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On E-J Hausdorff Transformations for Double Sequences

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Abstract. In 1933, Adams [1] developed Hausdorff transformations for double sequences. H. Şevli and R. Savaş [18] proved some result for the double Endl- Jakimovski (E-J) generalization. In this study, we consider some further results for E-J Hausdorff transformations for double sequences.

1. Introduction and Background

A generalization of Hausdorff matrices has been made independently by Endl [7] and Jakimovski [9] and this generalization is called the E-J generalization in the literature. Hausdorff transformations for double sequences were described by Adams [1]. Later than some researchers studied double Hausdorff matrices, see e.g. Ramanujan [12], Ustina [21], Rhoades [13] and further studied in [19, 20] to deal with some double summability problems.

Let $\{\mu_{ij}\}$ be a real or complex double sequence, and let Δ_1^m and Δ_2^n be the forward difference operators defined by $\Delta_1\mu_{ij} = \mu_{ij} - \mu_{i+1,j}$, $\Delta_1^{m+1}\mu_{ij} = \Delta_1\left(\Delta_1^m\mu_{ij}\right)$ and $\Delta_2\mu_{ij} = \mu_{ij} - \mu_{i,,j+1}$, $\Delta_2^{n+1}\mu_{ij} = \Delta_2\left(\Delta_2^n\mu_{ij}\right)$. A double Hausdorff matrix has entries

$$h_{mnij} = \binom{m}{i}\binom{n}{j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij},$$

where

$$\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij} = \sum_{s=0}^{m-i}\sum_{t=0}^{n-j}(-1)^{i+j}\binom{m-i}{s}\binom{n-j}{t}\mu_{i+s.j+t}.$$

For double Hausdorff matrices, the necessary and sufficient condition for H to be conservative is the existence of a function $\chi(s,t) \in BV[0,1] \times [0,1]$ such that

$$\int_0^1\int_0^1 |d\chi(s,t)| < \infty,$$

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and

$$\mu_{mn}=\int_0^1\int_0^1 s^m t^n d\chi(s,t).$$

Recently, the authors [18] considered the double E-J generalization. Let α and β be real numbers. The matrix $\delta^{(\alpha,\beta)}=\left(\delta_{mnij}^{(\alpha,\beta)}\right)$, whose elements are defined by

$$\delta_{mnij}^{(\alpha,\beta)}=\begin{cases}(-1)^{i+j}\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}, & i\le m,\ j\le n,\\ 0, otherwise.\end{cases}$$

is called doubly difference matrix. The matrix $\delta^{(\alpha,\beta)}=\left(\delta_{mnij}^{(\alpha,\beta)}\right)$ is its own inverse.

Let $\{\mu_{mn}\}$ be a given sequence and $\mu=\left(\mu_{mnij}\right)$ be a diagonal matrix whose only non-zero entries are $\mu_{mn}=\mu_{mnmn}$. The transformation matrix

$$H^{(\alpha,\beta)}=\delta^{(\alpha,\beta)}\mu\delta^{(\alpha,\beta)}$$

is called a double E-J generalized Hausdorff matrix corresponding to the sequence $\{\mu_{mn}\}$. A matrix $H^{(\alpha,\beta)}=\left(h_{mnij}^{(\alpha,\beta)}\right)$ is a double E-J generalized Hausdorff matrix corresponding to the sequence $\{\mu_{mn}\}$ if and only if its elements have the form

$$h_{mnij}^{(\alpha,\beta)}=\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij},$$

where

$$\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}=\sum_{r=0}^{m-i}\sum_{s=0}^{n-j}(-1)^{r+s}\binom{m-i}{r}\binom{n-j}{s}\mu_{i+r,j+s}.$$

For double E-J Hausdorff matrices, the necessary and sufficient condition for $H^{(\alpha,\beta)}$ to be conservative is the existence of a function, $\chi(s,t)\in BV[0,1]\times[0,1]$ such that

$$\int_0^1\int_0^1 |d^2\chi(s,t)|<\infty,$$

and

$$\mu_{mn}^{(\alpha,\beta)}=\int_0^1\int_0^1 s^{m+\alpha}t^{n+\beta}d^2\chi(s,t).$$

The function χ is called mass function associated with the moment generating sequence $\left\{\mu_{mn}^{(\alpha,\beta)}\right\}$. Given a function $\chi(s,t)\in BV[0,1]\times[0,1]$, bounded variation in the unit square, the corresponding double E-J Hausdorff transformation $\{t_{mn}\}$, of a sequence $\{s_{mn}\}$, may be defined by

$$t_{mn}=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}s_{ij}\int_0^1\int_0^1 s^{i+\alpha}(1-s)^{m-i}t^{j+\beta}(1-t)^{n-j}d\chi(s,t),$$

see e.g. [18] and the references contained there in.

Let $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}$ be an infinite double series with real or complex numbers, with partial sums

$$s_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}.$$

Denote by $\mathcal{A}_k^2$ the sequence space defined by,

$$\mathcal{A}_k^2 = \left\{ (s_{mn})_{m,n=0}^{\infty} \ : \ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (mn)^{k-1} |a_{mn}|^k < \infty \ ; a_{mn} = \Delta_{11} s_{m-1,n-1} \right\}$$

for $k \geq 1$, where

$$\Delta_{11} s_{m-1,n-1} = s_{m-1,n-1} - s_{m,n-1} - s_{m-1,n} + s_{mn}.$$

A four-dimensional matrix $T = \left(t_{mnij} : m, n, i, j = 0, 1, ...\right)$ is said to be absolutely k-th power conservative for $k \geq 1$, if $T \in B\left(\mathcal{A}_k^2\right)$; i.e., if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (mn)^{k-1} \left|\Delta_{11} s_{m-1,n-1}\right|^k < \infty,$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (mn)^{k-1} \left|\Delta_{11} t_{m-1,n-1}\right|^k < \infty,$$

where

$$t_{mn} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} t_{mnij} s_{ij} \qquad (m, n = 0, 1, ...),$$

see e.g. [16], [17] and the references contained there in.

Let H be a conservative double Hausdorff matrix. E. Savaş and B.E. Rhoades [14] proved that $H \in B\left(\mathcal{A}_k^2\right)$. Quite recently, the authors [18] proved the corresponding result of [15] for double E-J generalized Hausdorff matrices, i.e., they proved that $H^{(\alpha,\beta)} \in B\left(\mathcal{A}_k^2\right)$, $\alpha, \beta \geq 0$, where $H^{(\alpha,\beta)}$ is a conservative double E-J Hausdorff matrix.

In this study, we consider some further results for E-J Hausdorff transformations for double sequences.

2. Monotonicity-Preserving Matrices

For a given sequence $\{x_k\}$ let us define the difference operator of order r or r-th difference operator, $r \in \mathbb{N}_0$, as

$$\Delta^0 x_k = x_k, \quad \Delta^1 x_k = x_k - x_{k+1}, \quad \Delta^{r+1} x_k = \Delta^1 \left(\Delta^r x_k\right) x_k,$$

For an arbitrary $r \in \mathbb{N}_0$ and $k \in \mathbb{N}$ it hold

$$\Delta^r x_k = \sum_{j=0}^{n} (-1)^j \binom{r}{j} x_{j+k}.$$

A sequence $\{x_k\}$ is *r-convex* if its r-th differences, $\Delta^r x_k$, are all non-negative. Thus 0-convex sequences have positive entries, 1-convex sequences are decreasing, 2-convex sequences are convex.

Let X and Y be any two sequence spaces. If $x \in X$ implies $Ax \in Y$, then we say that the matrix $A = (a_{ni} : n, i = 0, 1, ...)$ maps X into Y. We denote the class of all matrices A which map X into Y by (X, Y). G. Bennett [6] studied on matrix transformations preserving $r-$convexity for all $r = 0, 1, 2, ...$, i.e. monotonicity-preserving.

For ordinary Hausdorff or $H(\mu)$ transformation, (see Hardy [8]), the transformation

$$\mathbf{y} = H(\mu)\,\mathbf{x}$$

can be exppressed in the following form:

$$\Delta^r y_0 = \mu_r \Delta^r x_0, \quad (r = 0, 1, 2, ...). \tag{1}$$

Lemma 2.1. ([6]) *Suppose that **a**, **b** and **c** are sequences of real (or complex) numbers satisfying*

$$\Delta^r a_0 = b_n \Delta^r c_0, \qquad (r = 0, 1, 2, ...).$$

Then, in fact,

$$\Delta^r a_k = \sum_{i=0}^{k} \left(\Delta^{k-i} b_{r+i}\right) \binom{k}{i} \Delta^r c_i, \qquad (r, k = 0, 1, 2, ...).$$

The above lemma lets us replace (1) by a more generous identity:

$$\Delta^r y_k = \sum_{i=0}^{k} \left(\Delta^{k-i} \mu_{r+i}\right) \binom{k}{i} \Delta^r x_i, \qquad (r, k = 0, 1, 2, ...).$$

This expresses all the rth differences of $\mathbf{y}$ as positive linear combinations of rth differences of $\mathbf{x}$, as a result of that $H(\mu)$ preserves r-convexity for each $r = 0, 1, 2, ...$ Hence any positive multiple $H(\mu)$ is monotonicity-preserving.

Theorem 2.2. ([6]) *The matrix transformation*

$$y_n = \sum_{i=0}^{n} a_{ni} x_i, \quad (n = 0, 1, 2, ...)$$

is monotonicity-preserving if and only if $a_{00} \geq 0$ and

$$a_{ni} = a_{00} \binom{n}{i} \int_0^1 \theta^i (1-\theta)^{n-i}\, d\mu(\theta), \quad (n, i = 0, 1, 2, ...),$$

where $d\mu$ is some probability measure on $[0, 1]$. In other words, A is a positive multiple of a Hausdorff mean.

A double sequence $x = \{x_{ij}\}$ of real or complex numbers is said to be bounded if

$$\|x\|_\infty = \sup_{i,j} |x_{ij}| < \infty,$$

and is said to be *convergent to the limit l in the Pringsheim sense* (shortly, *p-convergent to l*) if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{ij} - l| < \varepsilon$ whenever $i, j > N$. In this case l is called the *$p-$limit* of x. We denote by $\mathcal{L}_\infty$ and C_p, the space of bounded double sequences and the space of p-convergent sequences, respectively.

Note that, a convergent double sequence need not be bounded. The space of bounded p-convergent double sequences is denoted by C_{bp} [11].

Let $A = (a_{mnij} : m, n, i, j = 0, 1, ...)$ be a four dimensional infinite matrix of real numbers. The double series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{mnij}x_{ij}$$

is called $A-$transform of the double sequence $x = \{x_{ij}\}$ and denoted by Ax. We say that a sequence x is $A-$summable to the limit s if the the $A-$transform of x exists for all $m, n = 0, 1, ...$ and is convergent in the Pringsheim sense, i.e.,

$$p - \lim_{p,q\to\infty} \sum_{i=0}^{p}\sum_{j=0}^{q} a_{mnij}x_{ij} = y_{mn}$$

and

$$p - \lim_{m,n\to\infty} y_{mn} = s.$$

It is known that $A \in \left(C_{bp}, C_{bp}\right)$, that is, A is conservative if and only if (see [10])

$$p - \lim_{m,n} a_{mnij} = a_{ij} \text{ for each } i, j\,; \tag{2}$$

$$p - \lim_{m,n} \sum_{i}\sum_{j} a_{mnij} = a\,; \tag{3}$$

$$p - \lim_{m,n} \sum_{i} \left|a_{mnij}\right| = a_{0j} \text{ for each } j\,; \tag{4}$$

$$p - \lim_{m,n} \sum_{k} \left|a_{mnij}\right| = a_{i0} \text{ for each } i\,; \tag{5}$$

$$p - \lim_{m,n} \sum_{i}\sum_{j} \left|a_{mnij}\right| \text{ exists}\,; \tag{6}$$

$$\|A\| = \sup_{m,n} \sum_{i}\sum_{j} \left|a_{mnij}\right| < \infty. \tag{7}$$

If A is conservative and $p - \lim Ax = p - \lim x$ for all $x \in C_{bp}$, the matrix A is said to be $RH - regular$ and we denote $A \in \left(C_{bp}, C_{bp}\right)_{reg}$. Also, it is known that $A \in \left(C_{bp}, C_{bp}\right)_{reg}$ if and only if conditions (6), (7) hold, and also (2) with $a_{ij} = 0$, (3) with $a = 1$, (4) and (5) with $a_{0j} = a_{i0} = 0$ hold.

The double E-J Hausdorff method corresponding to the sequence $\{\mu_{mn}\}$ is regular if and only if

$$\mu_{mn}^{(\alpha,\beta)} = \int_0^1\int_0^1 u^{m+\alpha}v^{n+\beta}d^2g(u,v), \qquad m, n = 0, 1, 2, ...,$$

where $g(u, v)$ is a function of bounded variation in the unit square with

$$g(u, 0) = g(u, 0^+) = g(0^+, v) = g(0, v) = 0, \quad 0 \le u,\ v \le 1, \tag{8}$$

and

$$g(1, 1) - g(1, 0) - g(0, 1) + g(0, 0) = 1. \tag{9}$$

If, in addition, $H^{(\alpha,\beta)}$ has all nonnegative entries, then g is nonnegative and nonincreasing in each variable. This condition is equivalent to all of the forward differences in

$$h_{mnij}^{(\alpha,\beta)} = \binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}$$

being nonnegative.

The following lemma is double version of Lemma 2.1

Lemma 2.3. ([4]) *Suppose that **a**, **b** and **c** are real double sequences such that*

$$\Delta_1^r\Delta_2^s a_{00} = b_{rs}\Delta_1^r\Delta_2^s c_{00}, \qquad (r,s=0,1,2,...).$$

Then,

$$\Delta_1^r\Delta_2^s a_{kl} = \sum_{i=0}^{k}\sum_{j=0}^{l}\left(\Delta_1^{k-i}\Delta_2^{l-j}b_{r+i,s+j}\right)\binom{k}{i}\binom{l}{j}\Delta_1^r\Delta_2^s c_{ij},$$

$r,s,k,l=0,1,2,....$

Quite recently, Akgün and Rhoades [4] studied the following doubly matrix transformation

$$y_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{mnij}x_{ij}, \qquad (m,n=0,1,2,...) \tag{10}$$

such that all monotonicities are preserved. The matrix A is said to preserve all monotonicities if any kind of monotonic behavior is transferred to **y**. A sequence $\{x_{ij}\}$ is *rs-convex* if its *rs*-th differences, $\Delta_1^r\Delta_2^s x_{ij}$, are all non-negative. For all $r,s=0,1,2,...$ if $\Delta_1^r\Delta_2^s x_{ij}\geq 0$ then one must have $\Delta_1^r\Delta_2^s y_{mn}\geq 0$, for all $m,n=0,1,2,....$ We call such double matrices also monotonicity-preserving. In [4], they proved the result of Bennett [6] to double Hausdorff matrix with all nonnegative entries. Their theorem is as follows.

Theorem 2.4. ([4]) *A double triangular matrix A is monotonicity preserving if and only if it is a positive multiple of a double regular Hausdorff matrix with nonnegative entries.*

Let $\sum\limits_{m=0}^{\infty}\sum\limits_{n=0}^{\infty} a_{mn}$ be an infinite double series with real or complex numbers, with partial sums

$$s_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}.$$

The double E-J Hausdorff method $\left(H^{(\alpha,\beta)},\mu_{mn}\right)$ is defined by the transformation

$$\begin{aligned} t_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} h_{mnij}^{(\alpha,\beta)} s_{ij} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}s_{ij} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\left(1+\frac{\alpha}{m}\right)\left(1+\frac{\beta}{n}\right)...\left(1+\frac{\alpha}{i+1}\right)\left(1+\frac{\beta}{j+1}\right)\binom{m}{i}\binom{n}{j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}s_{ij} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\left(1+\frac{\alpha}{m}\right)\left(1+\frac{\beta}{n}\right)...\left(1+\frac{\alpha}{i+1}\right)\left(1+\frac{\beta}{j+1}\right)h_{mnij}^{(0,0)}s_{ij}, \end{aligned}$$

since

$$\binom{m+\alpha}{m-i} = \binom{m}{i}\left(1+\frac{\alpha}{m}\right)...\left(1+\frac{\alpha}{i+1}\right), \quad m>0$$

and

$$\binom{n+\beta}{n-j} = \binom{n}{j}\left(1+\frac{\beta}{n}\right)...\left(1+\frac{\beta}{j+1}\right), \quad n>0.$$

Therefore, the double E-J Hausdorff method $H^{(\alpha,\beta)}(\mu)$ proceeds from the ordinary double Hausdorff method $H^{(0,0)}(\mu)$ by multiplying the Hausdorff element $h^{(0,0)}_{mnij}$ by the factor

$$\left(1+\frac{\alpha}{m}\right)\left(1+\frac{\beta}{n}\right)...\left(1+\frac{\alpha}{i+1}\right)\left(1+\frac{\beta}{j+1}\right).$$

For $\alpha = -k$, $\beta = -l$, $(k,l = 1,2,...)$, there is a very simple interpretation. Since, if $m \geq k$ and $n \geq l$, for $i < k$ and $j < l$,

$$\binom{m-k}{m-i} = 0 \text{ and } \binom{n-l}{n-j} = 0,$$

we have for the transformed sequence t_{mn}

$$\begin{aligned} t_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m-k}{m-i}\binom{n-l}{n-j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}s_{ij} \\ &= \sum_{i=k}^{m}\sum_{j=l}^{n}\binom{m-k}{m-i}\binom{n-l}{n-j}\Delta_1^{m-i}\Delta_2^{n-j}\mu_{ij}s_{ij} \\ &= \sum_{i=0}^{m-k}\sum_{j=0}^{n-l}\binom{m-k}{i}\binom{n-l}{j}\Delta_1^{m-k-i}\Delta_2^{n-l-j}\mu_{i+k,j+l}s_{i+k,j+l}. \end{aligned}$$

Now, in the light of Theorem 2.2 and Theorem 2.4, we will express the following theorem.

Theorem 2.5. *The matrix transformation (10) is monotonicity-preserving if and only if* $a_{0000} \geq 0$ *and*

$$a_{mnij} = a_{0000}\binom{m+\alpha}{m-i}\binom{n+\beta}{n-j}\int_0^1\int_0^1 u^{i+\alpha}v^{j+\beta}(1-u)^{m-i}(1-v)^{n-j}\,d^2g(u,v),$$

$m,n,i,j = 0,1,2,...,$ *where* $g(u,v)$ *is a function of bounded variation in the unit square with (8) and (9).*

2.1. Equivalence for Double Summability

Let $A = (a_{ni} : n,i = 0,1,...)$ and $B = \left(b_{mj} : m,j = 0,1,...\right)$ be single infinite matrices and c_A and c_B denote the convergence domains of A and B, respectively, $c_A = \left\{x = \{x_i\} : \lim_n \sum_{i=0}^{\infty} a_{ni}x_i \text{ exists}\right\}$. A and B are said to be equivalent if they have the same convergence domain; i.e., if x is a sequence such that $\lim_n \sum_{i=0}^{\infty} a_{ni}x_i$ exists, then $\lim_m \sum_{j=0}^{\infty} b_{mj}x_j$ exists, and conversely. This equivalence will be shown as $A \equiv B$. If $c_A \subseteq c_B$, then we say that B is stronger than A. A lower triangular matrix with nonzero principal diagonal entries is called a triangle. If A and B are conservative triangles, then $c_A \subseteq c_B$ is equivalent to BA^{-1} being conservative. Consequently, if A and B are conservative triangles, then $A \equiv B$ is equivalent to AB^{-1} and BA^{-1} are equivalent to convergence; i.e., they each sum only convergent sequences. It should be noted that this definition of equivalence does not require AB^{-1} and BA^{-1} be regular.

Let $\left(H^{(\alpha)},\mu\right)$ and $\left(H^{(\beta)},\mu\right)$ be two E-J Hausdorff matrices with the same mass function. Assume that $\mu_n^{(\alpha)}.\mu_n^{(\beta)} \neq 0$ for each n, i.e., that the matrices are triangle. Thus

$$\begin{aligned}\left(\left(H^{(\alpha)},\mu\right)\left(H^{(\beta)},\mu\right)^{-1}\right)_{nk} &= \sum_{i=k}^{n}\left(H^{(\alpha)},\mu\right)_{ni}\left(H^{(\beta)},\lambda\right)_{ik} \\ &= \sum_{i=k}^{n}\binom{n+\alpha}{n-i}\Delta^{n-i}\mu_i^{(\gamma)}\binom{i+\beta}{i-k}\Delta^{i-k}\lambda_k,\end{aligned} \tag{11}$$

where $\lambda_n = \frac{1}{\mu_n^{(\beta)}}$. Therefore, to show that the matrices are equivalent, it will be necessary and sufficient to show that the matrix defined by (11), and the corresponding matrix with the roles of α and β interchanged, is equivalent to convergence. The matrix (C,δ) is an ordinary Hausdorff matrix with mass function χ defined by $\chi(t) = 1 - (1-t)^{\delta}$, $0 \leq t \leq 1$.

Quite recently, Albayrak and Rhoades [5] proved the following theorem.

Theorem 2.6. ([5]) *For each $\alpha > 0$, $\beta > 0$, the matrices $\left(C^{(\alpha)},\delta\right)$ and $\left(C^{(\beta)},\delta\right)$ are equivalent in $B(c)$.*

A double infinite matrix A is said to be the product of two single infinite matrices, A' and A'', written $A = A' \odot A''$, if $a_{mnij} = a'_{mi}a''_{nj}$, $m,n,i,j = 0,1,2,....$ If A' and A'' are any two regular matrices, then $A = A'.A''$ is regular for the class of double sequences which are bounded A [3]. Let A', A'', B' and B'' be row finite matrices. Adams [2] has proved that if $A = A' \odot A''$ and $B = B' \odot B''$, then $AB = (A'.B') \odot (A''.B'')$. If A' and A'' both have inverses, which we shall denote by D' and D'', respectively, then $A = A' \odot A''$ has as inverse $A^{-1} = D' \odot D''$.

A double infinite Cesàro matrix (C,γ,δ) is a special case of the double Hausdorff matrix with entries

$$h_{mnij} = \frac{\binom{m+\gamma-i-1}{n-i}\binom{n+\delta-j-1}{n-j}}{\binom{m+\gamma}{\gamma}\binom{n+\delta}{\delta}}, \quad \gamma,\delta \geq 0.$$

We use to denote the corresponding E-J generalizations of the (C,γ,δ).

$\left(C^{(\alpha,\beta)},\gamma,\delta\right)$ has the moment sequence

$$\mu_{mn}^{(\alpha,\beta)} = \int_0^1\int_0^1 u^{m+\alpha}v^{n+\beta}\gamma\delta(1-u)^{\gamma-1}(1-v)^{\delta-1}dudv,$$

where

$$\chi(u,v) = \gamma\delta\int_0^u\int_0^v (1-s)^{\gamma-1}(1-t)^{\delta-1}dsdt.$$

For $i \leq m$ and $j \leq n$,

$$\begin{aligned} h_{mnij}^{(\alpha,\beta)} &= \int_0^1 \int_0^1 \binom{m+\alpha}{m-i}\binom{n+\beta}{n-j} u^{i+\alpha} v^{j+\beta} (1-u)^{m-i} (1-v)^{n-j} \, d\chi(u,v) \\ &= \int_0^1 \int_0^1 \binom{m+\alpha}{m-i}\binom{n+\beta}{n-j} u^{i+\alpha} v^{j+\beta} (1-u)^{m-i} (1-v)^{n-j} \gamma\delta (1-u)^{\gamma-1} (1-v)^{\delta-1} dudv \\ &= \gamma\delta \binom{m+\alpha}{m-i}\binom{n+\beta}{n-j} \int_0^1 \int_0^1 u^{i+\alpha} (1-u)^{m-i+\gamma-1} v^{j+\beta} (1-v)^{n-j+\delta-1} dudv \\ &= \gamma\delta \binom{m+\alpha}{m-i}\binom{n+\beta}{n-j} B(i+\alpha+1, m-i+\gamma) B(j+\beta+1, n-j+\delta) \\ &= \frac{\gamma\Gamma(m+\alpha+1)\Gamma(m-i+\gamma)}{\Gamma(m-i+1)\Gamma(m+\alpha+\gamma+1)} \frac{\delta\Gamma(n+\beta+1)\Gamma(n-j+\delta)}{\Gamma(n-j+1)\Gamma(n+\beta+\delta+1)} \\ &= \frac{E_{m-i}^{\gamma-1} E_{n-j}^{\delta-1}}{E_{m+\alpha}^{\gamma} E_{n+\beta}^{\delta}}. \end{aligned}$$

For the special case $\gamma, \delta = 1$,

$$\left(C^{(\alpha,\beta)}, 1, 1\right) = \begin{cases} \frac{1}{(m+\alpha+1)(n+\beta+1)}, & i \leq m \text{ and } j \leq n, \\ 0, otherwise. \end{cases}$$

is a double E-J Hausdorff matrix.

Since $\left(C^{(\alpha,\beta)}, \gamma, \delta\right)$ is triangle, and can be written as the products $\left(C^{(\alpha,\beta)}, \gamma, \delta\right) = \left(C^{(\alpha)}, \gamma\right) \odot \left(C^{(\beta)}, \delta\right)$, now we feel ready to write the following theorem.

Theorem 2.7. *Let $\left(C^{(\alpha,\beta)}, \gamma, \delta\right)$ and $\left(C^{(\theta,\varphi)}, \gamma, \delta\right)$ be double infinite E-J Cesaro matrices with $\gamma, \delta > 0$. Then $\left(C^{(\alpha,\beta)}, \gamma, \delta\right)$ and $\left(C^{(\theta,\varphi)}, \gamma, \delta\right)$ are equivalent.*

3. Conclusion

In this study, we studied some further results for E-J Hausdorff transformations for double sequences. Therefore the present paper is filled up a gap in the existing literature.

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