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An Extension Problem of a Connectedness Preserving Map between Khalimsky Spaces

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Abstract. The goal of the present paper is to study an extension problem of a connected preserving (for short, *CP*-) map between Khalimsky (*K*- for brevity, if there is no ambiguity) spaces. As a generalization of a *K*-continuous map, for *K*-topological spaces the recent paper [13] develops a function sending connected sets to connected ones (for brevity, an *A*-map: see Definition 3.1 in the present paper). Since this map plays an important role in applied topology including digital topology, digital geometry and mathematical morphology, the present paper studies an extension problem of a *CP*-map in terms of both an *A*-retract and an *A*-isomorphism (see Example 5.2). Since *K*-topological spaces have been often used for studying digital images, this extension problem can contribute to a certain areas of computer science and mathematical morphology.

1. Introduction

In a topological category, it is well known that every continuous map sends connected spaces into connected ones. However, a function mapping connected spaces into connected ones (for brevity, a *CP*-map) need not be continuous. In both pure topology and applied topology, the study of a *CP*-map is meaningful because a *CP*-map is broader than a continuous map and further, in applied topology *CP*-maps can be more powerful than continuous maps. Thus the recent paper [13] develops an *A*-map as a *CP*-map (see Definition 3.1 in the present paper). Besides it turns out that the function is a generalized map of both a *K*-continuous map and a *Khalimsky adjacency* (*KA*- for brevity) map (see Theorem 3.3, Remark 3.4 and Corollary 3.5). In applied topology as well as digital topology and digital geometry, this kind of approach can be also interesting because an *A*-map can contribute to a certain area of computer science including image analysis, image processing, computer graphics, mathematical morphology, etc.

Let **N**, **Z** and **R** be the sets of natural numbers, integers and real numbers, respectively. Consider the set $\{p, q\}$ in a *K*-topological space. Since the notions of connectedness and Khalimsky adjacency of the set are proved to be equivalent [16], these notions have been often used for studying *K*-topological spaces. Hence both a *K*-continuous map and a *KA*-map have been substantially used in *K*-topology [15]. Hereafter, if there

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is no danger of ambiguity we use the terminology *K-space* and *K-adjacency* instead of *K-topological space* and *Khalimsky adjacency*, respectively.

Meanwhile, it turns out that both a K-continuous map and a KA-map have some limitations of geometrical transformations of objects [13]. More precisely, a K-continuous map does not even include all rotations of a K-space with 90° [13] (see the map q in Example 3.2(1) and Figure 1(a)) and cannot support a translation with an odd vector, and a K-adjacency map does not allow a constant map (see also Remark 2.4(3)). Hence the recent paper [13] develops a new map preserving connectedness (see also Definition 3.1), called an A-map, which is broader than both a K-continuous map and a KA-map. This means that none of a KA-map, an A-map and a K-map can be equivalent to the other (see Theorems 3.3 and 3.10, and Corollary 3.11).

Let us now recall an *extension problem* well known in classical topology [2] because it is also very important in applied topology including both digital topology and digital geometry [9, 20]. To be specific, for a map $f : X \to Y$ we say that a map $F : X' \to Y$ is an extension of f if the restriction map F on X, *for brevity* $F|_X$, is equal to f, where $X \subset X'$. Since the study of K-spaces can contribute to digital topology and digital geometry, we need to study an extension problem of a meaningful map between K-spaces. Motivated by the extension problem in [2], the paper [20] studied an extension problem of a continuous function from Khalimsky subspaces to the Khalimsky line. Furthermore, the recent paper [9] studied extension problems of several types of continuities of given maps in [9]. Up to now, although an extension problem has been studied with a continuous map between topological spaces, the present paper studies the issue with CP-maps such as A-maps. To be specific, the paper has a goal of studying an extension of an A-map and investigates its properties in terms of an A-retract. Since an A-map is broader than both a K-continuous map and a KA-map (see Theorem 3.3), it is also different from several continuous maps in [7, 20]. Thus the study of its extension problem has its own feature and further, it is different from those of [9, 20]. Indeed, this approach is an expansion of those of [9, 20] because an A-map is a generalization of both a K-continuous map and a KA-map. The paper has main results in Sections 4 and 5, and the rest of this paper proceeds as follows:

Section 2 recalls basic notions of a K-space and K-adjacency, and their properties. Section 3 investigates some properties of an A-map and an A-isomorphism. Section 4 studies some properties of a KA-map, a KA-isomorphism and an A-isomorphism related to an A-retract. Besides, we compare an A-retract and a k-retract in the computer topological category (or short CTC in [11]). Section 5 deals with an extension problem of an A-map in terms of an A-retract. Section 6 concludes the paper with a summary and a discussion of utilities of an extension problem of an A-map.

2. Khalimsky Adjacency and its Properties

Let us now recall basic notions and terminology for studying K-spaces. For two distinct points a and b in $\mathbf{Z}$ let $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \,|\, a \leq n \leq b\}$ [3]. The *Khalimsky line topology* on $\mathbf{Z}$ is induced by the set $\{[2n-1, 2n+1]_{\mathbf{Z}} : n \in \mathbf{Z}\}$ [1] as a subbase (see also [15, 16]). Furthermore, the usual product topology on $\mathbf{Z}^n$ induced by $(\mathbf{Z}, T)$ is called the *Khalimsky nD space* and is denoted by $(\mathbf{Z}^n, T^n)$. Indeed, $(\mathbf{Z}^n, T^n)$ is a semi-$T_{\frac{1}{2}}$ space [4]. In the present paper each space $X \subset \mathbf{Z}^n$ related to K-topology is considered to be a subspace (X, T^n_X) induced by $(\mathbf{Z}^n, T^n)$. Let us further recall some terms related to the structure of $(\mathbf{Z}^n, T^n)$. A point $x = (x_1, x_2, ..., x_n) \in \mathbf{Z}^n$ is called *pure open* if all coordinates are odd, and it is called *pure closed* if each of the coordinates is even [16]. These two points are called simply *pure* points. The other points in $\mathbf{Z}^n$ are called *mixed*. In the spaces of Figures 1, 2, 3, 5, 6 and 7, a black jumbo dot means a pure open point and further, the symbols ■ and • mean a pure closed point and a mixed point, respectively. We say that a point x is open if $SN(x) = \{x\}$, where $SN(x)$ means the smallest open neighborhood of $x \in \mathbf{Z}^n$.

To represent a K-adjacency neighbor, we need to recall the k-adjacency relation of $\mathbf{Z}^n$ [5, 6] which is a generalization of the digital k-connectivitiy on 2D and 3D digital spaces [19, 22], as follows:

For a natural number m, $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n,$$

are $k(m,n)$-(k-, for brevity)adjacent if

$$\text{at most } m \text{ of their coordinates differs by } \pm 1, \text{ and all others coincide.} \tag{2.1}$$

The number of such points is [8, 10]

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{2.2}$$

For consistency with the nomenclature "4-adjacent" and "8-adjacent" well established in the context of 2-dimensional integer grids, we will say that two points $p, q \in \mathbf{Z}^n$ are *k-adjacent* if they satisfy the property (2.1), *i.e.* $k := k(m,n)$ of (2.2) [5] (see also [8, 10]).

We say that a *k-path* from x to y in X is a sequence $(x = x_0, x_1, ..., x_{m-1}, x_m = y)$ in X such that each point x_i is k-adjacent to x_{i+1} for $m \geq 1$ and $1 \leq i \leq m$. The number m is called the *length* of this path [19]. If $x_0 = x_m$, then the k-path is said to be *closed*. Besides, if a k-path (or a k-sequence) is called *simple* if it satisfies the following: the points x_i and x_j of a k-path are k-adjacent if and only if $|\, i - j\,| = 1, i, j \in [0,m]_{\mathbf{Z}}$ [19].

By using the adjacency of (2.2), we can represent the *digital k-neighbor* of p in $\mathbf{Z}^n$ as the set $N_k(p) := \{q \,|\, p \text{ is } k\text{-adjacent to } q\}$ and further, $N_k^*(p) := N_k(p) \cup \{p\}$ [22]. More generally, for a digital space $X \subset \mathbf{Z}^n$ with a k-adjacency (X,k), the (digital) k-neighborhood of $x_0 \in X$ with radius ε is defined on X to be the following subset of X [5] (see also [6]):

$$N_k(x_0, \varepsilon) = \{x \in X \,|\, l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \tag{2.3}$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$.
For instance, for a set $X \subset \mathbf{Z}^n$ $N_k(x,1)$ can be represented by $N_k^*(x) \cap X$ [8]. To study K-adjacency of $(X, T_X^n) \subset (\mathbf{Z}^n, T^n)$, the present paper will often use $N_k(p)$ and $N_k(x_0, \varepsilon)$, where $p \in \mathbf{Z}^n$ and the number k is the digital k-connectivity of $\mathbf{Z}^n$ from (2.2).

In $(\mathbf{Z}^n, T^n)$, we say that two distinct points x and y are *(Khalimsky) adjacent* if $y \in SN(x)$ or $x \in SN(y)$ [16], where $SN(p)$ stands for a smallest open neighborhood of the point $p \in (\mathbf{Z}^n, T^n)$. More generally, for a point $p \in \mathbf{Z}^n$ $A(p)$ is represented [13] as follows:

$$A(p) = \{q \,|\; q \in SN(p) \text{ or } p \in SN(q)\} \tag{2.4}$$

Example 2.1. *[13, 21] We now characterize $A(p)$, as follows:*

If $n = 1$, then for a point $p \in \mathbf{Z}$ $A(p) = \{p-1, p+1\}$.
If $n = 2$, then for a point $p \in \mathbf{Z}^2$

$$\begin{cases} A(p) := N_4(p) \text{ if } p \text{ is a mixed point, and} \\ A(p) := N_8(p) \text{ if } p \text{ is a pure point.} \end{cases}$$

If $n \geq 3$ and $p \in \mathbf{Z}^n$ is a pure point, then $A(p) := N_{3^n-1}(p)$, and if a point $p := (p_i)_{i \in [1,n]_{\mathbf{Z}}} \in \mathbf{Z}^n$ is a mixed point, then according to the component of the given coordinates p_i, $A(p)$ can be represented by (2.4).

For a space $(X, T_X^n) := X$ we now define the notion of a *K-adjacency relation* of a point $p \in X$ as follows:

Definition 2.2. *[13] For $(X, T_X^n) := X$ put $A_X(p) := A(p) \cap X$. We say that two distinct points $p, q \in X$ are K-adjacent if $q \in A_X(p)$ or $p \in A_X(q)$.*

In view of Definition 2.2, K-adjacency holds only the *symmetric relation* without the reflexive relation. Hereafter, we consider (X, T_X^n) with the K-adjacency of Definition 2.2.

Let $f : (X, T_X^{n_0}) := X \to (Y, T_Y^{n_1}) := Y$ be a map. Then we can represent a KA-map at the point $p \in X$ [13] as follows:

$$\left\{ \begin{array}{l} \text{if for every } x \neq x' \text{ such that } x' \in SN(x) \text{ or } x \in SN(x') \\ \text{it holds that } f(x) \neq f(x'), \text{ and} \\ \text{one of the following is true } f(x') \in SN(f(x)) \text{ or } f(x) \in SN(f(x')). \end{array} \right\} \tag{2.5}$$

If f is a KA-map at every point $x \in X$, then f is called a KA-map in X.
By using the K-adjacency, we define the following terminology:

Definition 2.3. *[13] For a space $(X, T^n_X) := X$ we define the following:*
(1) Two distinct points $x, y \in X$ are called KA-connected if there is an injective sequence (or path) $(x_i)_{i\in[0,m]_{\mathbf{Z}}}$ on X with $\{x_0 = x, x_1, ..., x_m = y\}$ such that x_i and x_{i+1} are K-adjacent, $i \in [0, m-1]_{\mathbf{Z}}, m \geq 1$. This sequence is called a KA-path. Furthermore, the number m is called the length of this KA-path. Furthermore, a KA-path is called a closed KA-curve if $x_0 = x_m$.
(2) A simple KA-path on X is a KA-path such that x_i and x_j are K-adjacent if and only if $|\,i - j\,| = 1$. Furthermore, we say that a simple closed KA-curve with m elements $(x_i)_{i\in[0,m]_{\mathbf{Z}}}$ is a simple KA-path with $x_0 = x_m$ such that x_i and x_j are K-adjacent if and only if either $j = i + 1(mod\, m)$ or $i = j + 1(mod\, m), m \geq 4$. Hereafter, let $SC^{n,l}_{KA} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$ denote a simple closed KA-curve with l elements in $\mathbf{Z}^n, n \in \mathbf{N} - \{1\}, l \geq 4$.

Although both a K-continuous map and a KA-map play important roles in studying spaces (X, T^n_X), as we said in the previous part, they have some limitations of a transformation of objects as follows:

Remark 2.4. *(1) Consider the self-map $f : SC^{n,l}_{KA} \to SC^{n,l}_{KA}$ given by $f(c_i) = c_{i+1(mod\, l)}$, where $SC^{n,l}_{KA} := (c_i)_{i\in[0,l]_{\mathbf{Z}}}$. Then f need not be a K-continuous map [13].*
(2) Let us consider the map $g : (\mathbf{Z}, T) \to (\mathbf{Z}, T)$ given by $g(t) = t + (2n + 1), n \in \mathbf{Z}$ which is a parallel translation with an odd vector. Then g cannot be a K-continuous map [13].
(3) A KA-map does not allow a constant map with the following reason [13]: when we consider a K-adjacency relation of two points p and q in $\mathbf{Z}^n$, we always assume that the given two points are distinct.

In view of Remark 2.4, we strongly need to develop another map which can overcome the limitations suggested in Remark 2.4 (see Definition 3.1).

3. Some Properties of a *KA*-Map, a *KA*-Isomorphism, an *A*-Map and an *A*-Isomorphism

The recent paper [13] develops the notion of an *A-map* (see Definition 3.1 in the present paper) which is broader than both a K-continuous map and a KA-map. Besides, it establishes the notion of an *A-isomorphism* (see Definition 3.8 in the present paper) for K-spaces $(X, T^n_X) := X$. In relation to the establishment of an A-map, we will use the following K-adjacency neighborhood of a point $p \in X$. For a point $p \in X$, we define a KA-neighborhood of p to be the following set [21]

$$A_X(p) \cup \{p\} := AN_X(p). \tag{3.1}$$

Hereafter, we will use $AN(p)$ instead of $AN_X(p)$ if there is no danger of ambiguity. Indeed, the notions of an A-map and an A-isomorphism were developed in terms of $AN(p)$ [13] as follows. For $(X, T^n_X) := X$ and each point $x \in X$, since for every $x \in X$ there is always $AN(x) \subset X$, we establish a map preserving $AN(x)$ into $AN(f(x))$. In other words, this map sends K-connected sets containing a point x into those of $f(x)$, *i.e.* it is a kind of CP-map between K-spaces. Besides, it need not be a continuous map between K-spaces (see the map q in Example 3.2(1) and Theorem 3.3), which can play an important role in studying K-spaces.

Definition 3.1. *[13] For two spaces $(X, T^{n_0}_X) := X$ and $(Y, T^{n_1}_Y) := Y$, we say that a function $f : X \to Y$ is an A-map at a point $x \in X$ if*

$$f(AN(x)) \subset AN(f(x)).$$

Furthermore, we say that a map $f : X \to Y$ is an A-map if the map f is an A-map at every point $x \in X$.

Example 3.2. *(1) in Figure 1(a), let us consider the spaces $A := (a_i)_{i\in[0,3]_{\mathbf{Z}}}$ such that each a_i is a mixed point, $B := (b_i)_{i\in[0,3]_{\mathbf{Z}}}$ and $C := (c_i)_{i\in[0,3]_{\mathbf{Z}}}$. Assume the two maps $h : A \to B$ given by $h(a_i) = b_i$, and $j : A \to C$ given by $j(a_i) = c_i, i \in [0, 3]_{\mathbf{Z}}$. Then they are bijective A-maps because every point $a_i \in A, i \in [0, 3]_{\mathbf{Z}}$ has $SN(a_i) = \{a_i\} = AN(a_i)$. But each of the inverse maps can be neither an A-map nor a K-continuous map. Meanwhile, consider the map $q : B \to C$*

given by $q(b_i) = c_i, i \in [0,3]_{\mathbf{Z}}$. *While* q *is an A-map, it cannot be a K-continuous map (see the points* b_0 *and* b_2 *in B) [13].*

(2) Considering the maps in Remark 2.4, we observe that an A-map has strong merits of studying digital images from the viewpoint of digital geometry as follows: each of the following maps is an A-map.

$$\begin{cases} (2-1)\ f : SC_{KA}^{n,l} \to SC_{KA}^{n,l} \text{ given by } f(c_i) = c_{i+1(mod\, l)}, \text{ where } SC_{KA}^{n,l} := (c_i)_{i\in[0,l]_{\mathbf{Z}}};\\ (2-2)\ g : (\mathbf{Z}, T) \to (\mathbf{Z}, T) \text{ given by } g(t) = t + n, n \in \mathbf{Z}\,; \text{ and}\\ (2-3)\ c : (X, T_X^{n_0}) \to (Y, T_Y^{n_1}) \text{ which is a constant map.}\end{cases}$$

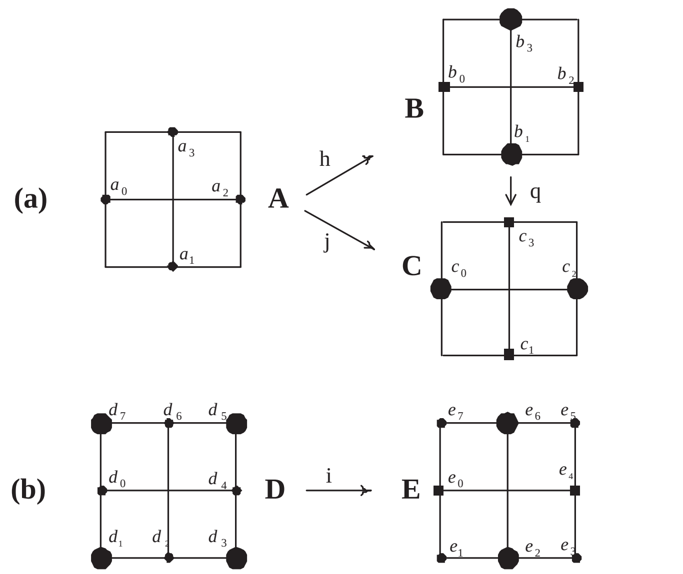

Figure 1: Explanation of a K-continuous map, an A-map and an A-isomorphism

Hereafter, the present paper will deal with only connected K-spaces.

Using spaces $(X, T_X^n) := X$ and A-maps, we can establish a K-adjacency category denoted by KAC [13] in terms of the following two sets.

$$\begin{cases} \bullet \text{ A set of spaces } (X, T_X^n) := X \text{ as objects of } KAC \text{ denoted by } Ob(KAC); \text{ and}\\ \bullet \text{ A set of } A\text{-maps between all pairs of elements in } Ob(KAC) \text{ as morphisms.}\end{cases}$$

Let KTC denote the K-topological category consisting of the following two sets [7]:

$$\begin{cases} \bullet \text{ A set of spaces } (X, T_X^n) := X \text{ as objects of } KTC \text{ denoted by } Ob(KTC); \text{ and}\\ \bullet \text{ A set of } K\text{-continuous maps between all pairs of elements in } Ob(KTC) \text{ as morphisms.}\end{cases}$$

The following theorem and corollary show that an A-map is a generalization of both a K-continuous map and a KA-map.

Theorem 3.3. *[13] Let* $f : (X, T_X^{n_0}) := X \to (Y, T_Y^{n_1}) := Y$ *be a map.*
A K-continuous map implies an A-map. But the converse does not hold.

By Theorem 3.3, we can observe the following:

Remark 3.4. *Let $f : (X, T_X^{n_0}) := X \to (Y, T_Y^{n_1}) := Y$ be a map. Although the given spaces X and Y are K-connected, an A-map does not imply a K-continuous map. In order to verify this assertion, consider the spaces B and C in Figure 1(a). Then, as discussed in Example 3.2(1), they are K-connected. Consider the map $q : B \to C$ given by $q(b_i) = c_i, i \in [0,3]_{\mathbf{Z}}$ in Figure 1(a) so that q is an A-map. But it cannot be a K-continuous map.*

In view of Remark 2.4 (3), we obtain the following:

Corollary 3.5. *Let $f : (X, T_X^{n_0}) := X \to (Y, T_Y^{n_1}) := Y$ be a map. A KA-map implies an A-map. But the converse does not hold.*

By Theorem 3.3, Remark 3.4, Corollary 3.5 and the property (2.5), we conclude that none of a *KA*-map, an *A*-map and a *K*-map can be equivalent to the other.

Definition 3.6. *We say that $f : (X, T_X^{n_0}) := X \to (Y, T_Y^{n_1}) := Y$ is a KA-isomorphism if f is a bijective KA-map and f^{-1} is a KA-map.*

To study simple closed *K*-continuous curves in $\mathbf{Z}^n$, we need to recall a *K*-homeomorphism as follows:

Definition 3.7. *[15] For two spaces $(X, T_X^{n_0}) := X$ and $(Y, T_Y^{n_1}) := Y$, a map $h : X \to Y$ is called a K-homeomorphism if h is a K-continuous bijection, and $h^{-1} : Y \to X$ is K-continuous.*

In $(\mathbf{Z}^n, T^n)$ we say that a simple closed *K*-curve with *l* elements in $\mathbf{Z}^n$ is a path $(x_i)_{i\in[0,l-1]_{\mathbf{Z}}} \subset \mathbf{Z}^n, l \geq 4$ that is *K*-homeomorphic to a quotient space of a Khalimsky line interval $[a,b]_{\mathbf{Z}}$ in terms of the identification of the only two end points *a* and *b* [18], where both the numbers *a* and *b* in $[a,b]_{\mathbf{Z}}$ are even numbers or odd numbers. We denote it by $SC_K^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$.

Thus we can say that a *K*-topological circle in $\mathbf{Z}^n$ is a finite space in $\mathbf{Z}^n$ which is locally *K*-homeomorphic to the Khalimsky line $(\mathbf{Z}, T)$.

Using an *A*-map, we establish the following notion:

Definition 3.8. *[13] For two spaces $(X, T_X^{n_0}) := X$ and $(Y, T_Y^{n_1}) := Y$, a map $h : X \to Y$ is called an A-isomorphism if h is a bijective A-map (for brevity, A-bijection) and if $h^{-1} : Y \to X$ is an A-map.*

Example 3.9. *(1) The map $q : B \to C$ in Example 3.2(1) is an A-isomorphism.*

(2) Consider the spaces D and E in Figure 1(b). Then they cannot be A-isomorphic to each other. More precisely, every point $d_i \in D$ has $AN(d_i) = \{d_{i-1(mod\,8)}, d_i, d_{i+1(mod\,8)}\}, i \in [0,7]_{\mathbf{Z}}$. But the points $e_i \in E, i \in \{0,2,4,6\}$ have $AN(e_0) = \{e_6, e_7, e_0, e_1, e_2\}$, $AN(e_2) = \{e_0, e_1, e_2, e_3, e_4\}$, etc. Thus, owing to the four points e_0, e_2, e_4 and e_6, the spaces D and E cannot be A-isomorphic to each other. More precisely, while there is a bijective A-map $i : D \to E$ given by $i(d_i) = e_i, i \in [0,7]_{\mathbf{Z}}$, the inverse map of i cannot be an A-map (see the points e_0, e_2, e_4 and e_6).

In Definitions 3.6, 3.7 and 3.8, we denote by $X \approx_{KA} Y$, $X \approx_K Y$ and $X \approx_A Y$ a *KA-isomorphism*, a *K-homeomorphism* and an *A-isomorphism*, respectively.

According to Theorem 3.3 and Remark 3.4, we obtain the following:

Theorem 3.10. *[13] Let $f : (X, T_X^{n_0}) := X \to (Y, T_Y^{n_1}) := Y$ be a map. If f is a K-homeomorphism, then it is an A-isomorphism. But the converse does not hold.*

For instance, the map $q : B \to C$ in Example 3.2 shows that an *A*-isomorphism need not imply a *K*-homeomorphism. While an *A*-isomorphism supports a constant map, a *KA*-isomorphism cannot comprise a constant map. Thus we obtain the following:

Corollary 3.11. *Let $f : (X, T_X^{n_0}) := X \to (Y, T_Y^{n_1}) := Y$ be a map. If f is a KA-isomorphism, then it is an A-isomorphism. But the converse does not hold.*

Comparing an *A*-map with both a *K*-continuous map and a *KA*-map, according to Theorem 3.3, Remark 3.4 and Corollary 3.5 (resp. Theorem 3.10 and Corollary 3.11), we can observe some utilities of an *A*-map (resp. an *A*-isomorphism).

4. Some Properties of an *A*-Retract Related to an *A*-Isomorphism

Compared with the several continuous maps in [7, 20], since an *A*-map has strong merits of studying *K*-spaces, an extension problem of an *A*-map can contribute to the study of *K*-spaces. Based on several continuities of maps in [7], some properties of several retracts have been investigated in [2, 3, 9, 20]. However, as discussed in Section 3, since *KAC* has its own features and benefits of studying *K*-spaces, we need to establish the notion of an *A*-retraction, as follows:

Definition 4.1. *[13] In KAC we say that an A-map* $r : (X', T^n_{X'}) \to (X, T^n_X)$ *is an A-retraction if*
(1) (X, T^n_X) *is a subspace of* $(X', T^n_{X'})$ *and*
(2) $r(a) = a$ *for all* $a \in (X, T^n_X)$.
Then we say that (X, T^n_X) *is an A-retract of* $(X', T^n_{X'})$. *Furthermore, we say that the point* $a \in X' \setminus X$ *is A-retractable.*

In view of Definition 4.1, it is clear that an *A*-retract holds the reflexivity and the transitivity because this retract was formulated in terms of an *A*-map instead of a *K*-map or a *KA*-map. Unlike the *A*-retraction of Definition 4.1, a retraction in *KTC* was established [9], as follows: we say that a *K*-continuous map $r : (X', T^n_{X'}) \to (X, T^n_X)$ is a *K*-retraction [9] if
(1) (X, T^n_X) is a subspace of $(X', T^n_{X'})$, and
(2) $r(a) = a$ for all $a \in (X, T^n_X)$.
Then we say that (X, T^n_X) is a *K*-retract of $(X', T^n_{X'})$. Furthermore, we say that the point $a \in X' \setminus X$ is *K*-retractable.

By Theorem 3.3 and Remark 3.4, it turns out that a *K*-retraction implies an *A*-retraction. But the converse does not hold. Thus the notion of an *A*-retract can be useful for studying *K*-spaces, as follows:

Example 4.2. *Consider the space* $X' := \{x_i \mid i \in [0, 14]_{\mathbf{Z}}\}$ *in Figure 2(a) and assume* $X = X' \setminus \{x_4, x_6, x_7, x_8, x_{10}, x_{12}, x_{13}\}$. *Assume the map* $r : X' \to X$ *given by* $r(x_4) = x_3, r(\{x_6, x_7\}) = \{x_5\}, r(\{x_8, x_{10}, x_{12}\}) = \{x_9\}, r(x_{13}) = x_{11}$ *and* $r(x_i) = x_i, i \in [0, 14]_{\mathbf{Z}} \setminus \{4, 6, 7, 8, 10, 12, 13\}$. *Then the map r is an A-retraction from X′ to X. But r cannot be a K-retraction because it cannot be a K-continuous map at the point* x_7. *Thus we speak out that an A-retraction is more flexible than a K-retraction.*
Similarly, we can observe that the description in Figure 2(b) is an A-retract which is not a K-retract.

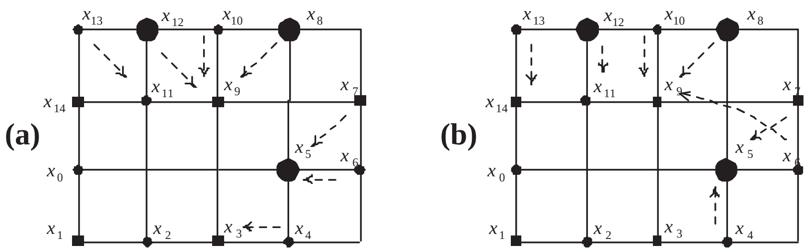

Figure 2: Configuration of an *A*-retract

To study a digital retract, the recent paper [9] studied a special kind of retract for studying *K*-spaces (see Definition 4.7). Indeed, the retract of Definition 4.7 is partially related to the present *A*-retract (see Definition 4.7 and Remark 4.9). It is represented by a special kind of continuity of Definition 4.4 and the neighborhood of Definition 4.3. Thus we need to review these notions and compare an *A*-retract with the retract of Definition 4.7. In view of the continuity of Definition 4.4 and the retract of Definition 4.7, we need to recall a Khalimsky topological *k*-neighborhood which can be used for establishing another continuity for *K*-spaces instead of both a *K*-continuous map and an *A*-map. Consider a space (X, T^n_X) with a digital *k*-connectivity which is denoted by $(X, k, T^n_X) := X_{n,k}$ [11].

Definition 4.3. *[5] (see also [7]) Consider a space $X_{n,k} := X$, $x, y \in X$, and $\varepsilon \in \mathbf{N}$.*

(1) A subset V of X is called a Khalimsky topological neighborhood of x if there exists the smallest open set $O_x \in T_X^n$ such that $x \in O_x \subseteq V$, as usual.

(2) If a digital k-neighborhood $N_k(x, \varepsilon)$ is a K-topological neighborhood of x in (X, T_X^n), then this set is called a Khalimsky topological k-neighborhood of x with radius ε and we use the notation $N_k^*(x, \varepsilon)$.

According to Definition 4.3, in $X_{n,3^n-1}$ it is clear that

$$N^*_{3^n-1}(x, 1) \text{ is equal to } N_{3^n-1}(x, 1) \text{ as a set.} \tag{4.1}$$

In view of Remark 2.4, to study the spaces $X_{n,k}$, we will use another continuity relating to the digital connectivity which is different from the digital continuity in [22], as follows:

Definition 4.4. *[5](see also [7, 11]) For two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$ we say that a function $f : X \to Y$ is (k_0, k_1)-continuous at a point $x \in X$ if $f(N^*_{k_0}(x, r)) \subset N^*_{k_1}(f(x), s)$, where the number r is the least element of $\mathbf{N}$ such that $N^*_{k_0}(x, r)$ contains an open set including the point x and s is the least element of $\mathbf{N}$ such that $N^*_{k_1}(f(x), s)$ contains an open set including the point $f(x)$.*
Furthermore, we say that a map $f : X \to Y$ is (k_0, k_1)-continuous if the map f is (k_0, k_1)-continuous at every point $x \in X$.

Let us consider the *computer topological category*, denoted by *CTC*, consisting of the following two sets [7]:

$$\begin{cases} \bullet \text{ A set of } K\text{-connected spaces } X_{n,k} \text{ as objects of } CTC; \text{ and} \\ \bullet \text{ A set of } (k_0, k_1)\text{-continuous maps of Definition 4.4 as morphisms.} \end{cases}$$

While the papers [7, 9] studied the category *CTC* with *Ob(CTC)* including *K*-connected or *K*-disconnected spaces, the present paper deals with only *K*-connected spaces.

To compare a (k_0, k_1)-continuous map in *CTC* with an *A*-map, let us recall $A(p)$ and $AN(p)$ of (2.2) and (3.1). Based on this review, in Figure 3(a) consider a mixed point $p \in \mathbf{Z}^2$. Then we can observe that each point t in $N_8(p) \setminus AN(p)$ is a mixed point (for this, see the point t in Figure 3(a)) and further, there is a pure point r, *e.g.* x_2 or x_3 in $AN(p)$ such that t is *K*-adjacent to r and $t \in N_4(r)$.
As another example, consider a mixed point in $\mathbf{Z}^3$ such as p and q in (b) and (c) of Figure 3, respectively. Then every point $t \in N_{26}(p) \setminus AN(p)$ (resp. $t \in N_{26}(q) \setminus AN(q)$) is a mixed point (see the point t in (b) and (c) of Figure 3) and further, there is a pure point $r \in AN(p)$, *e.g.* x_8, x_{10} (resp. $r \in AN(q)$) such that t and r are *K*-adjacent to each other and $t \in N_6(r)$.
In general, we can easily obtain the following property:

Lemma 4.5. *For a point $p \in \mathbf{Z}^n, n \in \{2, 3\}$ if p is a mixed point, then each point $t \in N_{3^n-1}(p) \setminus AN(p)$ is a mixed point and further, there is a pure point $r \in AN(p)$ such that t and r are K-adjacent to each other and $t \in N_{2n}(r)$.*

Reminding (2.2) and (4.1), for a pure point let us now compare a (k_0, k_1)-continuous map in *CTC* with an *A*-map (see Figure 3). In the following theorem, since the assertion is trivial in the case of the dimensional Kahlilmsky line, we study it for nD *K*-spaces, $n \in \{2, 3\}$.

Theorem 4.6. *(1) Assume a map $f : X_{n_0,k_0} := X \to Y_{n_1,k_1} := Y$ in CTC, where $n_i \in \{2, 3\}, i \in \{0, 1\}$. Let $k_i = 3^{n_i} - 1, i \in \{0, 1\}$, and let p and $f(p)$ be pure points in both X and Y, respectively. If f is a $(3^{n_0} - 1, 3^{n_1} - 1)$-continuous map at p, then f is an A-map at p. The converse also holds.*

In (1), if the hypothesis of the pure points p and $f(p)$ is omitted, then the assertion need not hold. More precisely, we obtain the following:
(2) Let p be a pure point and $f(p)$ a mixed point. If f is $(3^{n_0} - 1, 3^{n_1} - 1)$-continuous at p, then f need not be an A-map at p. Conversely, if f is an A-map at p, then f is $(3^{n_0} - 1, 3^{n_1} - 1)$-continuous at p.

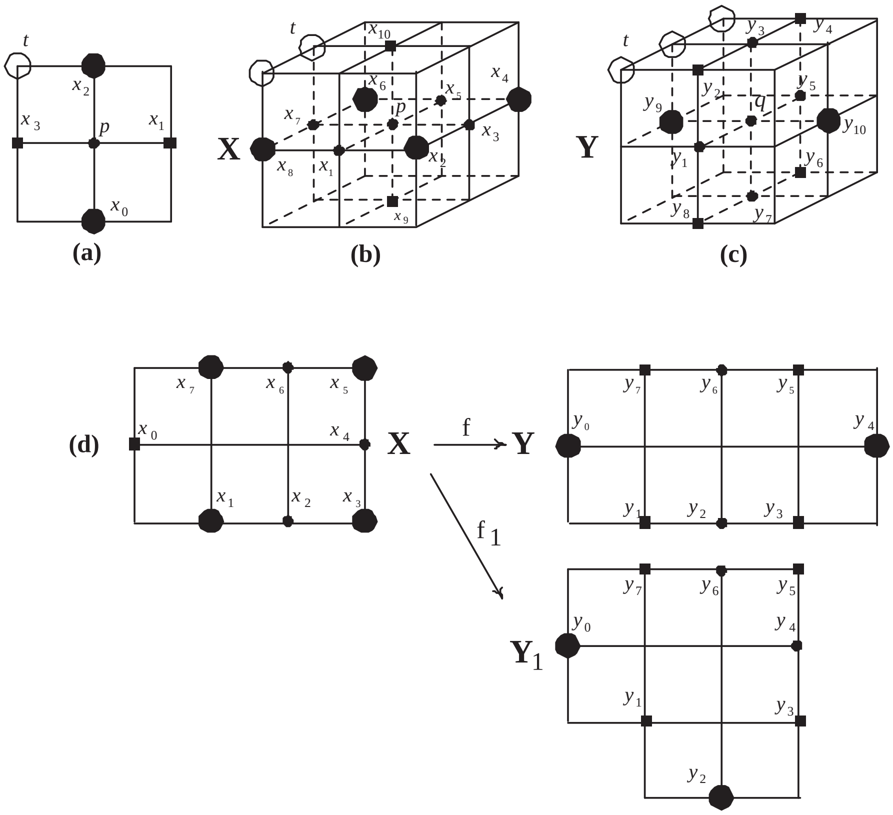

Figure 3: Comparison between an A-map and an 8-continuous map in $\mathbf{Z}^2$ and $\mathbf{Z}^3$

(3) Let p be a mixed point and $f(p)$ a pure point. If f is $(3^{n_0}-1, 3^{n_1}-1)$-continuous at p, then f is an A-map at p. Conversely, if f is an A-map at p, then f need not be $(3^{n_0}-1, 3^{n_1}-1)$-continuous at p.

(4) Let p be a mixed point and $f(p)$ a mixed point. If f is $(3^{n_0}-1, 3^{n_1}-1)$-continuous at p, then f need not be an A-map at p. Conversely, if f is an A-map at p, then f need not be $(3^{n_0}-1, 3^{n_1}-1)$-continuous at p either.

Proof: (1) Since the given spaces X and Y are K-connected, each pure point $x \in X$ (resp. $y \in Y$) has $N^*_{3^{n_0}-1}(x,1) \subset X$ (resp. $N^*_{3^{n_1}-1}(y,1) \subset Y$) such that $N^*_{3^{n_0}-1}(x,1) = AN(x)$ (resp. $N^*_{3^{n_1}-1}(y,1) = AN(y)$). Thus, for the pure points p and $f(p)$ a $(3^{n_0}-1, 3^{n_1}-1)$-continuous map $f : X \to Y$ at p is equivalent to an A-map at p because $AN(p) = N^*_{3^{n_0}-1}(p,1)$ and $AN(f(p)) = N^*_{3^{n_1}-1}(f(p),1)$.

(2) With the hypothesis, although $AN(p) = N^*_{3^{n_0}-1}(p,1)$, by (3.1) we obtain that $AN(f(p))$ need not be equal to $N^*_{3^{n_1}-1}(f(p),1)$ and $AN(f(p)) \subset N^*_{3^{n_1}-1}(f(p),1)$. Then we may have a point $r \in AN(p)$ such that $f(r) \in N^*_{3^{n_1}-1}(f(p),1) \setminus AN(f(p))$ instead of $f(r) \in AN(f(p)$, which implies that the $(3^{n_0}-1, 3^{n_1}-1)$-continuity of f at p need not support f as an A-map at p.
Conversely, with the hypothesis, we prove that f is $(3^{n_0}-1, 3^{n_1}-1)$-continuous at p. Since f is an A-map at the pure point p and $f(p)$ is a mixed point, we can observe that (see Example 2.1)

$$AN(f(p)) \subset N^*_{3^{n_1}-1}(f(p),1). \tag{4.2}$$

Owing to (4.2), the A-map property of f and the fact that $AN(p) = N^*_{3^{n_0}-1}(p,1)$, we obtain the following:

$$f(AN(p)) = f(N^*_{3^{n_0}-1}(p,1)) \subset AN(f(p)) \subset N^*_{3^{n_1}-1}(f(p),1), \tag{4.3}$$

which implies that the map f is also a $(3^{n_0}-1, 3^{n_1}-1)$-continuous map at p.

(3) Since the point p is a mixed point and $f(p)$ is a pure point, we can observe that

$$AN(p) \subset N^*_{3^{n_0}-1}(p,1) \text{ and } AN(f(p)) = N^*_{3^{n_1}-1}(f(p),1).$$

By (3.1) we obtain that $AN(p)$ need not be equal to $N^*_{3^{n_0}-1}(p,1)$. Owing to the $(3^{n_0}-1, 3^{n_1}-1)$-continuity of f at p, since $f(N^*_{3^{n_0}-1}(p,1)) \subset N^*_{3^{n_1}-1}(f(p),1)$, we obviously obtain that $f(AN(p)) \subset N^*_{3^{n_1}-1}(f(p),1)$, which implies that f is an A-map at p.
Conversely, take a point $t \in N^*_{3^{n_0}-1}(p,1) \setminus AN(p)$. By Lemma 4.5, the point t is a mixed point and further, there is a pure point $r \in AN(p)$ which is K-adjacent to t and $t \in N_{2n_0}(r,1)$. Then $f(t)$ need not be an element of $N^*_{3^{n_1}-1}(f(p),1)$. Thus f cannot be a $(3^{n_0}-1, 3^{n_1}-1)$-continuous map at p.
For instance, in Figure 3(d) consider the A-map $f : X \to Y$ given by $f(x_i) = y_i, i \in [0,7]_{\mathbf{Z}}$. Let us now examine if f is an 8-continuous map at the mixed point $x_4 \in X$. Take a point x_2 or x_6 in $N^*_8(x_4,1) \setminus AN(x_4)$. For convenience, let us investigate only the point x_2. Indeed, the point x_2 is a mixed point and further, by Lemma 4.5, there is a pure point $x_3 \in AN(x_4)$ which is K-adjacent to x_2 and $x_2 \in N_4(x_3,1)$. Then we can observe that $f(x_2) \notin N^*_8(f(x_4),1)$, which implies that f cannot be an 8-continuous map at x_4.

(4) Since the points p and $f(p)$ are mixed points, $AN(p) \subset N^*_{3^{n_0}-1}(p,1)$ and $AN(f(p)) \subset N^*_{3^{n_1}-1}(f(p),1)$. Owing to the $(3^{n_0}-1, 3^{n_1}-1)$-continuity of f at p, although $f(N^*_{3^{n_0}-1}(p,1)) \subset N^*_{3^{n_1}-1}(f(p),1)$, we obviously obtain that $f(AN(p))$ need not be a subset of $AN(f(p))$. Indeed, owing to the $(3^{n_0}-1, 3^{n_1}-1)$-continuity of f at p, the map f may map a point $x \in AN(p)$ into

$$f(x) \in N^*_{3^{n_1}-1}(f(p),1) \setminus AN(f(p)),$$

which implies that f is not an A-map at p.
Conversely, from the hypothesis although we have $f(AN(p)) \subset AN(f(p))$, it need not support the following property:

$$f(N^*_{3^{n_0}-1}(p,1)) \subset N^*_{3^{n_1}-1}(f(p),1).$$

To be specific, take a point $t \in N^*_{3^{n_0}-1}(x,1) \setminus AN(x)$. Then the point t is a mixed point and further, by Lemma 4.5, there is a pure point $r \in AN(x)$ which is K-adjacent to t and $t \in N_{2n_0}(r,1)$. Then $f(t)$ need not be an element of $N^*_{3^{n_1}-1}(f(x),1)$, which means that f cannot be a $(3^{n_0}-1, 3^{n_1}-1)$-continuous map at x.
For instance, in Figure 3(d) consider the A-map $f_1 : X \to Y_1$ given by $f_1(x_i) = y_i, i \in [0,7]_{\mathbf{Z}}$. Let us now examine if the given map f_1 is an 8-continuous map at the mixed point $x_4 \in X$. Take a point $x_2 \in N^*_8(x_4,1) \setminus AN(x_4)$. By Lemma 4.5, for the mixed point x_2 there is a pure point $x_3 \in AN(x_4)$ that is K-adjacent to x and $x_2 \in N_4(x_3,1)$. Then $f_1(x_2) = y_2 \notin N^*_8(f_1(x_4),1)$. Thus the map f_1 need not be an 8-continuous map at x_4.
In view of this example, we can observe that an A-map need not be a $(3^{n_0}-1, 3^{n_1}-1)$-continuous map at the mixed point x. □

The assertion of Theorem 4.6 can be similarly considered in nD K-spaces, $n \geq 4$.

By using the method similar to that of Definition 4.1, in *CTC* we obtain the following:

Definition 4.7. *[9] In CTC, we say that a k-continuous map $r : X'_{n,k} \to X_{n,k}$ is a k-retraction if*
(1) $X_{n,k} \subset X'_{n,k}$, and
(2) $r(x) = x$ for all $x \in X_{n,k}$.
Then we say that $X_{n,k}$ is a k-retract of $X'_{n,k}$.

Example 4.8. *In Figure 5, assume the map $r : X' \to X$ given by $r(x_3) = x_2, r(x_1) = x_0, r(x_9) = x_8, r(x_7) = x_6$ and $r(x_i) = x_i, i \in \{0,2,4,5,6,8\}$. Then X is an 8-retract of X′ in CTC. However, this map r cannot be a 4-retraction in CTC.*

In view of Theorem 4.6(1) and (4), depending on a point $x \in X_{n,k}$ in *CTC*, we can obtain a relation between an A-retract and a (3^n-1)-retract as follows:

p	$f(p)$	$(3^{n_0}-1, 3^{n_1}-1)$-continuity of f implies an A-map	A-map implies $(3^{n_0}-1, 3^{n_1}-1)$-continuity of f
pure point	pure point	Yes	Yes
pure point	mixed point	No	Yes
mixed point	pure point	Yes	No
mixed point	mixed point	No	No

Figure 4: Comparison between a $(3^{n_0}-1, 3^{n_1}-1)$-continuous map and an A-map

Remark 4.9. *(1) At a pure point $x \in X_{n,k}$, a $(3^n - 1)$-retraction in CTC is equivalent to an A-retract.*
(2) At a mixed point $x \in X_{n,k}$, none of a $(3^n - 1)$-retraction in CTC and an A-retract implies to the other.

In terms of the following remark, we can observe some difference between the present k-retract in *CTC* and the A-retract in *KAC*, where $k \neq 3^n - 1$.

Remark 4.10. *Consider the K-connected space $X' := \{x_i \,|\, i \in [0, 9]_{\mathbf{Z}}\}$ in Figure 5. Let us compare a k-retract in CTC with an A-retract at $x_0 \in X'$ in Example 4.8 (see Figure 5). Recall that the smallest open set of x_0, denoted by $SN(x_0)$, is the set $\{x_0, x_1, x_2, x_5, x_8, x_9\}$. Thus there is no K-topological 4-neighborhood of x_0 in X', denoted by $N_4^*(x_0, \varepsilon)$, because there is no $N_4(x_0, \varepsilon)$ containing $SN(x_0)$. Thus, in CTC we cannot consider a 4-retraction at x_0. However, in this case we can consider an A-retraction from X' to X in terms of the following mapping: $r : X' \to X$ given by $r(\{x_1, x_3\}) = \{x_2\}, r(x_7) = x_6, r(x_9) = x_0, r(x_i) = x_i, i \in \{0, 2, 4, 5, 6, 8\}$.*

In relation to the A-isomorphic property of an A-retract, we obtain the following:

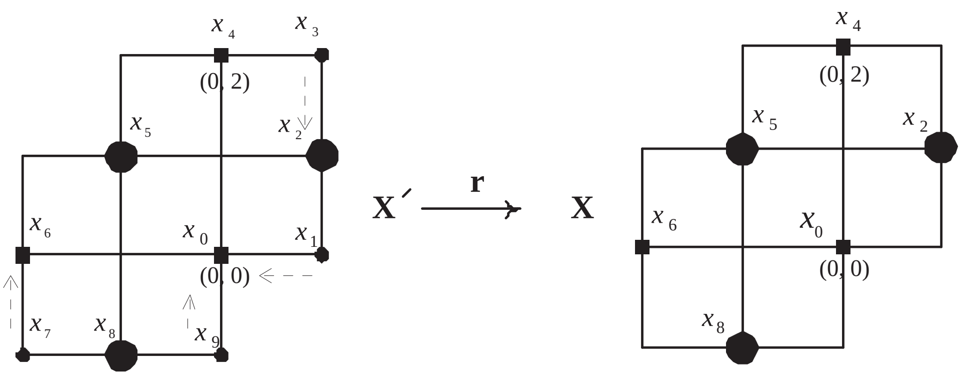

Figure 5: Explanation of an 8-retract in *CTC* as an example

Proposition 4.11. *Let $(X, T_X^{n_0})$ be an A-retract of $(X', T_{X'}^{n_0})$ and let $h : (X', T_{X'}^{n_0}) := X' \to (Y, T_Y^{n_1}) := Y$ be an A-isomorphism. Then $h(X)$ is an A-retract of Y.*

Proof: Let $r : X' \to X$ be an A-retraction. Then $h \circ r \circ h^{-1} : Y \to h(X)$ is a A-retraction because the composition of A-maps is also an A-map. □

Example 4.12. *Assume $X = X' \setminus \{x_5, x_6, x_7\}$ (see Figure 6). Then we observe that X is an A-retract of X' in such a way: Assume the map $r : X' \to X$ given by $r(\{x_6, x_7\}) = \{x_8\}, r(x_5) = x_4$ and $r(x_i) = x_i, i \in [0, 10]_{\mathbf{Z}} \setminus \{5, 6, 7\}$. Then X is an A-retract of X'. Consider an A-isomorphism $h : X' \to Y$ defined by $h(x_i) = y_i, i \in [1, 10]_{\mathbf{Z}}$. Then $h(X)$ is an A-retract of Y, where $h(X) = Y \setminus \{y_5, y_6, y_7\}$.*

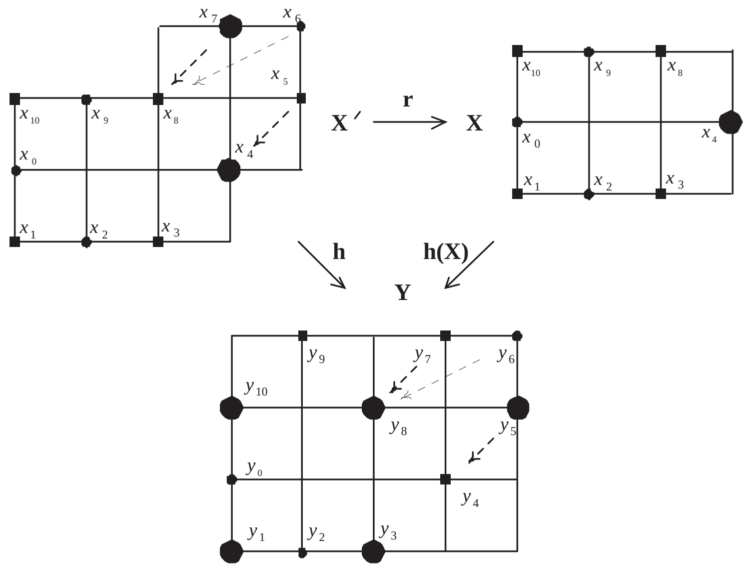

Figure 6: Some property of an A-isomorphism related to an A-retract

5. An Extension Problem of an A-Map

In classical topology [2], we recall the following: let (X', T') be a topological space and (X, T) a subspace of (X', T'). Then (X, T) is a retract of (X', T') if and only if every continuous map $f : (X, T) \to (Y, T)$ has a continuous map $F : X' \to Y$ such that the restriction map on X $F|_X = f$ for any (Y, T) [2]. Consequently, in *KTC* every K-continuous map $f : (X, T_X) \to (Y, T_Y^n)$ has an extension $F : (X', T_{X'}) \to (Y, T_Y^n)$ such that $F|_X = f$ for any (Y, T_Y^n) if and only if X is a K-retraction of X'.
Unlike this extension property, as discussed in Theorem 3.3 and Example 4.8, since an A-map is an expansion of both a K-continuous map and a A-map and further, it includes a non-continuous map between K-spaces, an extension problem of an A-map has its intrinsic feature (see Theorem 5.1). Thus we need to study an extension problem of an A-map using an A-retract.

Theorem 5.1. *In KAC, $(X, T_X^n) := X$ is an A-retract of $(X', T_{X'}^n) := X'$ if and only if every A-map $f : (X, T_X^n) \to (Y, T_Y^{n_1}) := Y$ has an A-map $F : X' \to Y$ such that $F|_X = f$ for any Y.*

Proof: If X is an A-retract of X', then let $g : X' \to X$ such that $1_X \subset g$ and note that for each $f : X \to Y$, $f = 1_X \circ f \subset g \circ f : X' \to Y$.

Conversely, if each $f : X \to Y$ has an extension $F : X' \to Y$ for every Y, then as a special case, $1_X : X \to X$ does. Namely, there is a map $g : X' \to X$ such that $1_X \subset g$. Thus X is an A-retract of X'. □

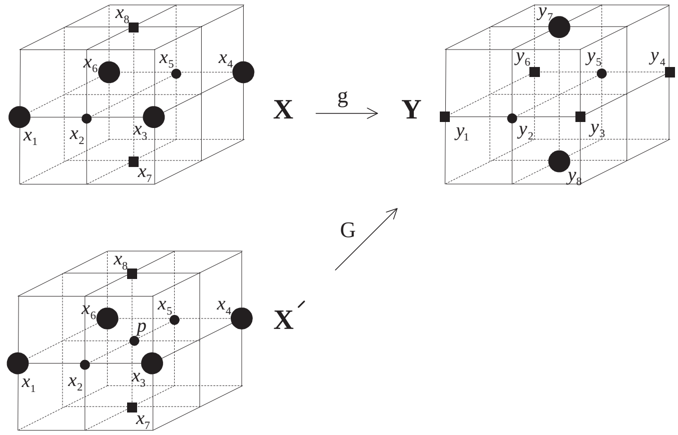

Figure 7: Non-existence of an extension of an A-map

Example 5.2. *Consider the spaces* $X = \{x_i \,|\, i \in [1,8]_{\mathbf{Z}}\}$, $X' = X \cup \{p\}$ *and* $Y = \{y_i \,|\, i \in [1,8]_{\mathbf{Z}}\}$ *(see Figure 7). Then they are K-connected.*
First, let us prove that X is not an A-retract of X'*. Consider the point* $p \in X'$*. Then we obtain* $AN(p) = X'$*. Suppose that X is an A-retract of* X'*. If we assume that the point p is mapped into a certain point* $x_i \in X$ *in terms of the A-retraction from* X' *into X, then* $AN(p)$ *should be mapped into* $AN(x_i)$*. However, there is no element* $x_i \in X$ *such that* $AN(x_i) = X'$*, which the mapping is contrary to an A-retract of* X' *onto X.*
Second, assume the map $g : (X, T^3_X) \to (Y, T^3_Y)$ *defined by* $g(x_i) = y_i, i \in [1,8]_{\mathbf{Z}}$ *so that g is an A-map. Then there is no extension of g into the map* $G : (X', T^3_{X'}) \to (Y, T^3_Y)$ *such that* $G|_X = g$ *because X is not an A-retract of* X'*.*

6. Summary and Further Works

Using both an A-isomorphism and an A-retract, we have studied an extension problem of a CP-map. Since a (k_0, k_1)-continuous map in CTC is also another non-continuous map between K-spaces, we have compared an A-map and a (k_0, k_1)-continuous map in CTC. Compared with an extension problems of K-continuity in [20] and several continuities in computer topology in [9], it turns out that the present extension problem can be substantially used for studying K-spaces. As discussed in Theorem 3.3, Remark 3.4, Corollary 3.5 and the property (2.5), since none of a KA-map, an A-map and a K-map can be equivalent to the other and further, the present extension problem is an expansion of an extension problem of a K-continuous map. Moreover, compared with the extension problems of several continuities of [9], the present research can also contribute to the study of K-spaces and mathematical morphology. As further works, by using the digitizing methods in the papers [12, 14], we can a new type of thinning method. Furthermore, by using the pasting method in the paper [17], we can study pasting properties of A-maps.

References

[1] P. Alexandorff, Diskrete Räume, *Mat. Sb.* **2** (1937) 501-519.
[2] K. Borsuk, Theory of retracts, Polish Scientific publisher, Warsaw, 1967.
[3] L. Boxer, Digitally continuous functions, *Pattern Recognition Letters* **15** (1994) 833-839.
[4] V.A. Chatyrko, S.E. Han, Y, Hattori, Some remarks concerning semi-$T_{\frac{1}{2}}$ spaces *Filomat* **28**(1) (2014) 21-25.
[5] S.E. Han, Comparison between digital continuity and computer continuity, *Honam Math. Jour.* **26** (3) (2004) 331-339.
[6] S.E. Han, Non-product property of the digital fundamental group, *Information Sciences* **171** (1-3) (2005) 73-91.
[7] S.E. Han, Continuities and homeomorphisms in computer topology, *Journal of Korean Mathematical Society* **45** (4) (2008) 923-952.
[8] S.E. Han, The k-homotopic thinning and a torus-like digital image in $\mathbf{Z}^n$, *Journal of Mathematical Imaging and Vision* **31** (1) (2008) 1-16.

[9] S.E. Han, Extension problem of several continuities in computer topology, *Bulletin of Korean Mathematical Society*, **47**(5) (2010) 915-932.
[10] S.E. Han, KD-(k_0, k_1)-homotopy equivalence and its applications *Journal of Korean Mathematical Society* **47**(5) (2010) 1031-1054.
[11] S.E. Han, Homotopy equivalence which is suitable for studying Khalimsky nD spaces, *Topology Appl.* **159** (2012) 1705-1714.
[12] S.E. Han, A digitization method of subspaces of the Euclidean nD space associated with the Khalimsky adjacency structure, *Comp. Appl. Math.* (DOI 10.1007/s40314-015-0223.6) (2015).
[13] S.E. Han, A. Sostak, A compression of digital images derived from a Khalimsky topological structure *Comp. Appl. Math.* **32** (2013) 521-536.
[14] Jeong Min Kang, Sang-Eon Han, Kyung Chan Min, Digitizations associated with several types of digital topological approaches *Comp. Appl. Math.* (DOI 10.1007/s40314-015-0245.0) (2015).
[15] E.D. Khalimsky, Applications of connected ordered topological spaces in topology, *Conference of Math. Department of Provoia* (1970).
[16] E. Khalimsky, R. Kopperman, P.R. Meyer, Computer graphics and connected topologies on finite ordered sets, *Topology and its Applications*, **36** (1) (1991) 1-17.
[17] I.S. Kim, S.E. Han and C.J. Yoo, The almost pasting property of digital continuity, *Acta Applicandae Mathematicae*, **110** (2010) 399-408.
[18] C. O. Kiselman, Digital Geometry and mathematical morphology, Lecture notes, Uppsala University, 2004.
[19] T.Y. Kong, A. Rosenfeld, Topological Algorithms for the Digital Image Processing, Elsevier Science, Amsterdam, 1996.
[20] E. Melin, Extension of continuous functions in digital spaces with the Khalimsky topology, *Topology and its Applications* **153** (2005) 52-65.
[21] E. Melin, Digital Khalimsky manifold, *Journal of Mathematical Imaging and Vision* **33** (2009) 267-280.
[22] A. Rosenfeld, Arcs and curves in digital pictures, *Jour. of the ACM* **20** (1973) 81-87.