, ..., n$, there exists $j_m \in I$ such that

$$z^* \in \Gamma_{j_m}^{-1}(z_{j_m}^m),$$

where $z^m_{j_m}$ *is the* j_m*th projection of* $(z^m_j)_{j\in I}$. *Thus*

$$z^m_{j_m} \in \Gamma_{j_m}(z^*) \subseteq conv\Gamma_{j_m}(z^*),\ m = 1, 2, ..., n.$$

Applying $z^* = (z^*_j)_{j\in I} \in convB$, *it follows that*

$$z^*_{j_m} \in conv\{z^1_{j_m}, z^2_{j_m}, ..., z^n_{j_m}\} \subseteq conv\Gamma_{j_m}(z^*),$$

which is a contradiction to (*i*) *and this completes the proof of the assertion. Moreover, it follows from condition* (*iii*) *that*

$$\bigcap_{(f_j)_{j\in I}\in\Pi_{j\in I}E_j} \Gamma((f_j)_{j\in I}) \subseteq D.$$

Indeed, if $g \in \bigcap_{(f_j)_{j\in I}\in\Pi_{j\in I}E_j} \Gamma((f_j)_{j\in I})$, *then it follows that*

$$g \notin \bigcup_{j\in I} \Gamma^{-1}_j(f_j), \forall f_j \in E_j.$$

This immediately implies that $\Gamma_j(g) \bigcap E_j = \emptyset, \forall j \in I$, *and so* $g \in D$.
Since $\bigcap_{f\in\Pi_{j\in I}E_j} \Gamma(f)$ *is a closed subset of the compact set* D *(note that the values of* Γ *are closed), we get that* $\bigcap_{f\in\Pi_{j\in I}E_j} \Gamma(f)$ *is a compact subset of* D, *and so* Γ *satisfies all the assumptions of Lemma 1.6. Hence* $\bigcap_{f\in\Pi_{j\in I}K_j} \Gamma(f) \neq \emptyset$. *Thus there exists* $f^* = (f^*_j)_{j\in I} \in \Pi_{j\in I}K_j$ *such that*

$$f^* \in \Gamma(f), \forall f = (f_j)_{j\in I} \in \Pi_{j\in I}K_j.$$

This implies that

$$f^* \notin \bigcup_{j\in I} \Gamma^{-1}_j(f_j), \forall j \in I, \forall f_j \in K_j.$$

Thus

$$f_j \notin \Gamma_j(f^*), \forall j \in I, \forall f_j \in K_j.$$

Therefore $\Gamma_j(f^*) = \emptyset$, *for each* $j \in I$. *This completes the proof.* □

Following the same arguments as in the proof of Theorem 2.1, we can get the following result.

Theorem 2.2. *Let all assumptions of Theorem 2.1 and the following conditions hold:*

(i) $\forall i \in I$ *and* $\forall f \in \Pi_{j\in I}K_j$; $\Gamma_i(f)$ *is convex;*

(ii) $\forall i \in I$ *and* $\forall g_i \in K_i$, $\Gamma^{-1}_i(g_i)$ *is open in* $\Pi_{j\in I}K_j$;

(iii) There exist a nonempty compact subset D *of* $\Pi_{j\in I}K_j$ *and a nonempty compact convex subset* $E_i \subseteq K_i$, $\forall i \in I$, *such that* $\forall f \in \Pi_{j\in I}K_j \setminus D$ *there exists* $i \in I$ *with* $\Gamma_i(f) \cap E_i \neq \emptyset$.

Then

(a) if $\exists i \in I$ *such that* $\Gamma_i(f) \neq \emptyset, \forall f \in \Pi_{j\in I}K_j$, *then there exist* $i \in I$ *and* $f \in \Pi_{j\in I}K_j$ *such that* $f_i \in \Gamma_i(f)$.

(b) if $\forall i \in I$ *and* $\forall f \in \Pi_{j\in I}K_j, f_i \notin \Gamma_i(f)$, *then there exists* $f^* \in \Pi_{j\in I}K_j$ *such that* $\Gamma_i(f^*) = \emptyset$, *for each* $i \in I$.

Now applying the properties of nonlinear scalarization mapping and Lemma 2.1, we prove the following existence theorem for SGOEP.

Theorem 2.3. *For each* $i \in I$, *let* K_i *be nonempty and convex subset of* $L(X_i, Y_i)$ *and* $F_i : \Pi_{j\in I}K_j \times K_i \longrightarrow 2^{Y_i}$ *be a mapping satisfying the following conditions:*

(i) $\forall i \in I$ *and* $\forall (f_j)_{j\in I} \in \Pi_{j\in I}K_j$, $F_i((f_j)_{j\in I}, f_i) \nsubseteq -C((f_j)_{j\in I})$, *where* f_i *is the ith component of* $(f_j)_{j\in I}$;

(ii) $\forall i \in I$ *and* $\forall (f_j)_{j\in I} \in \Pi_{j\in I}K_j$, *the mapping* $g_i \longrightarrow \xi_{e_i} o F_i((f_j)_{j\in I}, g_i)$ *is* $\mathbb{R}_+$*-natural quasi convex;*

(iii) $\forall i \in I$ *and* $\forall g_i \in K_i$, *the set*

$$\{(f_j)_{j\in I} \in \Pi_{j\in I}K_j : F_i((f_j)_{j\in I}, g_i) \nsubseteq -C((f_j)_{j\in I})\},$$

is closed in $\Pi_{j\in I}K_j$;

(iv) *There exist a nonempty compact subset* D *of* $\Pi_{j\in I}K_j$ *and a nonempty compact convex subset* $E_i \subseteq K_i$, $\forall i \in I$ *such that* $\forall (f_j)_{j\in I} \in \Pi_{j\in I}K_j \setminus D$; *there exist* $i \in I$ *and* $g_i^* \in E_i$ *with*

$$F_i((f_j)_{j\in I}, g_i^*) \subset -C_i((f_j)_{j\in I}).$$

Then the solution set of SGOEP is nonempty and relatively compact.

Proof. For each $i \in I$, define a set-valued mapping

$$\Gamma_i : \Pi_{j\in I}K_j \longrightarrow 2^{K_i},$$

as follows

$$\Gamma_i((f_j)_{j\in I}) = \{g_i \in K_i : \xi_{e_i}(F_i((f_j)_{j\in I}, g_i)) \cap (0, \infty) = \emptyset\},$$

for all $(f_j)_{j\in I} \in \Pi_{j\in I}K_j$.
We claim that, $\Gamma_i((f_j)_{j\in I})$ is convex, $\forall i \in I$ and $(f_j)_{j\in I} \in \Pi_{j\in I}K_j$. Let $i \in I$ and $(f_j)_{j\in I} \in \Pi_{j\in I}K_j$ be arbitrary and fixed. Let $g_i^1, g_i^2 \in \Gamma_i((f_j)_{j\in I})$ and $\lambda \in [0, 1]$. Then

$$\xi_{e_i}(F_i(((f_j)_{j\in I}), g_i^m)) \cap (0, \infty) = \emptyset, \quad m = 1, 2. \tag{3}$$

Since $\xi_{e_i} o F_i((f_j)_{j\in I}, .)$ is $\mathbb{R}_+$-natural quasi convex mapping, there exists $\mu \in [0, 1]$ such that

$$\begin{aligned} \xi_{e_i} o F_i((f_j)_{j\in I}, \lambda g_i^1 + (1-\lambda) g_i^2) \subseteq\ & \mu \xi_{e_i} o F_i((f_j)_{j\in I}, g_i^1) \\ & + (1-\mu)\xi_{e_i} o F_i((f_j)_{j\in I}, g_i^2) - \mathbb{R}_+. \end{aligned}$$

Now, inclusion (3) and Lemma 1.4, imply that

$$\xi_{e_i} o F_i((f_j)_{j\in I}, \lambda g_i^1 + (1-\lambda) g_i^2) \cap (0, \infty) = \emptyset.$$

Hence

$$\lambda g_i^1 + (1-\lambda) g_i^2 \in \Gamma_i((f_j)_{j\in I}).$$

Therefore $\Gamma_i((f_j)_{j\in I})$ is convex.
Applying (iii), we have condition (ii) of Theorem 2.2. Indeed, $\forall i \in I$ and $\forall g_i \in K_i$, the complement of $\Gamma_i^{-1}(g_i)$ in K can be defined as

$$\begin{aligned} (\Gamma_i^{-1}(g_i))^c &= \{(f_j)_{j\in I} \in \Pi_{j\in I}K_j : g_i \notin \Gamma_i((f_j)_{j\in I})\} \\ &= \{(f_j)_{j\in I} \in \Pi_{j\in I}K_j : \xi_{e_i} o F_i((f_j)_{j\in I}, g_i) \cap (0, +\infty) \neq \emptyset\} \\ &= \{(f_j)_{j\in I} \in \Pi_{j\in I}K_j : F_i((f_j)_{j\in I}, g_i) \nsubseteq -C_i((f_j)_{j\in I})\}. \end{aligned}$$

Applying condition (iii), $(\Gamma_i^{-1}(g_i))^c$ is closed in $\Pi_{j\in I}K_j$. That is, $\Gamma_i^{-1}(g_i)$ is open in $\Pi_{j\in I}K_j$.
Now, we show that $\forall i \in I$ and $(f_j)_{j\in I} \in \Pi_{j\in I}K_j$, $f_i \notin \Gamma_i((f_j)_{j\in I})$.
Condition (i) and Lemma 1.4, imply that

$$\xi_{e_i} o F_i((f_j)_{j\in I}, f_i) \nsubseteq (-\infty, 0], \ \forall i \in I, \ \forall (f_j)_{j\in I} \in \Pi_{j\in I}K_j.$$

Thus

$$f_i \notin \Gamma_i((f_j)_{j\in I}).$$

Applying (iv), *we have condition* (iv) *of Theorem 2.2. Hence, applying Theorem 2.2, there exists* $f^* \in \Pi_{j\in I}K_j$ *such that* $\Gamma_i(f^*) = \emptyset$, *for each* $i \in I$. *Then applying Lemma 1.4, we get*

$$F_i(f^*, g_i) \nsubseteq -C_i(f^*), \forall g_i \in K_i, \forall i \in I. \tag{4}$$

Thus f^* *is a solution of SGOEP.*
To prove the relatively compactness of the solution set, we claim that the solution set, that is

$$\{f \in \Pi_{j\in I}K_j : F_i(f, g_i) \nsubseteq -C_i(f), \forall i \in I, \forall g_i \in K_i\}$$

is a subset of D. *Otherwise, there exists* $f^* \in K \setminus D$ *such that*

$$F_i(f^*, g_i) \nsubseteq -C_i(f^*),\ \forall i \in I,\ \forall g_i \in K_i.$$

Applying (iv), *there exists* $i \in I$ *and* $g_i^* \in E_i$, *such that*

$$F_i(f^*, g_i^*) \subseteq -C_i(f^*),$$

which is a contradiction. This completes the proof. □

The next result is a special case of Theorem 2.3 when I is singelton.

Theorem 2.4. *Let* X *and* Y *be two t.v.s. and* K *be a nonempty convex subset of* $L(X, Y)$, C *be a closed, pointed convex cone in* Y *with* $e \in intC$ *and also* $F : K \times K \longrightarrow 2^Y$ *be a set-valued mapping with nonempty values. Assume that the following conditions hold:*

(i) for all $f \in K$, $F(f, f) \nsubseteq -C(f)$;

(ii) for all fixed $f \in K$, *the mapping* $g \longrightarrow \xi_e o F(f, g)$ *is* $\mathbb{R}_+-$*natural quasi convex;*

(iii) for all $g \in K$, *the set*

$$\{f \in K : F(f, g) \subseteq -C(f)\},$$

is open in K;

(iv) there exist a nonempty compact convex subset D *of* K *and a nonempty compact subset* E *of* K *such that for each* $f \in K \setminus D$, *there exists* $g \in E$ *satisfying* $F(f, g) \subseteq -C(f)$.

Then the solution set of GOEP is nonempty and relatively compact.

Remark 2.5. *If* $F(f, .)$ *is* $C(.)-$*natural quasi convex, then the mapping* $g \longrightarrow \xi_e o F(f, g)$ *is* $\mathbb{R}_+-$*natural quasi convex. Therefore Theorem 2.4 is valid when one replaces* $\xi_e o F(f, .)$ *by* $F(f, .)$.

The next example shows that although Theorem 2.4 is true when $F(f, .)$ is $C(.)-$natural quasi convex but condition (ii) is sharper than it.

Example 2.6. *Assume that*

$$f(x) = \begin{cases} |x| & x \in Q \cap [-1, 1] \\ 2|x| + 1 & x \in Q^c \cap [-1, 1]. \end{cases}$$

Define the mapping $F : [-1, 1] \longrightarrow 2^{\mathbb{R}^2}$ *by*

$$F(x) = [f(x), f(x) + 1] \times [3, 4],$$

$\xi_e o F(x) = [3, 4]$, *where* $C = \{(x, y) : x, y \geq 0\}$ *and* $e = (1, 1) \in intC$.

Remark 2.7. *If the set-valued mapping $f \longrightarrow F(f,g)$ is u.s.c., then the set*

$$S = \{f \in K : F(f,g) \subseteq -intC\},$$

is open. Indeed, let $f_0 \in S$. Since F is u.s.c. mapping in its first variable, there exists an open set V of f_0 such that for each $f \in V$, we have $F(f,g) \subseteq -intC$. This means that f_0 is an interior point of S. Hence S is open. Therefore, if one replace $-C(f)$ by $-intC$ in the condition (iii) in Theorem 2.4, one can underestand that the solution set of the following problem is nonempty:
Find $f^* \in \Pi_{j\in I} K_j$ such that for each $i \in I$,

$$F_i(f^*, g_i) \nsubseteq -intC, \forall g_i \in K_i. \tag{5}$$

Remark 2.8. *It seems, reviewing the proof of Lemma 3.1 and Lemma 3.3 given in [8], which are important in the proof of Theorem 3.4 of [8], some parts of them are not clear. Similarly, we can say the above statements about Lemma 3.1 and Theorem 3.2 of [1].*
Next we prove the following lemma which is a new type of Lemma 3.3 given in [8] and Theorem 3.2 in [1] which is necessary to prove an existence result.

Lemma 2.9. *Let X and Y be two topological vector spaces and K be a nonempty convex subset of $L(X,Y)$. Suppose that the set-valued mapping $F : K \times K \longrightarrow 2^Y$ satisfies the following conditions:*

(i) *F is $C(.)-$pseudomonotone;*

(ii) $F(f,f) \nsubseteq Y \setminus C(f), \quad \forall f \in K;$

(iii) *if $F((1-t)g + tf, f) \nsubseteq Y \setminus C((1-t)g + tf), \forall t \in [0,1]$, then $F(f,g) \nsubseteq -C(f)$;*

(iv) *for each $f \in K$, the mapping $g \longrightarrow F(f,g)$ is $C(.)$-convex, i.e.,*

$$F(f,(1-t)g + th) \subseteq (1-t)F(f,g) + tF(f,h) - C(f), \forall g,h \in K,\ \forall t \in [0,1].$$

Then the following are equivalent:

(a) $\exists f \in K$ *such that* $F(f,g) \nsubseteq -C(f),\ \forall g \in K;$

(b) $\exists f \in K$ *such that* $F(g,f) \subseteq -C(g),\ \forall g \in K.$

Proof. $(a) \Rightarrow (b)$: *It is obvious from the definition of $C(.)-$pseudomonotonicity.*
$(b) \Rightarrow (a)$: *Assume that g is an arbitrary element of K. For each $t \in (0,1)$, define $h_t = g + t(f-g)$. Applying (b), we have*

$$F(h_t, g) \subseteq -C(h_t), \forall t \in (0,1).$$

We assert that

$$F(h_t, f) \nsubseteq Y \setminus C(h_t), \quad \forall t \in (0,1).$$

Otherwise, there exists $t \in (0,1)$ such that $F(h_t, f) \subseteq Y \setminus C(h_t)$.
It follows from (iv) that

$$\begin{aligned} F(h_t, h_t) &\subseteq (1-t)F(h_t, g) + tF(h_t, f) - C(h_t) \\ &\subseteq -C(h_t) + Y \setminus C(h_t) - C(h_t) \\ &\subseteq Y \setminus C(h_t) - C(h_t) \subseteq Y \setminus C(h_t), \end{aligned}$$

which is a contradiction to (ii). Therefore, for all $t \in (0,1)$, we have

$$F(h_t, f) \nsubseteq Y \setminus C(h_t).$$

Now, applying (iii), we get $F(f,g) \nsubseteq -C(f)$. This complets the proof. □

Proposition 2.10. *Under the hypothesis of the previous lemma the solution set of the generalized operator equilibrium problem(GOEP) is convex.*

Proof. Let f_1, f_2 be solutions of GOEP. Applying the previous lemma, we have

$$F(g, f_i) \subseteq -C(g), \ \forall g \in K, \ i = 1, 2.$$

Applying (*iv*) of Lemma 2.9, for all $t \in (0, 1)$, we have

$$F(g, (1-t)f_1 + tf_2) \subseteq (1-t)F(g, f_1) + tF(g, f_2) - C(g) \subset -C(g), \forall g \in K.$$

Applying Lemma 2.9, we get

$$F((1-t)f_1 + tf_2, g) \nsubseteq -C((1-t)f_1 + tf_2), \forall g \in K.$$

This means $(1-t)f_1 + tf_2$ is a solution of GOEP and the proof is complete. □

Remark 2.11. *(a) Condition (ii) of Lemma 2.9 implies the following assumption which there exists in page 1 in [8].*

$$F(g, g) \nsubseteq -C(g), \ \forall g \in K.$$

(b) If $0 \notin C(f)$, for each $f \in K$, then condition (iii) of the previous lemma is still valid where the set-valued mapping F has compact values and F is v-hemicontinuous at the first variable, that is the set valued mapping $\lambda \longrightarrow F(f + \lambda g, h)$ is u.s.c. at $\lambda = 0^+$, for all $f, g, h \in K$, (see, Lemma 3.3 of [8]) and the graph of $C : K \longrightarrow 2^Y$ is closed. Indeed, let

$$F((1-t)g + tf, f) \nsubseteq Y \setminus C((1-t)g + tf), \ \forall t \in (0, 1).$$

Hence

$$\exists h_t \in F((1-t)g + tf, f) \cap C((1-t)g + tf),$$

and applying u.s.c. of the mapping $t \longrightarrow F((1-t)g + tf, f)$ at 0^+, there exist a subnet $\{h_{t_i}\}$ of $\{h_t\}$ and $h \in F(g, f)$ such that $h_{t_i} \longrightarrow h$.
On the other hand, since the graph of $C : K \longrightarrow 2^Y$ is closed and $h_{t_i} \in C((1-t)g + tf)$, we get $h \in C(g)$. Consequently, $h \in F(g, f)$ and $h \notin -C(f)$. This completes the proof of the assertion. Therefore, Lemma 2.9 improves Lemma 3.3 in [8].

By a similar argument as given for Lemma 2.9 and using Remark 2.11, we can deduce the following result.

Lemma 2.12. *Let X and Y be two topological vector spaces and K be a nonempty convex subset of $L(X, Y)$. Let the set-valued mapping $F : K \times K \longrightarrow 2^Y$ satisfies the following conditions:*

(i) for each $g \in K$, the set-valued mapping $f \longrightarrow F(f, g)$ is upper semicontinuous with compact values;

(ii) F is strongly $C(.)$-pseudomonotone;

(iii) $F(f, f) \nsubseteq -intC(f), \forall f \in K$;

(iv) for each $f \in K$, the set-valued mapping $g \longrightarrow F(f, g)$ is $C(f)$-convex;

(v) The mapping $f \longrightarrow Y \setminus -intC(f)$, for each $f \in K$, has closed graph.

Then the following are equivalent:

(a) $\exists f \in K$ such that $F(f, g) \nsubseteq -intC(f), \forall g \in K$;

(b) $\exists f \in K$ such that $F(g, f) \subseteq -C(g), \forall g \in K$.

Proof. $(a) \Rightarrow (b)$: *It is clear from* (ii) *that* (a) *implies* (b).
$(b) \Rightarrow (a)$: *Assume that* (b) *holds and* $g \in K$ *is an arbitrary element of* K. *For each* $t \in [0,1]$, *put* $h_t = (1-t)f + tg$. *It follows from* (b) *that*

$$F(h_t, f) \subseteq -C(h_t), \forall t \in [0,1]. \tag{6}$$

Now, (iv) *implies that*

$$F(h_t, h_t) \subseteq (1-t)F(h_t, f) + tF(h_t, g) - C(h_t),$$

and so (6) implies that

$$F(h_t, h_t) \subseteq (1-t)(-C(h_t)) + tF(h_t, g) - C(h_t) \subseteq tF(h_t, g) - C(h_t).$$

Since $F(h_t, h_t) \nsubseteq -intC(h_t)$ *(condition (iii)) and* $-intC(h_t) - C(h_t) \subseteq -intC(h_t)$, *we have*

$$F(h_t, g) \nsubseteq -intC(h_t), \forall t \in [0,1].$$

Then we can choose $t \in (0,1]$ *such that* $t \longrightarrow 0^+$ *and*

$$w_t \in F(h_t, g) \cap Y \setminus -intC(h_t).$$

Applying (i), there exist subnet $\{w_{t_i}\}$ *of* $\{w_t\}$ *and* $w \in F(f,g)$ *and* $w \in F(f,g)$ *such that* $w_{t_i} \longrightarrow w$. *Since* $h_{t_i} \longrightarrow f$ *(as* $t \longrightarrow 0^+$ *), applying (v), we get* $(f, w) \in Y \setminus -intC(.)$. *This means that* $w \notin -int(C(f))$. *This completes the proof.* □

Note that if for each $f \in K$, $C(f)$ is open and convex cone, then the previous lemma is a new version of Lemma 3.3 given in [8].

The following example shows that the $C(.)-$pseudomonotoneity in Lemma 2.12 is essentional.

Example 2.13. *If we take* $X = Y = \mathbb{R}$, $K = [0,1]$, $C(f) = [0, +\infty]$, $L(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}$ *and we define* $F : K \times K \longrightarrow 2^Y$ *by* $F(f,g) = \{f + g\}$, $\forall f, g \in K$, *then it is easy to verify that all the hypothesis of Lemma 2.12 are satisfied except (ii). Also, one can check that each solution of the problem given by (a) in Lemma 2.12 is not a solution of part (b).*

Theorem 2.14. *Let all the assumptions of Lemma 2.9 hold and for each* $f \in K \subseteq L(X,Y)$, *the mapping* $g \longrightarrow F(f,g)$ *is lower semicontinuous. If there exist a nonempty compact subset* B *of* K *and a nonempty convex compact subset* D *of* K *such that for all* $f \in K \setminus B$ *there exists* $g \in D$ *such that* $F(g,f) \nsubseteq -C(g)$, *then the solution set GOEP is nonempty and compact.*

Proof. Define $S, T : K \subseteq L(X,Y) \longrightarrow 2^Y$ by

$$S(g) = \{f \in K : F(f,g) \nsubseteq -C(f) \setminus \{0\}\},$$
$$T(g) = \{f \in K : F(g,f) \subseteq -C(g)\}.$$

It is obvious from (i) of Lemma 2.9 that $S(g) \subseteq T(g)$, for each $g \in K$.
We claim that S is KKM-mapping. Otherewise, there exist $f_1, f_2, ..., f_n \in K$ and $t_i \geq 0, \sum_{i=1}^n t_i = 1$, such that $f = \sum_{i=1}^n t_i f_i$, and

$$F(f, f_i) \subseteq -C(f) \setminus \{0\}, \; i = 1, 2, ..., n.$$

Since F is convex in the second variable (see condition (iv) of Lemma 2.9), we get

$$\begin{aligned} F(f,f) = F(f, \sum_{i=1}^n t_i f_i) &\subseteq \sum_{i=1}^n t_i F(f, f_i) - C(f) \\ &\subseteq \sum_{i=1}^n t_i(-C(f) \setminus \{0\}) - C(f) \\ &\subseteq -C(f) \setminus \{0\}, \end{aligned}$$

which is a contradiction to (ii) of Lemma 2.9. Hence S is a KKM-mapping, and so T is a KKM. The values of T are closed, because of the lower semicotinuity of F. By the hypothesis of theorem, it is clear that $\bigcap_{f\in D} T(f)$ is a closed sbset of B, and so $\bigcap_{f\in D} T(f)$ is compact. Now, we can apply the Ky Fan's lemma. Hence that there exists $f^* \in \bigcap_{f\in K} T(f)$. It is obvious that the solution set of GOEP is equal to the set $\bigcap_{f\in K} T(f)$. Consequently, the solution set of GOEP is nonempty and a compact subset of B. This completes the proof. □

Remark 2.15. *Theorem 2.14 is a new version of Theorem 3.4 of [8] by relaxing the compactness of values of the mapping F and replacing upper semicontinuity of F by lower semicontinuity. Further, the coercivity condition given is Theorem 2.14 improves the coercivity condition presented by Definition 2.8 in [8]. Moreover, Theorem 2.14 provides conditions for which the solution set of GOEP is compact. Finally, it seems that the proofs of Theorems 3.4 and 3.5 of [8] based on Lemma 3.3 contain some gaps, for instance, see line 6 of page 6 and line 5 from below in the proof of Theorem 3.5, where the authors assumed that the set* $\{f_\alpha\} \cup \{f\}$ *is compact.*

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