Filomat 32:13 (2018), zzz–zzz
(will be added later)

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

EP Elements and the Solutions of Equation in Rings with Involution

Ruju Zhao[a], Hua Yao[b], Junchao Wei[c]

[a]*School of Mathematical Science, Yangzhou University, Yangzhou, 225002, P. R. China*
[b]*School of Mathematical Science, Yangzhou University, Yangzhou, 225002, P. R. China*
[c]*School of Mathematical Science, Yangzhou University, Yangzhou, 225002, P. R. China*

Abstract. The aim of this paper is to describe the elements in rings with involution which are EP. Especially, contact is established between EP elements and the solutions of certain equations. In addition, we reduce the preliminary requirements met in some existing results.

1. Introduction

Throughout this paper, R will denote a unital ring with involution, i.e., a ring R with a map $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$, for all $a, b \in R$. The notion of Moore-Penrose inverse (or MP-inverse) has been investigated by many authors (see, for example, [6, 8, 9]). We say that $b = a^\dagger$ is the Moore-Penrose inverse (or MP-inverse) of a, if the following conditions hold: $aba = a$, $bab = b$, $(ab)^* = ab$, and $(ba)^* = ba$. There is at most one b such that the above conditions hold. We write $R^\dagger$ for the set of all MP-inverses of R. a is said to be group invertible if there is $a^\# \in R$ such that $aa^\#a = a; a^\#aa^\# = a^\#; aa^\# = a^\#a$. $a^\#$ is called a group inverse of a and it is uniquely determined by these equations. Denote by $R^\#$ the set of all group invertible elements of R.

An element $a \in R$ is said to be an EP element if $a \in R^\dagger \cap R^\#$ and $a^\dagger = a^\#$ [3]. The set of all EP elements of R will be denoted by R^{EP}. Mosić et al. in [12, Theorem 2.1] gave several equivalent conditions under which an element in R is an EP element. Patrćio and Puystjens in [14, Proposition 2] proved that for an element $a \in R$, a is EP if and only if $a \in R^\dagger$ (or $a \in R^\#$) and $aR = a^*R$ if and only if $a \in R^\dagger$ and $aa^\dagger = a^\dagger a$. It is known by [10, Theorem 7.3] that $a \in R$ is EP if and only if a is group invertible and $aa^\#$ is symmetric. More results on EP elements can also be found in [2, 4, 5, 7, 11, 15].

This paper considers the characterizations of EP elements, from the perspective of the solutions of equations. Let $a \in R^\# \cap R^\dagger$ and $\chi_a = \{a, a^*, a^\dagger, a^\#, (a^\#)^*, (a^\dagger)^*\}$. It will be proved that the equation $axa^\dagger = xa^\dagger a$ has at least one solution in χ_a if and only if $a \in R^{EP}$. In Corollary 2.6, we reduce the condition $aa^\dagger = a^2(a^\dagger)^2$ in [1, Proposition 2.3]. In Theorem 2.10, we change [16, Theorem 2.2] by using the condition $x - ax^2 \in J(R) \cap comm(a)$ to replace the requirement $x = ax^2$, where $J(R)$ is the Jacobson radical of R. As we know, $aa^\dagger = a^2(a^\dagger)^2$ and $a^\dagger a = (a^\dagger)^2a^2$ are equivalent [1], and so do $aa^\dagger = (a^\dagger)^2a^2$ and $a^\dagger a = a^2(a^\dagger)^2$ which is proved in Proposition 2.11. Finally, with the help of the results mentioned above, we show that even if the restrictive conditions $(a^\dagger)^2a^\# = a^\#(a^\dagger)^2$ in [12, Conjecture 1] and $a^\dagger \in R^\#$ in [1, Corollary 2.5] are reduced, the relevant conclusions are still established.

2010 *Mathematics Subject Classification*. 15A09, 16U99, 16W10
Keywords. EP element, Group inverse, Moore-Penrose inverse, solution of equation.
Received: 29 12 2007; Accepted: 04 05 2018
Communicated by Dijana Mosić
Research supported by the National Natural Science Foundation of China (No.11471282).
Email addresses: zrj0115@126.com (Ruju Zhao), dalarston@126.com (Hua Yao), jcwei@yzu.edu.cn (Junchao Wei)

2. EP elements and the solutions of equation

In this section, we let R be a ring with involution $*$. For any $a \in R$, denote

$$comm(a) = \{x \in R | xa = ax\}, \; l(a) = \{z \in R | za = 0\}, \text{ and}$$
$$comm^2(a) = \{y \in R | xy = yx, \text{ for all } x \in comm(a)\}.$$

Lemma 2.1. *Let $a, b \in R^{\#}$ and $x \in R$. Then $ax = xb$ if and only if $a^{\#}x = xb^{\#}$.*

Proof. $\Rightarrow$ It follows from $ax = xb$ that

$$a^{\#}x = (a^{\#})^2ax = (a^{\#})^2xb = (a^{\#})^2xb^2b^{\#} = (a^{\#})^2axbb^{\#} = a^{\#}xbb^{\#}.$$

On the other hand, it is obvious that

$$xb^{\#} = xb(b^{\#})^2 = ax(b^{\#})^2 = a^{\#}a^2x(b^{\#})^2 = a^{\#}axb(b^{\#})^2 = a^{\#}axb^{\#} = a^{\#}xbb^{\#}.$$

Hence, $a^{\#}x = xb^{\#}$.

$\Leftarrow$ Since $a^{\#}x = xb^{\#}$, we get

$$ax = a^2a^{\#}x = a^2xb^{\#} = a^2x(b^{\#})^2b = a^2a^{\#}xb^{\#}b = axb^{\#}b.$$

What is left is to show that $xb = axb^{\#}b$. In fact,

$$xb = xb^{\#}b^2 = a^{\#}xb^2 = a(a^{\#})^2xb^2 = aa^{\#}xb^{\#}b^2 = aa^{\#}xb = axb^{\#}b.$$

Hence $ax = xb$. $\square$

Especially, choosing $a = b$ in Lemma 2.1, we have the following corollary.

Corollary 2.2. *[13, P733] If $a \in R^{\#}$, then $a^{\#} \in comm^2(a)$ and $a \in comm^2(a^{\#})$.*

Let R be a ring and write $ZE(R) = \{x \in R | xe = ex, \text{ for all } e \in E(R)\}$, where $E(R)$ is the set of all idempotents in R. Then $ZE(R)$ is a subring of R.

Theorem 2.3. *Let R be a ring and $a \in R^{\#}$. If $a \in ZE(R)$, then $a \in ZE(R)^{\#}$.*

Proof. Note that $a \in R^{\#}$. Then $a^{\#}$ exists and $a^{\#} \in R$. For each $e \in E(R)$, we have $e \in comm(a)$ because $a \in ZE(R)$. By Corollary 2.2, $ea^{\#} = a^{\#}e$. Hence $a^{\#} \in ZE(R)$, it follows that $a \in ZE(R)^{\#}$. $\square$

Lemma 2.4. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if one of the following conditions holds:*
(1) $a^{\dagger}R \subseteq aR$;
(2) $Ra \subseteq Ra^{\dagger}$.

Proof. It is well-known that $a^{\dagger}R = a^*R$ and $Ra^* = Ra^{\dagger}$. Consequently, Lemma 2.4 follows by [1, Theorem 3.6]. $\square$

Let $a \in R^{\#} \cap R^{\dagger}$. Write $\chi_a = \{a, a^*, a^{\dagger}, a^{\#}, (a^{\#})^*, (a^{\dagger})^*\}$. Now, we consider the relations between EP elements and the solutions of the equation $axa^{\dagger} = xa^{\dagger}a$ in χ_a.

Theorem 2.5. *Let $a \in R^{\#} \cap R^{\dagger}$. If the equation $axa^{\dagger} = xa^{\dagger}a$ has at least one solution in χ_a, then $a \in R^{EP}$.*

Proof. (1) When $x = a$, then $a^2a^\dagger = aa^\dagger a = a$. It follows from [11, Theorem 2.1(xviii)] that a is an EP element.

(2) When $x = a^*$, then $aa^*a^\dagger = a^*a^\dagger a = a^*(a^\dagger a)^* = (a^\dagger a^2)^*$. By applying involution on the above equation, we have $a^\dagger a^2 = (a^\dagger)^*aa^*$. According to the last equation, we have

$$a^\dagger R = a^\dagger aR = a^\dagger a^2a^\# R \subseteq a^\dagger a^2R = (a^\dagger)^*aa^*R \subseteq (a^\dagger)^*R = aa^\dagger(a^\dagger)^*R \subseteq aR.$$

It follows from Lemma 2.4 that a is an EP element.

(3) When $x = a^\#$, then $aa^\# a^\dagger = a^\# a^\dagger a$. That a is an EP element follows from [12, Theorem 2.1(xix)].

(4) When $x = (a^\#)^*$, then $a(a^\#)^*a^\dagger = (a^\#)^*a^\dagger a = (a^\dagger aa^\#)^*$. Applying involution on the equation, we get $a^\dagger aa^\# = (a^\dagger)^*a^\# a^*$. This means that

$$a^\dagger R = a^\dagger aR = a^\dagger aa^\# aR \subseteq a^\dagger aa^\# R = (a^\dagger)^*a^\# a^*R \subseteq (a^\dagger)^*R = aa^\dagger(a^\dagger)^*R \subseteq aR.$$

Hence a is an EP element by Lemma 2.4.

(5) When $x = (a^\dagger)^*$, then $a(a^\dagger)^*a^\dagger = (a^\dagger)^*a^\dagger a = (a^\dagger aa^\dagger)^* = (a^\dagger)^*$. It follows that $a^\dagger = (a^\dagger)^*a^\dagger a^*$. This gives $a^\dagger R \subseteq (a^\dagger)^*R \subseteq aR$. Therefore, a is an EP element by Lemma 2.4.

(6) When $x = a^\dagger$, then $aa^\dagger a^\dagger = a^\dagger a^\dagger a$. We deduce that

$$a^*a^\dagger = (aa^\dagger a)^*a^\dagger = a^*aa^\dagger a^\dagger = a^*a^\dagger a^\dagger a.$$

Applying involution on the last equation, $(a^\dagger)^*a = (a^\dagger a)^*(a^\dagger)^*a = a^\dagger a(a^\dagger)^*a$. Then

$$a^\dagger a^2 = (a^\dagger a)^*a = a^*(a^\dagger)^*a = a^*a^\dagger a(a^\dagger)^*a = a^*(a^\dagger a)^*(a^\dagger)^*a = (a^\dagger a^\dagger aa)^*a.$$

Post-multiplying by $a^\#$, we get $a^\dagger a = (a^\dagger a^\dagger aa)^*aa^\#$. On the other hand, $a^\dagger a = (a^\dagger a)^*$. We assert that $a^\dagger a = (aa^\#)^*a^\dagger a^\dagger aa = (aa^\#)^*a^2a^\dagger a^\dagger$. Therefore, $Ra = Ra^\dagger a = R(aa^\#)^*a^2a^\dagger a^\dagger \subseteq Ra^\dagger$. By Lemma 2.4, we obtain that a is an EP element.

□

Let $a \in R^\# \cap R^\dagger$. Then $a^\# a^\dagger a^\dagger = a^\dagger a^\dagger a^\#$ if and only if $aa^\dagger a^\dagger = a^\dagger a^\dagger a$, by Corollary 2.2. Thus we have the following corollary which illustrates that the condition $aa^\dagger = a^2(a^\dagger)^2$ in [1, Proposition 2.3] is superfluous.

Corollary 2.6. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if $a^\# a^\dagger a^\dagger = a^\dagger a^\dagger a^\#$.*

Since a is an EP element, we get $a^* \in R^{EP}$. Take $b = a^*$. If $axa^\dagger = xa^\dagger a$ for any $x \in R$, then $bb^\dagger y = b^\dagger yb$, where $y = x^*$. By Theorem 2.5, we have the following theorem.

Theorem 2.7. *Let $a \in R^\# \cap R^\dagger$. If the equation $aa^\dagger x = a^\dagger xa$ has at least one solution in χ_a, then $a \in R^{EP}$.*

Next, we give an example in which $a \in R^{EP}$ but $a^*, (a^\#)^*$ and $(a^\dagger)^*$ are not the solutions of equation $axa^\dagger = xa^\dagger a$.

Example 2.8. *Let ring $S = M_2(\mathbb{R})$ of which the involution $*$ is the transposition of a matrix in S. Write $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. We have*

$$A^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

This implies that $A^\dagger = A^\# = A^{-1}$. However,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

That is, $AA^*A^\dagger \neq A^*A^\dagger A$. Similarly, we have

$$A(A^\#)^*A^\dagger = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = (A^\#)^*A^\dagger A.$$

The last inequation now leads to $A(A^\dagger)^*A^\dagger \neq (A^\dagger)^*A^\dagger A$.

Let $\chi_a^{'} = \{b \in \chi_a | aba^\dagger = ba^\dagger a\}$. Denote by $|\chi_a^{'}|$ the number of all elements in $\chi_a^{'}$. By Example 2.8, we know that $a \in R^{EP}$ cannot imply $|\chi_a^{'}| = |\chi_a|$. In the following corollary, we show that $\{a, a^\dagger, a^\#\} \subseteq \chi_a^{'}$.

Corollary 2.9. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if $|\chi_a^{'}| \geq 3$.*

Proof. From Theorem 2.5, the sufficiency is obvious. Conversely, if a is an EP element, then $a^\dagger = a^\#$. It follows that $a^2a^\dagger = a^2a^\# = a = aa^\dagger a$, $aa^\dagger a^\dagger = aa^\# a^\# = a^\# a^\# a = a^\dagger a^\dagger a$, and $aa^\# a^\dagger = aa^\# a^\# = a^\# a^\# a = a^\# a^\dagger a$. Thus $\{a, a^\dagger, a^\#\} \subseteq \chi_a^{'}$. That is, $|\chi_a^{'}| \geq 3$. □

As usual, we let $J(R)$ be the Jacobson radical of R and $U(R)$ be the set of all invertible elements of R. In [16, Theorem 2.2], it is shown that $a \in R^{EP}$ if and only if the system of equations $a = xa^2, (xa)^* = xa$, and $x = ax^2$ has at least one solution in R. Motivated by the result, we have the following theorem.

Theorem 2.10. *$a \in R^{EP}$ if and only if there exists $x \in R$ such that*

$$a = xa^2,\ (xa)^* = xa, \text{ and } x - ax^2 \in J(R) \cap comm(a).$$

Proof. If a is an EP element, then $a^\dagger = a^\#$. It is easy to show that $a^\dagger$ satisfies the above three relations. Conversely, writing $t = x - ax^2$, we deduce that

$$a = xa^2 = (ax^2 + t)a^2 = axxa^2 + ta^2 = axa + ta^2.$$

That is, $a - ta^2 = axa$. It follows from $t \in J(R) \cap comm(a)$ that $a - a^2t = axa$. Thus, $a(1 - at) = axa$. Note that $1 - at \in U(R) \cap comm(a)$ for $at \in J(R)$. So $(1 - at)^{-1}a = a(1 - at)^{-1}$. The last equation but one, post-multiplied by $(1 - at)^{-1}$, we get

$$a = ax(1 - at)^{-1}a = a(ax^2 + t)(1 - at)^{-1}a = a^2x^2(1 - at)^{-1}a + at(1 - at)^{-1}a.$$

This gives $a(1 - t(1 - at)^{-1}a) = a^2x^2(1 - at)^{-1}a$. Since $t(1 - at)^{-1}a \in J(R)$, we get $1 - t(1 - at)^{-1}a \in U(R)$. It follows immediately that

$$a = a^2x^2(1 - at)^{-1}a(1 - t(1 - at)^{-1}a)^{-1} \in a^2R.$$

On the other hand, $a = xa^2 \in Ra^2$. Thus $a \in R^\#$. It is easy to verify that

$$aa^\# = xa^2a^\# = xa = (xa)^* = (xaa^\# a)^* = (xa^2a^\#)^* = (aa^\#)^*.$$

It yields that $a^\dagger = a^\#$. This completes the proof. □

Motivated by [12, Conjecture 1] and [1, Proposition 2.2, Corollary 2.5], we have the following results about EP elements.

Proposition 2.11. *Let $a \in R^\# \cap R^\dagger$. Then the following conditions are equivalent:*
(1) $a \in R^{EP}$;
(2) $aa^\dagger = (a^\dagger)^2a^2$;
(3) $a^\dagger a = a^2(a^\dagger)^2$.

Proof. If a is an EP element, then it implies that $aa^{\dagger} = (a^{\dagger})^2a^2$. Suppose the condition (2) hold. Then we have $aa^{\dagger} = (a^{\dagger})^2a^2$. Post-multiplying by $a^{\#}a^{\dagger}$, we claim that

$$a^{\#}a^{\dagger} = a(a^{\#})^2a^{\dagger} = aa^{\dagger}a(a^{\#})^2a^{\dagger} = aa^{\dagger}a^{\#}a^{\dagger} = (a^{\dagger})^2a^2a^{\#}a^{\dagger} = (a^{\dagger})^2aa^{\dagger} = (a^{\dagger})^2.$$

Indeed, pre-multiplying by a, we get $a^{\#}aa^{\dagger} = aa^{\#}a^{\dagger} = a(a^{\dagger})^2$. Finally, post-multiplying by a, we obtain

$$a^{\#}a = a^{\#}aa^{\dagger}a = a(a^{\dagger})^2a = (aa^{\dagger})a^{\dagger}a = (a^{\dagger})^2a^2a^{\dagger}a = (a^{\dagger})^2a^2 = aa^{\dagger}.$$

This means that a is an EP element. We thus get

$$a^2(a^{\dagger})^2 = a^2(a^{\#})^2 = aa^{\#} = a^{\#}a = a^{\dagger}a.$$

The implication (3) $\Rightarrow$ (1) is similar to (2) $\Rightarrow$ (3). $\square$

Theorem 2.12. *Let $a \in R^{\#} \cap R^{\dagger}$ and $m \geq 2$ be a positive integer. Then the following conditions are equivalent:*
(1) $a \in R^{EP}$;
(2) $aa^{\dagger} = a^k(a^{\dagger})^m a^{m-k}$, for $0 \leq k < m$;
(3) $aa^{\dagger} = (a^{\dagger})^k a^m (a^{\dagger})^{m-k}$, for $1 \leq k \leq m$.

Proof. If a is an EP element, then $a^{\dagger} = a^{\#}$. It is clear that the conditions (2) and (3) hold. Conversely, assume that $aa^{\dagger} = a^k(a^{\dagger})^m a^{m-k}$, for $0 \leq k < m$. Then $a = aa^{\dagger}a = a^k(a^{\dagger})^m a^{m-k}a = a^k(a^{\dagger})^m a^{m-k+1}$, for $0 \leq k < m$. Post-multiplying by $a^{\#}$, we get

$$aa^{\#} = a^k(a^{\dagger})^m a^{m-k+1}a^{\#} = a^k(a^{\dagger})^m a^{m-k} = aa^{\dagger}.$$

Hence, a is an EP element.

Suppose that $aa^{\dagger} = (a^{\dagger})^k a^m (a^{\dagger})^{m-k}$, for $1 \leq k \leq m$. Then we have

$$aR = aa^{\dagger}R = (a^{\dagger})^k a^m (a^{\dagger})^{m-k}R \subseteq a^{\dagger}R.$$

It follows from Lemma 2.4 that a is an EP element. $\square$

Particularly, taking $m = 2, k = 0, 1$ in (2) and $k = 1$ in (3), we have the following corollary.

Corollary 2.13. *$a \in R^{EP}$ if and only if $a \in R^{\#} \cap R^{\dagger}$ and one of the following conditions holds:*
(1) $aa^{\dagger} = a(a^{\dagger})^2a$;
(2) $aa^{\dagger} = a^{\dagger}a^2a^{\dagger}$.

Corollary 2.14. *$a \in R^{EP}$ if and only if $a \in R^{\#} \cap R^{\dagger}$ and one of the following conditions holds:*
(1) $a(a^{\dagger})^2a = a^{\dagger}a^2a^{\dagger}$;
(2) $a^{\dagger} = (a^{\dagger})^2a$;
(3) $a^{\dagger} = a(a^{\dagger})^2$;
(4) $a = a^{n+1}(a^{\dagger})^n$, for $n \geq 1$.

Proof. If a is an EP element, then the conditions (1)-(4) are clearly satisfied. Conversely, if the condition (1) hold, then $a(a^{\dagger})^2a = a^{\dagger}a^2a^{\dagger}$. Pre-multiplying by a, we have $a^2(a^{\dagger})^2a = a(a(a^{\dagger})^2a) = a(a^{\dagger}a^2a^{\dagger}) = a^2a^{\dagger}$. Then

$$a(a^{\dagger})^2a = (a^{\#}a^2)(a^{\dagger})^2a = a^{\#}(a^2(a^{\dagger})^2a) = a^{\#}(a^2a^{\dagger}) = aa^{\dagger}.$$

From Corollary 2.13, we see that a is an EP element.

Assume that the condition (2) hold. Then we have $aa^{\dagger} = a((a^{\dagger})^2a) = a(a^{\dagger})^2a$. It follows from Corollary 2.13 that a is an EP element.

Suppose that $a^{\dagger} = a(a^{\dagger})^2$. Then we get $a = aa^{\dagger}a = a^2(a^{\dagger})^2a$. Pre-multiplying by $a^{\#}$, we obtain

$$a^{\#}a = a^{\#}a^2(a^{\dagger})^2a = a(a^{\dagger})^2a = a^{\dagger}a.$$

It yields that a is an EP element.

If the condition (4) hold, then we get

$$a^{\#}a = a^{\#}(a^{n+1}(a^{\dagger})^n) = a^{\#}a^2(a^{n-1}(a^{\dagger})^n) = a^n(a^{\dagger})^n = a^n(a^{\dagger})^n aa^{\dagger} = aa^{\#}aa^{\dagger} = aa^{\dagger}.$$

It follows that a is an EP element. $\square$

Acknowledgement

The authors thank the anonymous referee for his/her valuable comments.

References

[1] W. X. Chen, On EP elements, normal elements and paritial isometries in rings with involution, Electron. J. Linear Algebra 23 (2012) 553–561.
[2] D. Drivaliaris, S. Karanasios, D. Pappas, Factorizations of EP operators, Linear Algebra Appl 429 (2008) 1555–1567.
[3] R. E. Hartwig, Block generalized inverses, Arch. Retion. Mech. Anal 61 (1976) 197–251.
[4] R. E. Hartwig, Generalized inverses, EP elements and associates, Rev. Roumaine Math. Pures Appl 23 (1978) 57–60.
[5] R. E. Hartwig, I. J. Katz, Products of EP elements in reflexive semigroups, Linear Algebra Appl 14 (1976) 11–19.
[6] R. E. Harte, M. Mbekhta, On generalized inverses in C^*-algebras, Studia Math 103 (1992) 71–77.
[7] S. Karanasios, *EP* elements in rings and semigroup with involution and C^*-algebras, Serdica Math. J 41 (2015) 83–116.
[8] J. J. Koliha, D. Cvetković, D. S. Djordjević, Moore-Penrose inverse in rings with involution, Linear Algebra Appl 426 (2007) 371–381.
[9] J. J. Koliha, The Drazin and Moore-Penrose inverse in C^*-algebras, Math. Proc. R. Ir. Acad 99A (1999) 17–27.
[10] J. J. Koliha, P. Patrílcio, Elements of rings with equal spectral idempotents, J. Aust. Math. Soc 72 (2002) 137–152.
[11] D. Mosić, D. S. Djordjević, New characterizations of EP, generalized normal and generalized Hermitian elements in rings, Appl. Math. Comput 218 (2012) 6702–6710.
[12] D. Mosić, D. S. Djordjević, J. J. Koliha, EP elements in rings, Linear Algebra Appl 431 (2009) 527–535.
[13] D. Mosić, D. S. Djordjević, Moore-Penrose-invertible normal and Hermitian elements in rings, Linear Algebra Appl 431 (2009) 732–745.
[14] P. Patrício, R. Puystjens, Drazin-Moore-Penrose invertibility in rings, Linear Algebra Appl 389 (2004) 159–173.
[15] Y. Tian, H. Wang, Characterizations of EP matrices and weighted-EP matrices, Linear Algebra Appl 434 (2011) 1295–1318.
[16] S. Z. Xu, J. L. Chen, J. Benítez, EP elements in rings with involution, arXiv: 1602.08184v2, 2017.