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Abel Statistical Quasi Cauchy Sequences

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Abstract.
In this paper, we investigate the concept of Abel statistical quasi Cauchy sequences. A real function f is called Abel statistically ward continuous if it preserves Abel statistical quasi Cauchy sequences, where a sequence (α_k) of point in $\mathbb{R}$ is called Abel statistically quasi Cauchy if $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta\alpha_k|\geqslant\varepsilon} x^k = 0$ for every $\varepsilon > 0$, where $\Delta\alpha_k = \alpha_{k+1} - \alpha_k$ for every $k \in \mathbb{N}$. Some other types of continuities are also studied and interesting results are obtained. It turns out that the set of Abel statistical ward continuous functions is a closed subset of the space of continuous functions.

1. Introduction

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer sciences, information theory, biological science, economics, and dynamical systems.

Throughout the paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of non negative integers and the set of real numbers, respectively. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if it preserves Cauchy sequences. Using the idea of continuity of a real function in this manner, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: ward continuity ([1], and [5]), p-ward continuity ([9]), statistical ward continuity, ([8]), λ-statistically ward continuity ([18]), ρ-statistical ward continuity ([11]), slowly oscillating continuity ([2], and [35]), quasi-slowly oscillating continuity ([23]), lacunary statistical ward continuity ([14]), and Abel continuity ([13]) which enabled some authors to obtain interesting results related to uniform continuity via one of the following types of sequences: quasi-Cauchy sequences, p-quasi-Cauchy sequences, statistical quasi-Cauchy sequences [12], lacunary statistical quasi-Cauchy sequences, ρ-statistical quasi-Cauchy sequences, ideal quasi-Cauchy sequences, strongly lacunary quasi-Cauchy sequences, slowly oscillating sequences. A real sequence (α_k) of points in $\mathbb{R}$ is called statistically convergent to an $\ell \in \mathbb{R}$ if $\lim_{n\to\infty}\frac{1}{n}|\{k \leqslant n : |\alpha_k - \ell| \geqslant \varepsilon\}| = 0$ for every $\varepsilon > 0$, and this is denoted by $st - \lim \alpha_k = \ell$ ([4, 7, 16, 21, 22, 26, 28, 29]). A sequence (α_k) is called lacunary statistically convergent ([30]) to an $\ell \in \mathbb{R}$ if $\lim_{r\to\infty}\frac{1}{h_r}|\{k \in I_r : |\alpha_k - \ell| \geqslant \varepsilon\}| = 0$ for every $\varepsilon > 0$, where $I_r = (k_{r-1}, k_r]$, and $k_0 = 1$, $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $\theta = (k_r)$ is an increasing sequence of positive integers, and this is denoted by $S_\theta - \lim \alpha_n = \ell$. Throughout this paper we assume that $\liminf_r \frac{k_r}{k_{r-1}} > 1$. A sequence (α_k) is slowly oscillating, if for any given $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon) > 0$ and an $N = N(\varepsilon)$ such that $|\alpha_m - \alpha_n| < \varepsilon$ whenever

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$n \geqslant N(\varepsilon)$ and $n \leqslant m \leqslant (1+\delta)n$. A sequence (α_k) is quasi-slowly oscillating if $(\Delta\alpha_k)$ is slowly oscillating, where $\Delta\alpha_k = \alpha_{k+1} - \alpha_k$ for each $k \in \mathbb{N}$ ([23]).

The purpose of this paper is to introduce and investigate the concept of Abel statistical ward continuity and present interesting results.

2. Abel Statistical Ward Continuity

A sequence (α_k) of real numbers is called Abel convergent (or Abel summable) to ℓ if the series

$$\Sigma_{k=0}^{\infty}\alpha_k x^k$$

is convergent for $0 \leqslant x < 1$ and

$$\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty}\alpha_k x^k = \ell$$

(see for example [13]). In this case, we write $Abel - \lim \alpha_k = \ell$. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to 0 and specially speaking, than that the distance between successive terms is Abel convergent to zero. Nevertheless, sequences which satisfy this weaker property, i.e. Abel quasi Cauchy sequences satisfying $Abel - lim\Delta\ \alpha_k = 0$, are interesting in their own right. In other words, a sequence (α_k) of points in $\mathbb{R}$ is called Abel quasi-Cauchy if $(\Delta\alpha_k)$ is Abel convergent to 0, i.e. the series

$$\sum_{k=0}^{\infty}\Delta\alpha_k x^k$$

is convergent for $0 \leqslant x < 1$ and

$$\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty}\Delta\alpha_k x^k = 0.$$

A and ΔA will denote the set of Abel convergent sequences and the set of Abel quasi Cauchy sequences, respectively.

Recently the concept of Abel statistical convergence of a sequence is investigated in [34] (see also [20]) in the sense that a sequence (α_k) is called Abel statistically convergent to a real number L if

$$\lim_{x\to 1^-}(1-x)\sum_{k:|\alpha_k - L|\geqslant\varepsilon} x^k = 0$$

for every $\varepsilon > 0$, and denoted by $Abel_{st} - lim\ \alpha_k = L$.

Now we introduce the concept of Abel statistical quasi Cauchyness in the following:

Definition 2.1. A sequence of points in a subset E of $\mathbb{R}$ is called Abel statistically quasi Cauchy if $Abel_{st} - lim\Delta\ \alpha_k = 0$, i.e. $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta\alpha_k|\geqslant\varepsilon} x^k = 0$ for every $\varepsilon > 0$, and

$$\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta\alpha_k|\geqslant\varepsilon} x^k = 0$$

for every $\varepsilon > 0$, where $\Delta\alpha_k = \alpha_{k+1} - \alpha_k$ for every $k \in \mathbb{N}$.

For any fixed constant $c \in \mathbb{R}$, the sequence $(c\alpha_k)$ is Abel statistically quasi Cauchy whenever (α_k) is, and the sum of two Abel statistically quasi-Cauchy sequence is Abel statistically quasi-Cauchy. Thus the set of all Abel statistically quasi Cauchy sequences is a vector space of the space of all sequences. The product of two Abel statistical quasi-Cauchy sequences need not be Abel statistically quasi-Cauchy as it can be seen by considering the product of the sequence $(\sqrt{k})$ itself. Cauchy sequences have the property that any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for Abel statistical quasi Cauchy sequences. A counter example is the sequence $(\sqrt{k})$ with the subsequence (k) ([20]). Any convergent sequence is Abel statistically quasi Cauchy: let (α_k) be a convergent sequence with limit L, and $\varepsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $|\alpha_k - L| < \frac{\varepsilon}{2}$ for $k \geqslant N$. Hence

$$\{k \in \mathbb{N} : |\alpha_k - L| \geqslant \frac{\varepsilon}{2}\} \subseteq \{1, 2, ..., N\}$$

for every $\varepsilon > 0$. Therefore

$$\sum_{k \in \mathbb{N}: |\alpha_k - L| \geqslant \frac{\varepsilon}{2}} x^k \leqslant \sum_{k=1}^{N} x^k$$

for every $\varepsilon > 0$. On the other hand,

$$\sum_{k: |\Delta\alpha_k| \geqslant \varepsilon} x^k \leqslant \sum_{k: |\alpha_{k+1} - L| \geqslant \frac{\varepsilon}{2}} x^k + \sum_{k: |L - \alpha_k| \geqslant \frac{\varepsilon}{2}} x^k \leqslant 2 \sum_{k=1}^{N} x^k$$

for every $\varepsilon > 0$. Therefore

$$\lim_{x \to 1^-} (1 - x) \sum_{k: |\Delta\alpha_k| \geqslant \varepsilon} x^k \leqslant 2 \lim_{x \to 1^-} (1 - x) \sum_{k=1}^{N} x^k = 0$$

for every $\varepsilon > 0$.
Furthermore, a slowly oscillating sequence is Abel statistically quasi-Cauchy, so is a Cauchy sequence.

Definition 2.2. A subset E of $\mathbb{R}$ is called Abel statistically ward compact if any sequence of points in E has an Abel statistical quasi Cauchy subsequence, i.e. whenever $\boldsymbol{\alpha} = (\alpha_n)$ is a sequence of points in E, there is an Abel statistical quasi Cauchy subsequence $\boldsymbol{\xi} = (\xi_k) = (\alpha_{n_k})$ of $\boldsymbol{\alpha}$.

According to this definition, any bounded subset of $\mathbb{R}$ is Abel statistically ward compact. The union of two Abel statistical ward compact subsets of $\mathbb{R}$ is Abel statistically ward compact, therefore it is seen inductively that any finite union of Abel statistical ward compact subsets of $\mathbb{R}$ is Abel statistically ward compact, whereas the union of the infinite family of Abel statistical ward compact subsets of $\mathbb{R}$ is not always Abel statistically ward compact. The intersection of any family of Abel statistical ward compact subsets of $\mathbb{R}$ is Abel statistically ward compact. The sum of two Abel statistical ward compact subsets of $\mathbb{R}$ is Abel statistically ward compact. These observations above suggest to us the following.

Theorem 2.3. *A subset of $\mathbb{R}$ is Abel statistically ward compact if and only if it is bounded.*

Proof. Since Abel statistical sequential method is regular, it is clear that any bounded subset of $\mathbb{R}$ is Abel statistically ward compact. Suppose now that E is unbounded. First pick an element α_0 of E so that $\alpha_0 > 1$. Then choose an element α_1 of E so that $\alpha_1 > \alpha_0 + 1$. Similarly choose an element α_2 of E so that $\alpha_2 > \alpha_1 + 2^1$. We can inductively choose elements of E so that $\alpha_{k+1} > \alpha_k + 2^k$ for each $k \in \mathbb{N}$. Take any subsequence α_{n_k} of the sequence (α_n). Thus

$$\sum_{k \in \mathbb{N}: |\Delta\alpha_k| \geqslant 1} x^k = \sum_{k=1}^{\infty} x^k = \frac{1}{1 - x}.$$

Hence

$$\lim_{x \to 1^-} (1 - x) \sum_{k \in \mathbb{N}: |\Delta\alpha_k| \geqslant 1} x^k = \lim_{x \to 1^-} (1 - x) \sum_{k=1}^{\infty} x^k = 1 \neq 0.$$

Thus the sequence α_k has no Abel statistical quasi Cauchy subsequence as well. If it is unbounded below, then similarly we construct a sequence of points in E which has no Abel statistical quasi Cauchy subsequence. This completes the proof of the theorem. □

We note that Abel statistical ward compactness coincides with not only Abel ward compactness, but also statistical ward compactness, λ-statistical ward compactness, ρ-statistical ward compactness, lacunary statistical ward compactness, strongly lacunary ward compactness. We note that a subset of $\mathbb{R}$ is Abel statistically ward compact if and only if it is statistically upward half compact and statistically downward half compact, which follows from [13, Theorem 5] and [10, Corollary 3.9]); and a subset of $\mathbb{R}$ is Abel statistically ward compact if and only if it is p-ward compact for a $p \in \mathbb{N}$, which follows from [13, Theorem 5] and [9, Theorem 2.3].

We now introduce a new type of continuity defined via Abel statistical quasi-Cauchy sequences.

Definition 2.4. A function f is called Abel statistically ward continuous on a subset E of $\mathbb{R}$ if it preserves Abel statistical quasi Cauchy sequences of points in E, i.e. $(f(\alpha_k))$ is Abel statistically quasi Cauchy whenever (α_k) is an Abel statistical quasi Cauchy sequence of points in E.

We note that this definition of continuity can not be obtained by $\mathtt{A}$-continuity for any regular summability matrix $\mathtt{A}$ considered in [24] (see also [3], [6], [17], and [31]).

Proposition 2.5. *The sum of two Abel statistical ward continuous functions is Abel statistically ward continuous.*

Proof. Let f and g be Abel statistical ward continuous functions on a subset E of $\mathbb{R}$, and (α_k) be an Abel statistical quasi Cauchy sequence of points in E. Take any $\varepsilon > 0$. Since f is Abel statistically ward continuous on E, we have

$$\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta f(\alpha_k)|\geq \frac{\varepsilon}{2}} x^k = 0;$$

since g is Abel statistically ward continuous on E,

$$\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta g(\alpha_k)|\geq \frac{\varepsilon}{2}} x^k = 0.$$

Now it follows from the inequality

$$\sum_{k:|(f+g)(\alpha_{k+1})-(f+g)(\alpha_k)|\geq\varepsilon} x^k \leq \sum_{k:|\Delta f(\alpha_k)|\geq \frac{\varepsilon}{2}} x^k + \sum_{k:|\Delta g(\alpha_k)|\geq \frac{\varepsilon}{2}} x^k$$

that

$$\lim_{x\to 1^-}(1-x)\sum_{k:|(f+g)(\alpha_{k+1})-(f+g)(\alpha_k)|\geq \frac{\varepsilon}{2}} x^k \leq \lim_{x\to 1^-}(1-x)\sum_{k:|\Delta f(\alpha_k)|\geq \frac{\varepsilon}{2}} x^k + \lim_{x\to 1^-}\sum_{k:|\Delta g(\alpha_k)|\geq \frac{\varepsilon}{2}} x^k = 0+0=0.$$

This completes the proof. □

The composite of two Abel statistical ward continuous functions is Abel statistically ward continuous but the product of two Abel statistical ward continuous functions need not be Abel statistically ward continuous as it can be seen by considering the product of the Abel statistical ward continuous function $f(t) = t$ with itself, and the Abel statistical quasi Cauchy sequence $(\sqrt{n})$. If f is an Abel statistical ward continuous function, then cf is also Abel statistically ward continuous for any constant $c \in \mathbb{R}$. $max\{f, g\}$ is an Abel statistical ward continuous function, whenever f and g are Abel statistically ward continuous functions.

Theorem 2.6. *Abel statistical ward continuous image of any Abel statistical ward compact subset of* $\mathbb{R}$ *is Abel statistically ward compact.*

Proof. Let $f : E \to \mathbb{R}$ be an Abel statistical ward continuous function and B be an Abel statistical ward compact subset of E. Take any sequence $\boldsymbol{\eta} = (\eta_k)$ of points in $f(B)$. Write $\eta_k = f(\alpha_k)$ for each $k \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_k)$. Since B is Abel statistically ward compact, there exists an Abel statistical quasi Cauchy subsequence $\boldsymbol{\xi} = (\xi_k)$ of the sequence $\boldsymbol{\alpha}$. Since f is Abel statistically ward continuous, $f(\boldsymbol{\xi}) = (f(\xi_k))$ is Abel statistically quasi Cauchy, which is a subsequence of the sequence $\boldsymbol{\eta}$. This completes the proof of the theorem. □

In connection with Abel statistical quasi Cauchy sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbb{R}$.

(ΔA_{st}) $(\alpha_k) \in \Delta A_{st} \Rightarrow (f(\alpha_k)) \in \Delta A_{st}$

($\Delta A_{st}c$) $(\alpha_k) \in \Delta A_{st} \Rightarrow (f(\alpha_k)) \in c$

(A_{st}) $(\alpha_k) \in A_{st} \Rightarrow (f(\alpha_k)) \in A_{st}$

($c\Delta A_{st}$) $(\alpha_k) \in c \Rightarrow (f(\alpha_k)) \in \Delta A_{st}$

(c) $(\alpha_k) \in c \Rightarrow (f(\alpha_k)) \in c$

We see that ΔA_{st} is Abel statistical ward continuity of f, (A_{st}) states the Abel statistical continuity of f, and (c) states the ordinary continuity of f. We easily see that (A_{st}) implies ($c\Delta A_{st}$); (ΔA_{st}) implies (cΔA_{st}); (A_{st}) implies (c), but (c) does not imply (A_{st}) ([13, Theorem 7]); and (ΔA_{st}c) implies (ΔA_{st}).

Now we give the implication (ΔA_{st}) implies (c), i.e. any Abel statistical ward continuous function is continuous in the ordinary sense.

Theorem 2.7. *If a function is Abel statistically ward continuous on a subset E of $\mathbb{R}$, then it is continuous on E.*

Proof. Suppose that a function f is not continuous on E so that there exists a convergent sequence (α_n) with $\lim_{n\to\infty} \alpha_n = \ell$ such that $(f(\alpha_n))$ is not convergent to $f(\ell)$. If $(f(\alpha_n))$ is bounded, then either $(f(\alpha_n))$ has a limit different from $f(\ell)$, or there are at least two convergent subsequences of $(f(\alpha_n))$ with different limits. In both cases it is not difficult to fall in a contradiction. If $(f(\alpha_n))$ is unbounded above. Then we can find an n_1 such that $f(\alpha_{n_1}) > f(\alpha_0) + 1$. There exists a positive integer an $n_2 > n_1$ such that $f(\alpha_{n_2}) > f(\alpha_{n_1}) + 2$. Suppose that we have chosen an $n_{k-1} > n_{k-2}$ such that $f(\alpha_{n_{k-1}}) > f(\alpha_{n_{k-2}}) + 2^{k-2}$. Then we can choose an $n_k > n_{k-1}$ such that $f(\alpha_{n_k}) > f(\alpha_{n_{k-1}}) + 2^{k-1}$. Inductively we can construct a subsequence $(f(\alpha_{n_k}))$ of $(f(\alpha_n))$ such that $f(\alpha_{n_{k+1}}) > f(\alpha_{n_k}) + 2^k$ for each $k \in \mathbb{N}$. Since the sequence (α_{n_k}) is a subsequence of (α_n), the subsequence (α_{n_k}) is convergent so is Abel statistically quasi Cauchy. But $(f(\alpha_{n_k}))$ is not Abel statistically quasi-Cauchy as we see below. For each $k \in \mathbb{N}$ we have $\Delta f(\alpha_{n_k}) > 2^k$. The series $\sum_{k:|\Delta f(\alpha_{n_k})|\geqslant 1} x^k$ is convergent and equal to $\frac{1}{1-x}$ for any x satisfying $0 < x < 1$, so $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta f(\alpha_{n_k})|\geqslant 1} x^k = 1 \neq 0$. Thus the sequence $(f(\alpha_{n_k}))$ is not Abel statistically quasi Cauchy. If $(f(\alpha_n))$ is unbounded below, similarly $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta f(\alpha_{n_k})|\geqslant 1} x^k$ is found not to be 0. The contradiction for all possible cases to the Abel statistical ward continuity of f completes the proof of the theorem. □

The converse of the preceding theorem is not always true. As a counterexample consider the function defined by $f(x) = x^2$ and the Abel statistical quasi Cauchy sequence defined by $(\sqrt{n})$. We note that Abel statistical ward continuity implies not only ordinary continuity, but also statistical continuity, which follows from [4, Corollary 4], Lemma 1 and Theorem 8 in [6]; lacunary statistical sequential continuity, which was observed in [14] (see also [36]), λ-statistical continuity ([18]), ρ-statistical continuity, and ([11]); G-sequential continuity for any regular subsequential method G, which follows from Theorem 8 in [6](see also [31])

It is well-known that any continuous function on a compact subset E of $\mathbb{R}$ is uniformly continuous on E. We have an analogous theorem for an Abel statistical ward continuous function defined on an Abel statistical ward compact subset of $\mathbb{R}$.

Theorem 2.8. *Any Abel statistical ward continuous function on an Abel statistical ward compact subset E of $\mathbb{R}$ is uniformly continuous.*

Proof. Let f be a function defined on an Abel statistical ward compact subset E of $\mathbb{R}$ into $\mathbb{R}$. Suppose that f is not uniformly continuous on E so that there exist an $\epsilon_0 > 0$ and sequences (α_n) and (β_n) of points in E such that $|\alpha_n - \beta_n| < 1/n$ and $|f(\alpha_n) - f(\beta_n)| \geqslant \epsilon_0$ for all $n \in \mathbb{N}$. By Theorem 2.3, E is Abel statistically ward compact, there is a subsequence (α_{n_k}) of (α_n) that is Abel statistically quasi-Cauchy. On the other hand there is a subsequence $(\beta_{n_{k_j}})$ of (β_{n_k}) that is Abel statistically quasi-Cauchy. The corresponding subsequence $(\alpha_{n_{k_j}})$ is also Abel statistically quasi-Cauchy, which follows from the following inclusion

$$\{k \in \mathbb{N} : |\alpha_{n_{k_{j+1}}} - \alpha_{n_{k_j}}| \geqslant \varepsilon\} \subseteq \{k \in \mathbb{N} : |\alpha_{n_{k_{j+1}}} - \beta_{n_{k_{j+1}}}| \geqslant \frac{\varepsilon}{3}\} \cup \{k \in \mathbb{N} : |\beta_{n_{k_{j+1}}} - \beta_{n_{k_j}}| \geqslant \frac{\varepsilon}{3}\}$$

$$\cup \{k \in \mathbb{N} : |\beta_{n_{k_j}} - \alpha_{n_{k_j}}| \geqslant \frac{\varepsilon}{3}\}$$

for every $\varepsilon > 0$, that implies

$\lim_{x\to 1^-}(1-x)\sum_{k\in\mathbb{N}:|\beta_{n_{k_{j+1}}}-\alpha_{n_{k_{j+1}}}|\geqslant\varepsilon} x^k$

$$\leqslant \lim_{x\to 1^-}(1-x)\sum_{k\in\mathbb{N}:|\beta_{n_{k_{j+1}}}-\beta_{n_{k_j}}|\geqslant\frac{\varepsilon}{3}} x^k + \lim_{x\to 1^-}(1-x)\sum_{k\in\mathbb{N}:|\beta_{n_{k_j}}-\alpha_{n_{k_j}}|\geqslant\frac{\varepsilon}{3}} x^k + \lim_{x\to 1^-}(1-x)\sum_{k\in\mathbb{N}:|\alpha_{n_{k_j}}-\alpha_{n_{k_{j+1}}}|\geqslant\frac{\varepsilon}{3}} x^k$$

$= 0 + 0 + 0 = 0$
for every $\varepsilon > 0$. Now the sequence

$$(\eta_j) = (\alpha_{n_{k_1}}, \beta_{n_{k_1}}, ..., \alpha_{n_{k_j}}, \beta_{n_{k_j}}, ...)$$

is Abel statistically quasi-Cauchy while the sequence

$$(f(\eta_j)) = (f(\alpha_{n_{k_1}}), f(\beta_{n_{k_1}}), ..., f(\alpha_{n_{k_j}}), f(\beta_{n_{k_j}}), ...)$$

is not Abel statistically quasi-Cauchy since

$$\lim_{x\to 1^-}(1-x)\sum_{k\in\mathbb{N}:|f(\eta_j)-f(\eta_{j+1})|\geqslant\varepsilon_0} x^k = \lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty} x^k = (1-x)\frac{1}{(1-x)} = 1 \neq 0.$$

Hence this establishes a contradiction so completes the proof of the theorem. □

3. Conclusion

In this paper we investigate the concept of Abel statistical ward continuity, and present theorems related to this kind of continuity, and some other kinds of continuities. One may expect this investigation to be a useful tool in the field of analysis in modeling various problems occurring in many areas of science, dynamical systems, computer science, information theory, and biological science. On the other hand, we suggest to investigate the concept of fuzzy Abel statistical quasi Cauchy sequences of fuzzy points or soft points (see [15] for the definitions and related concepts in fuzzy setting, and see [25] related concepts in soft setting). However due to the change in settings, the definitions and methods of proofs will not always be the same. An investigation of Abel ward continuity and Abel ward compactness can be done for double sequences (see [33] for basic concepts in the double sequences case). However due to the change in settings, the definitions and methods of proofs will not always be the same. For some further study, we suggest to investigate Abel statistical quasi Cauchy sequences of points in a topological vector space valued cone metric space (see [19], and [32]) or in 2-normed spaces ([27]).

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