Filomat 29:7 (2015), 1429–1444
DOI 10.2298/FIL1507429M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Anti-Invariant Riemannian Submersions from Cosymplectic Manifolds onto Riemannian Manifolds

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Abstract. We introduce anti-invariant Riemannian submersions from cosymplectic manifolds onto Riemannian manifolds. We survey main results of anti-invariant Riemannian submersions defined on cosymplectic manifolds. We investigate necessary and sufficient condition for an anti-invariant Riemannian submersion to be totally geodesic and harmonic. We give examples of anti-invariant submersions such that characteristic vector field ξ is vertical or horizontal. Moreover we give decomposition theorems by using the existence of anti-invariant Riemannian submersions.

1. Introduction

In [19], O'Neill defined a Riemannian submersion, which is the "dual" notion of isometric immersion, and obtained some fundamental equations corresponding to those in Riemannian submanifold geometry, that is, Gauss, Codazzi and Ricci equations. Subspaces of generalized Riemannian spaces were studied in [15]-[17]. We have also the following submersions: semi-Riemannian submersion and Lorentzian submersion [7], Riemannian submersion [8], slant submersion [5], [25], almost Hermitian submersion [27], contact-complex submersion [12], quaternionic submersion [11], almost h-slant submersion and h-slant submersion [21], semi-invariant submersion [26], h-semi-invariant submersion [22], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory [3], [28], Kaluza-Klein theory [4], [9], supergravity and superstring theories [10], [29]. In [24], Sahin introduced anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds.

In this paper we consider anti-invariant Riemannian submersions from cosymplectic manifolds. The paper is organized as follows: In section 2, we present the basic information about Riemannian submersions needed for this paper. In section 3, we mention about cosymplectic manifolds. In section 4, we give definition of anti-invariant Riemannian submersions and introduce anti-invariant Riemannian submersions from cosymplectic manifolds onto Riemannian manifolds. We survey main results of anti-invariant submersions defined on cosymplectic manifolds. We give examples of anti-invariant submersions such that characteristic vector field ξ is vertical or horizontal. Moreover we give decomposition theorems by using the existence of anti-invariant Riemannian submersions and observe that such submersions put some restrictions on the geometry of the total manifold.

2010 *Mathematics Subject Classification*. Primary 53C25, 53C43, 53C55; Secondary 53D15
Keywords. Riemannian submersion, Cosymplectic manifold, Anti-invariant submersion
Received: 26 November 2013; Accepted: 17 July 2014
Communicated by Ljubica Velimirović
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2. Riemannian Submersions

In this section we recall several notions and results which will be needed throughout the paper.

Let (M, g_M) be an m-dimensional Riemannian manifold , let (N, g_N) be an n-dimensional Riemannian manifold. A Riemannian submersion is a smooth map $F : M \to N$ which is onto and satisfies the following three axioms:

$S1$. F has maximal rank.

$S2$. The differential F_* preserves the lenghts of horizontal vectors.

The fundamental tensors of a submersion were defined by O'Neill ([19],[20]). They are $(1,2)$-tensors on M, given by the formula:

$$\mathcal{T}(E,F) = \mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \quad (1)$$

$$\mathcal{A}(E,F) = \mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \quad (2)$$

for any vector field E and F on M. Here ∇ denotes the Levi-Civita connection of (M, g_M). These tensors are called integrability tensors for the Riemannian submersions. Note that we denote the projection morphism on the distributions $\ker F_*$ and $(\ker F_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.The following Lemmas are well known ([19],[20]).

Lemma 2.1. *For any U, W vertical and X, Y horizontal vector fields, the tensor fields $\mathcal{T}$, $\mathcal{A}$ satisfy:*

$$i)\mathcal{T}_U W = \mathcal{T}_W U, \quad (3)$$

$$ii)\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X,Y]. \quad (4)$$

It is easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$.

For each $q \in N$, $F^{-1}(q)$ is an $(m-n)$ dimensional submanifold of M. The submanifolds $F^{-1}(q)$, $q \in N$, are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and F-related to a vector field X on N, i. e., $F_* X_p = X_{*F(p)}$ for all $p \in M$.

Lemma 2.2. *Let $F : (M, g_M) \to (N, g_N)$ be a Riemannian submersion. If X, Y are basic vector fields on M, then:*

i) $g_M(X,Y) = g_N(X_*, Y_*) \circ F$,

ii) $\mathcal{H}[X,Y]$ is basic, F-related to $[X_*, Y_*]$,

iii) $\mathcal{H}(\nabla_X Y)$ is basic vector field corresponding to $\nabla^*_{X_*} Y_*$ where ∇^* is the connection on N.

iv) for any vertical vector field V, $[X,V]$ is vertical.

Moreover, if X is basic and U is vertical then $\mathcal{H}(\nabla_U X) = \mathcal{H}(\nabla_X U) = \mathcal{A}_X U$. On the other hand, from (1) and (2) we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W \quad (5)$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X \quad (6)$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V \quad (7)$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \quad (8)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$.

Notice that $\mathcal{T}$ acts on the fibres as the second fundamental form of the submersion and restricted to vertical vector fields and it can be easily seen that $\mathcal{T} = 0$ is equivalent to the condition that the fibres are totally geodesic. A Riemannian submersion is called a Riemannian submersion with totally geodesic fiber if $\mathcal{T}$ vanishes identically. Let $U_1, ..., U_{m-n}$ be an orthonormal frame of $\Gamma(\ker F_*)$. Then the horizontal vector field $H = \frac{1}{m-n}\sum_{j=1}^{m-n} \mathcal{T}_{U_j} U_j$ is called the mean curvature vector field of the fiber. If $H = 0$ the Riemannian submersion is said to be minimal. A Riemannian submersion is called a Riemannian submersion with totally umbilical fibers if

$$\mathcal{T}_U W = g_M(U, W)H \tag{9}$$

for $U, W \in \Gamma(\ker F_*)$. For any $E \in \Gamma(TM), \mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on $(\Gamma(TM), g_M)$ reversing the horizontal and the vertical distributions. By Lemma 2.1. horizontally distribution $\mathcal{H}$ is integrable if and only if $\mathcal{A}$ =0. For any $D, E, G \in \Gamma(TM)$ one has

$$g(\mathcal{T}_D E, G) + g(\mathcal{T}_D G, E) = 0, \tag{10}$$

$$g(\mathcal{A}_D E, G) + g(\mathcal{A}_D G, E) = 0. \tag{11}$$

We recall the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \to N$ is a smooth map between them. Then the differential φ_* of φ can be viewed a section of the bundle $Hom(TM, \varphi^{-1}TN) \to M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X, Y) = \nabla^{\varphi}_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y) \tag{12}$$

for $X, Y \in \Gamma(TM)$, where ∇^{φ} is the pullback connection. It is known that the second fundamental form is symmetric. If φ is a Riemannian submersion it can be easily prove that

$$(\nabla\varphi_*)(X, Y) = 0 \tag{13}$$

for $X, Y \in \Gamma((\ker F_*)^{\perp})$. A smooth map $\varphi : (M, g_M) \to (N, g_N)$ is said to be harmonic if $trace(\nabla\varphi_*) = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = div\varphi_* = \sum_{i=1}^{m} (\nabla\varphi_*)(e_i, e_i), \tag{14}$$

where $\{e_1, ..., e_m\}$ is the orthonormal frame on M. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$, for details, [1].

Let g be a Riemannian metric tensor on the manifold $M = M_1 \times M_2$ and assume that the canonical foliations D_{M_1} and D_{M_2} intersect perpendicularly everywhere. Then g is the metric tensor of a usual product of Riemannian manifolds if and only if D_{M_1} and D_{M_2} are totally geodesic foliations [23].

3. Cosymplectic Manifolds

A $(2m + 1)$-dimensional C^{∞}-manifold M is said to have an almost contact structure if there exist on M a tensor field ϕ of type $(1, 1)$, a vector field ξ and 1-form η satisfying:

$$\phi^2 = -I + \eta \otimes \xi, \;\; \phi\xi = 0, \; \eta \circ \phi = 0, \;\; \eta(\xi) = 1. \tag{15}$$

There always exists a Riemannian metric g on an almost contact manifold M satisfying the following conditions

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \; \eta(X) = g(X, \xi) \tag{16}$$

where X, Y are vector fields on M.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $M \times R$ is given by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

where f is the C^∞-function on $M \times R$ has no torsion i.e., J is integrable. The condition for normality in terms of ϕ, ξ and η is $\left[\phi,\phi\right] + 2d\eta \otimes \xi = 0$ on M, where $\left[\phi,\phi\right]$ is the Nijenhuis tensor of ϕ. Finally, the fundamental two-form Φ is defined by $\Phi(X,Y) = g(X,\phi Y)$.

An almost contact metric structure (ϕ,ξ,η,g) is said to be cosymplectic, if it is normal and both Φ and η are closed ([2], [14]), and the structure equation of a cosymplectic manifold is given by

$$(\nabla_X \phi)Y = 0 \tag{17}$$

for any X, Y tangent to M, where ∇ denotes the Riemannian connection of the metric g on M. Moreover, for cosymplectic manifold is

$$\nabla_X \xi = 0. \tag{18}$$

The canonical example of cosymplectic manifold is given by the product $B^{2n} \times \mathbb{R}$ Kahler manifold $B^{2n}(J,g)$ with $\mathbb{R}$ real line. Now we will introduce a well known cosymplectic manifold example on $\mathbb{R}^{2n+1}$.

Example 3.1 ([18]). *We consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates (x_i, y_i, z) $(i = 1, \cdots, n)$ and its usual contact form*

$$\eta = dz.$$

The characteristic vector field ξ is given by $\frac{\partial}{\partial z}$ and its Riemannian metric g and tensor field ϕ are given by

$$g = \sum_{i=1}^{n} ((dx_i)^2 + (dy_i)^2) + (dz)^2, \ \ \phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ i = 1, \cdots, n.$$

This gives a cosymplectic structure on $\mathbb{R}^{2n+1}$. The vector fields $E_i = \frac{\partial}{\partial y_i}$, $E_{n+i} = \frac{\partial}{\partial x_i}$, ξ form a ϕ-basis for the cosymplectic structure. On the other hand, it can be shown that $\mathbb{R}^{2n+1}(\phi,\xi,\eta,g)$ is a cosymplectic manifold.

Example 3.2 ([13]). *We denote Cartesian coordinates in $\mathbb{R}^5$ by $(x_1, x_2, x_3, x_4, x_5)$ and its Riemannian metric g*

$$g = \begin{pmatrix} 1+\tau^2 & 0 & \tau^2 & 0 & -\tau \\ 0 & 1 & 0 & 0 & 0 \\ \tau^2 & 0 & 1+\tau^2 & 0 & -\tau \\ 0 & 0 & 0 & 1 & 0 \\ -\tau & 0 & -\tau & 0 & 1 \end{pmatrix},$$

where $\tau = \sin(x_1 + x_3)$. We define an almost contact structure (ϕ,ξ,η) on $\mathbb{R}^5$ by

$$\phi = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\tau & 0 & -\tau & 0 \end{pmatrix}, \ \eta = -\tau dx_1 - \tau dx_3 + dx_5, \ \xi = \frac{\partial}{\partial x_5}.$$

The fundamental 2-form Φ have the form

$$\Phi = dx_1 \wedge dx_2 + dx_3 \wedge dx_4.$$

This gives a cosymplectic structure on $\mathbb{R}^5$. If we take vector fields $E_1 = \frac{\partial}{\partial x_1} + \tau\frac{\partial}{\partial x_5}, E_2 = \frac{\partial}{\partial x_3}, \phi E_1 = E_3 = \frac{\partial}{\partial x_2}, \phi E_2 = E_4 = \frac{\partial}{\partial x4}$ and $E_5 = \frac{\partial}{\partial x_5}$ then these vector fields form a frame field in $\mathbb{R}^5$.

4. Anti-invariant Riemannian submersions

Definition 4.1. *Let $M(\phi, \xi, \eta, g_M)$ be a cosymplectic manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ is called an anti-invariant Riemannian submersion if* $\ker F_*$ *is anti-invariant with respect to ϕ, i.e. $\phi(\ker F_*) \subseteq (\ker F_*)^\perp$.*

Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion from a cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). First of all, from Definition 4.1., we have $\phi(\ker F_*) \cap (\ker F_*)^\perp \neq \{0\}$. We denote the complementary orthogonal distribution to $\phi(\ker F_*)$ in $(\ker F_*)^\perp$ by μ. Then we have

$$(\ker F_*)^\perp = \phi \ker F_* \oplus \mu. \tag{19}$$

Now we will introduce some examples.

Example 4.2. *Let $\mathbb{R}^5$ has got a cosymplectic structure as in Example 3.1. and let $F : \mathbb{R}^5 \to \mathbb{R}^2$ be a map defined by $F(x_1, x_2, y_1, y_2, z) = (\frac{x_1+y_2}{\sqrt{2}}, \frac{x_2+y_1}{\sqrt{2}})$. Then, by direct calculations we have*

$$\ker F_* = span\{V_1 = \frac{1}{\sqrt{2}}(E_1 - E_4),\ V_2 = \frac{1}{\sqrt{2}}(E_2 - E_3),\ V_3 = E_5 = \xi\}$$

and

$$(\ker F_*)^\perp = span\{H_1 = \frac{1}{\sqrt{2}}(E_1 + E_4),\ H_2 = \frac{1}{\sqrt{2}}(E_2 + E_3)\}.$$

Then it is easy to see that F is a Riemannian submersion. Moreover, $\phi V_1 = H_2$, $\phi V_2 = H_1$, $\phi V_3 = 0$ imply that $\phi(\ker F_*) = (\ker F_*)^\perp$. As a result, F is an anti-invariant Riemannian submersion such that ξ is vertical.

Example 4.3. *Let $\mathbb{R}^5$ be a cosymplectic manifold as in Example 3.2., and let $F : \mathbb{R}^5 \to \mathbb{R}^2$ be a map defined by $F(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_3 + x_4)$. After some calculations we have*

$$\ker F_* = span\{V_1 = E_1 - E_3,\ V_2 = E_2 - E_4, V_3 = \xi\}$$

and

$$(\ker F_*)^\perp = span\{H_1 = E_1 + E_3,\ H_2 = E_2 + E_4\}$$

Then it is easy to see that F is a Riemannian submersion. Moreover, $\phi V_1 = H_1$, $\phi V_2 = H_2$, $\phi V_3 = 0$ imply that $\phi(\ker F_*) = (\ker F_*)^\perp$. As a result, F is an anti-invariant Riemannian submersion such that ξ is vertical.

Example 4.4. *Let $\mathbb{R}^7$ be a cosymplectic manifold as in Example 3.1., and let $F : \mathbb{R}^7 \to \mathbb{R}^4$ be a map defined by $F(x_1, x_2, x_3, y_1, y_2, y_3, z) = (\frac{1}{\sqrt{2}}(x_1 + y_1), \frac{1}{\sqrt{2}}(x_2 + y_2), \frac{1}{\sqrt{2}}(x_3 + y_3), \frac{1}{\sqrt{2}}(x_3 - y_3))$ After some calculations we have*

$$\ker F_* = span\{V_1 = \frac{1}{\sqrt{2}}(E_1 - E_4), V_2 = \frac{1}{\sqrt{2}}(E_2 - E_5), V_3 = \xi\}$$

and

$$(\ker F_*)^\perp = span\{H_1 = \frac{1}{\sqrt{2}}(E_1 + E_4),\ H_2 = \frac{1}{\sqrt{2}}(E_2 + E_5),\ H_3 = \frac{1}{\sqrt{2}}(E_3 - E_6), H_4 = \frac{1}{\sqrt{2}}(E_3 + E_6)\}.$$

Then it is easy to see that F is a Riemannian submersion. Moreover, $\phi V_1 = H_1$, $\phi V_2 = H_2$ imply that $\phi(\ker F_*) \subset (\ker F_*)^\perp = \phi(\ker F_*) \oplus span\{H_3, H_4\}$. Hence F is an anti-invariant Riemannian submersion such that $\xi$is vertical.

Example 4.5. *Let $\mathbb{R}^5$ be a cosymplectic manifold as in Example 3.1., and let $F : \mathbb{R}^5 \to \mathbb{R}^3$ be a map defined by $F(x_1, x_2, y_1, y_2, z) = (\frac{x_1+y_2}{\sqrt{2}}, \frac{x_2+y_1}{\sqrt{2}}, z)$. After some calculations we have*

$$\ker F_* = span\{V_1 = \frac{1}{\sqrt{2}}(E_1 - E_4),\ V_2 = \frac{1}{\sqrt{2}}(E_2 - E_3)\}$$

and

$$(\ker F_*)^\perp = span\{H_1 = \frac{1}{\sqrt{2}}(E_1 + E_4),\ H_2 = \frac{1}{\sqrt{2}}(E_2 + E_3),\ H_3 = E_5 = \xi\}.$$

Then it is easy to see that F is a Riemannian submersion. Moreover, $\phi V_1 = H_2$, $\phi V_2 = H_1$ imply that $\phi(\ker F_*) \subset (\ker F_*)^\perp = \phi(\ker F_*) \oplus \{\xi\}$. Thus F is an anti-invariant Riemannian submersion such that ξ is horizontal.

4.1. Anti-invariant submersions admitting vertical structure vector field

In this section, we will study anti-invariant submersions from a cosymplectic manifold onto a Riemannian manifold such that the characteristic vector field ξ is vertical.

It is easy to see that μ is an invariant distribution of $(\ker F_*)^\perp$, under the endomorphism ϕ. Thus, for $X \in \Gamma((\ker F_*)^\perp)$, we write

$$\phi X = BX + CX, \tag{20}$$

where $BX \in \Gamma(\ker F_*)$ and $CX \in \Gamma(\mu)$. On the other hand, since $F_*((\ker F_*)^\perp) = TN$ and F is a Riemannian submersion, using (20) we derive $g_N(F_*\phi V, F_*CX) = 0$, for every $X \in \Gamma((\ker F_*))^\perp$ and $V \in \Gamma(\ker F_*)$, which implies that

$$TN = F_*(\phi(\ker F_*)) \oplus F_*(\mu). \tag{21}$$

Theorem 4.6. *Let $M(\phi, \xi, \eta, g_M)$ be a cosymplectic manifold of dimension $2m + 1$ and (N, g_N) is a Riemannian manifold of dimension n. Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion such that $\phi(\ker F_*) = (\ker F_*)^\perp$. Then the characteristic vector field ξ is vertical and $m = n$.*

Proof. By the assumption $\phi(\ker F_*) = (\ker F_*)^\perp$, for any $U \in \Gamma(\ker F_*)$ we have $g_M(\xi, \phi U) = -g_M(\phi\xi, U) = 0$, which shows that the structure vector field is vertical. Now we suppose that $U_1, ..., U_{k-1}, \xi = U_k$ be an orthonormal frame of $\Gamma(\ker F_*)$, where $k = 2m - n + 1$. Since $\phi(\ker F_*) = (\ker F_*)^\perp$, $\phi U_1, ..., \phi U_{k-1}$ form an orthonormal frame of $\Gamma((\ker F_*)^\perp)$. So, by help of (21) we obtain $k = n + 1$ which implies that $m = n$. □

Remark 4.7. *We note that Example 4.2., and Example 4.3., satisfy Theorem 4.6.*

From (15) and (20) we have following Lemma.

Lemma 4.8. *Let F be an anti-invariant Riemannian submersion from a cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then we have*

$$\begin{aligned} BCX &= 0,\ \eta(BX) = 0, \\ C^2X &= -X - \phi BX, \\ C\phi V &= 0,\ C^3X + CX = 0, \\ B\phi V &= -V + \eta(V)\xi \end{aligned}$$

for any $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\ker F_*))$.

Using (17) one can easily obtain

$$\nabla_X Y = -\phi\nabla_X\phi Y \tag{22}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Lemma 4.9. *Let F be an anti-invariant Riemannian submersion from a cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{A}_X \xi = 0, \tag{23}$$

$$\mathcal{T}_U \xi = 0, \tag{24}$$

$$g_M(CX, \phi U) = 0, \tag{25}$$

$$g_M(\nabla_X CY, \phi U) = -g_M(CY, \phi \mathcal{A}_X U) \tag{26}$$

for any $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma((\ker F_*))$.

Proof. By virtue of (5) and (18) we have (23). Using (7) and (18) we get (24).
For $X \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$, by virtue of (16) and (20) we get

$$\begin{aligned} g_M(CX, \phi U) &= g_M(\phi X - BX, \phi U) \\ &= g_M(X, U) - \eta(X)\eta(U) + g_M(\phi BX, U). \end{aligned} \tag{27}$$

Since $\phi BX \in \Gamma((\ker F_*)^\perp)$ and $\xi \in \Gamma(\ker F_*)$, (27) implies (25).
Then using (7), (17) and (25), we have

$$g_M(\nabla_X CY, \phi U) = -g_M(CY, \phi \mathcal{A}_X U) - g_M(CY, \phi(\mathcal{V}\nabla_X U)).$$

Since $\phi(\mathcal{V}\nabla_X U) \in \Gamma(\phi \ker F_*) = \Gamma((\ker F_*)^\perp)$, we obtain (26). □

Theorem 4.10. *Let $M(\phi, \xi, \eta, g_M)$ be a cosymplectic manifold of dimension $2m + 1$ and (N, g_N) is a Riemannian manifold of dimension n. Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion. Then the fibers are not proper totally umbilical.*

Proof. If the fibers are proper totally umbilical, then we have $\mathcal{T}_U V = g_M(U, V)H$ for any vertical vector fields U, V where H is the mean curvature vector field of any fibre. Since $\mathcal{T}_\xi \xi = 0$, we have $H = 0$, which shows that fibres are minimal. Hence the fibers are totally geodesic. This completes proof of the theorem. □

Since the distribution $\ker F_*$ is integrable, we only study the integrability of the distribution $(\ker F_*)^\perp$ and then we investigate the geometry of leaves of $\ker F_*$ and $(\ker F_*)^\perp$.

Theorem 4.11. *Let F be an anti-invariant Riemannian submersion from a cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other:*

i) $(\ker F_*)^\perp$ *is integrable,*
ii) $g_N((\nabla F_*)(Y, BX), F_*\phi V) = g_N((\nabla F_*)(X, BY), F_*\phi V) + g_M(CY, \phi \mathcal{A}_X V) - g_M(CX, \phi \mathcal{A}_Y V)$,
iii) $g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, \phi V) = g_M(CY, \phi \mathcal{A}_X V) - g_M(CX, \phi \mathcal{A}_Y V)$
for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Proof. Using (22), for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$ we get

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X Y, V) - g_M(\nabla_Y X, V) \\ &= g_M(\nabla_X \phi Y, \phi V) - g_M(\nabla_Y \phi X, \phi V). \end{aligned}$$

Then from (20) we have

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - g_M(\nabla_Y BX, \phi V) \\ &\quad - g_M(\nabla_Y CX, \phi V). \end{aligned}$$

Using (2), (7) and (26) and if we take into account that F is a Riemannian submersion, we obtain

$$\begin{aligned} g_M([X,Y],V) &= g_N(F_*\nabla_X BY, F_*\phi V) - g_M(CY, \phi\mathcal{A}_X V) \\ &\quad - g_N(F_*\nabla_Y BX, F_*\phi V) + g_M(CX, \phi\mathcal{A}_Y V). \end{aligned}$$

Thus, from (12) we have

$$\begin{aligned} g_M([X,Y],V) &= g_N(-(\nabla F_*)(X,BY) + (\nabla F_*)(Y,BX), F_*\phi V) \\ &\quad + g_M(CX, \phi\mathcal{A}_Y V) - g_M(CY, \phi\mathcal{A}_X V) \end{aligned}$$

which proves (*i*) ⇔ (*ii*). On the other hand using (12) we get

$$(\nabla F_*)(Y,BX) - (\nabla F_*)(X,BY) = -F_*(\nabla_Y BX - \nabla_X BY).$$

Then (7) implies that

$$(\nabla F_*)(Y,BX) - (\nabla F_*)(X,BY) = -F_*(\mathcal{A}_Y BX - \mathcal{A}_X BY).$$

From (2) $\mathcal{A}_Y BX - \mathcal{A}_X BY \in \Gamma((\ker F_*)^\perp)$, this shows that (*ii*) ⇔ (*iii*). □

Remark 4.12. *If $\phi(\ker F_*) = (\ker F_*)^\perp$ then we get $C = 0$ and morever* (21) *implies that $TN = F_*(\phi(\ker F_*))$.*

Hence we have the following Corollary.

Corollary 4.13. *Let $F : M(\phi,\xi,\eta,g_M) \to (N,g_N)$ be an anti-invariant Riemannian submersion such that $\phi(\ker F_*) = (\ker F_*)^\perp$, where $M(\phi,\xi,\eta,g_M)$ is a cosymplectic manifold and (N,g_N) is a Riemannian manifold. Then following assertions are equivalent to each other:*
i) $(\ker F_*)^\perp$ *is integrable,*
ii) $(\nabla F_*)(Y,\phi X) = (\nabla F_*)(X,\phi Y)$ *for* $X,Y \in \Gamma((\ker F_*)^\perp)$,
iii) $\mathcal{A}_X\phi Y = \mathcal{A}_Y\phi X$.

Theorem 4.14. *Let F be an anti-invariant Riemannian submersion from a cosymplectic manifold $M(\phi,\xi,\eta,g_M)$ to a Riemannian manifold (N,g_N). Then the following assertions are equivalent to each other:*

i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on M,*
ii) $g_M(\mathcal{A}_X BY, \phi V) = g_M(CY, \phi\mathcal{A}_X V)$,
iii) $g_N((\nabla F_*)(X,\phi Y), F_*\phi V) = -g_M(CY, \phi\mathcal{A}_X V)$
for $X,Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

Proof. From (16) and (17) we obtain

$$g_M(\nabla_X Y, V) = g_M(\nabla_X \phi Y, \phi V)$$

for $X,Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.Using (7) and (20)

$$g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X BY + \mathcal{V}\nabla_X BY, \phi V) - g_M(CY, \phi\mathcal{A}_X V).$$

The last equation shows (*i*) ⇔ (*ii*).
For $X,Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$,

$$\begin{aligned} g_M(\mathcal{A}_X BY, \phi V) &= g_M(CY, \phi\mathcal{A}_X V) \\ &\overset{(26)}{=} -g_M(\nabla_X CY, \phi V) \\ &\overset{(20)}{=} -g_M(\nabla_X \phi Y, \phi V) + g_M(\nabla_X BY, \phi V) \end{aligned} \tag{28}$$

Since differential F_* preserves the lenghts of horizontal vectors the relation (28) forms

$$g_M(\mathcal{A}_X BY, \phi V) = g_N(F_*\nabla_X BY, F_*\phi V) - g_M(\nabla_X \phi Y, \phi V) \tag{29}$$

Using, (17), (16), (12) and (13) in (29) respectively, we obtain

$$g_M(\mathcal{A}_X BY, \phi V) = g_N(-(\nabla F_*)(X, \phi Y), F_*\phi V)$$

which tells that (*ii*) ⇔ (*iii*). □

Corollary 4.15. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant Riemannian submersion such that* $\phi(\ker F_*) = (\ker F_*)^{\perp}$, *where* $M(\phi, \xi, \eta, g_M)$ *is a cosymplectic manifold and* (N, g_N) *is a Riemannian manifold. Then the following assertions are equivalent to each other:*

i) $(\ker F_*)^{\perp}$ *defines a totally geodesic foliation on M,*

ii) $\mathcal{A}_X \phi Y = 0$,

iii) $(\nabla F_*)(X, \phi Y) = 0$ *for* $X, Y \in \Gamma((\ker F_*)^{\perp})$ *and* $V \in \Gamma(\ker F_*)$.

Theorem 4.16. *Let F be an anti-invariant Riemannian submersion from a cosymplectic manifold* $M(\phi, \xi, \eta, g_M)$ *to a Riemannian manifold* (N, g_N). *Then the following assertions are equivalent to each other:*

i) $\ker F_*$ *defines a totally geodesic foliation on M,*

ii) $g_N((\nabla F_*)(V, \phi X), F_*\phi W) = 0$ *for* $X \in \Gamma((\ker F_*)^{\perp})$ *and* $V, W \in \Gamma(\ker F_*)$,

iii) $\mathcal{T}_V BX + \mathcal{A}_{CX} V \in \Gamma(\mu)$.

Proof. Since $g_M(W, X) = 0$ we have $g_M(\nabla_V W, X) = -g(W, \nabla_V X)$. From (16) and (17) we get $g_M(\nabla_V W, X) = -g_M(\phi W, H\nabla_V \phi X)$. Then Riemannian submersion F and (12) imply that

$$g_M(\nabla_V W, X) = g_N(F_*\phi W, (\nabla F_*)(V, \phi X))$$

which is (*i*) ⇔ (*ii*). By direct calculation, we derive

$$g_N((F_*\phi W, (\nabla F_*)(V, \phi X)) = -g_M(\phi W, \nabla_V \phi X).$$

Using (20) we have

$$g_N((F_*\phi W, (\nabla F_*)(V, \phi X)) = -g_M(\phi W, \nabla_V BX + \nabla_V CX).$$

Hence we get

$$g_N((F_*\phi W, (\nabla F_*)(V, \phi X)) = -g_M(\phi W, \nabla_V BX + [V, CX] + \nabla_{CX} V).$$

Since $[V, CX] \in \Gamma(\ker F_*)$, using (5) and (7), we obtain

$$g_N((F_*\phi W, (\nabla F_*)(V, \phi X)) = -g_M(\phi W, \mathcal{T}_V BX + \mathcal{A}_{CX} V).$$

This shows (*ii*) ⇔ (*iii*). □

Corollary 4.17. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant Riemannian submersion such that* $\phi(\ker F_*) = (\ker F_*)^{\perp}$, *where* $M(\phi, \xi, \eta, g_M)$ *is a cosymplectic manifold and* (N, g_N) *is a Riemannian manifold. Then following assertions are equivalent to each other:*

i) $\ker F_*$ *defines a totally geodesic foliation on M,*

ii) $(\nabla F_*)(V, \phi X) = 0$, *for* $X \in \Gamma((\ker F_*)^{\perp})$ *and* $V, W \in \Gamma(\ker F_*)$,

iii) $\mathcal{T}_V \phi W = 0$.

We note that a differentiable map F between two Riemannian manifolds is called totally geodesic if $\nabla F_* = 0$. For an anti-invariant Riemannian submersion such that $\phi(\ker F_*) = (\ker F_*)^{\perp}$ we have the following characterization.

Theorem 4.18. *Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion such that $\phi(\ker F_*) = (\ker F_*)^\perp$, where $M(\phi, \xi, \eta, g_M)$ is a cosymplectic manifold and (N, g_N) is a Riemannian manifold. Then F is a totally geodesic map if and only if*

$$\mathcal{T}_W\phi V = 0, \quad \forall W,\ V \in \Gamma(\ker F_*) \tag{30}$$

and

$$\mathcal{A}_X\phi W = 0, \ \ \forall X \in \Gamma((\ker F_*)^\perp), \forall W \in \Gamma(\ker F_*). \tag{31}$$

Proof. First of all, we recall that the second fundamental form of a Riemannian submersion satisfies (13). For $W, V \in \Gamma(\ker F_*)$, by using (6), (12), (15) and (17), we get

$$(\nabla F_*)(W, V) = F_*(\phi\mathcal{T}_W\phi V). \tag{32}$$

On the other hand by using (12) and (17) we have

$$(\nabla F_*)(X, W) = F_*(\phi\nabla_X\phi W)$$

for $X \in \Gamma((\ker F_*)^\perp)$. Then from (8) and (15), we obtain

$$(\nabla F_*)(X, W) = F_*(\phi\mathcal{A}_X\phi W). \tag{33}$$

Since ϕ is non-singular, using (10) and (11) proof comes from (13), (32) and (33). □

Finally, we give a necessary and sufficient condition for an anti-invariant Riemannian submersion such that $\phi(\ker F_*) = (\ker F_*)^\perp$ to be harmonic.

Theorem 4.19. *Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion such that $\phi(\ker F_*) = (\ker F_*)^\perp$, where $M(\phi, \xi, \eta, g_M)$ is a cosymplectic manifold and (N, g_N) is a Riemannian manifold. Then F is harmonic if and only if Trace $\phi\mathcal{T}_V = 0$ for $V \in \Gamma(\ker F_*)$.*

Proof. From [6] we know that F is harmonic if and only if F has minimal fibres. Thus F is harmonic if and only if $\sum_{i=1}^{k} \mathcal{T}_{e_i}e_i = 0$, where k is dimension of $\ker F_*$. On the other hand, from (5), (6) and (17), we get

$$\mathcal{T}_V\phi W = \phi\mathcal{T}_V W \tag{34}$$

for any $W, V \in \Gamma(\ker F_*)$. Using (34), we get

$$\sum_{i=1}^{k} g_M(\mathcal{T}_{e_i}\phi e_i, V) = -\sum_{i=1}^{k} g_M(\mathcal{T}_{e_i}e_i, \phi V)$$

for any $V \in \Gamma(\ker F_*)$. The equation (10) implies that

$$\sum_{i=1}^{k} g_M(\phi e_i, \mathcal{T}_{e_i}V) = \sum_{i=1}^{k} g_M(\mathcal{T}_{e_i}e_i, \phi V).$$

Then, using (3) we have

$$\sum_{i=1}^{k} g_M(\phi e_i, \mathcal{T}_{Ve_i}) = \sum_{i=1}^{k} g_M(\mathcal{T}_{e_i}e_i, \phi V).$$

Hence, proof comes from (16). □

Using [23], Theorem 4.14., and Theorem 4.16., we will give our first decomposition theorem for an anti invariant Riemannian submersion.

Theorem 4.20. *Let F be an anti-invariant Riemannian submersion from a cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then M is a locally product manifold if and only if*

$$g_N((\nabla F_*)(X, \phi Y), F_*\phi V) = -g_M(CY, \phi\mathcal{A}_X V)$$

and

$$g_N((\nabla F_*)(V, \phi X), F_*\phi W) = 0$$

for $W, V \in \Gamma(\ker F_*)$, $X, Y \in \Gamma((\ker F_*)^\perp)$.

From Corollary 4.15., and Corollary 4.17., we obtain following decomposition theorem.

Theorem 4.21. *Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion such that $\phi(\ker F_*) = (\ker F_*)^\perp$, where $M(\phi, \xi, \eta, g_M)$ is a cosymplectic manifold and (N, g_N) is a Riemannian manifold. Then M is a locally product manifold if and only if $\mathcal{A}_X\phi Y = 0$ and $\mathcal{T}_V\phi W = 0$ for $W, V \in \Gamma(\ker F_*)$, $X, Y \in \Gamma((\ker F_*)^\perp)$.*

4.2. Anti-invariant submersions admitting horizontal structure vector field

In this section, we will study anti-invariant submersions from a cosymplectic manifold onto a Riemannian manifold such that the characteristic vector field ξ is horizontal. Using (19), we have $\mu = \phi\mu \oplus \{\xi\}$. For any horizontal vector field X we put

$$\phi X = BX + CX, \tag{35}$$

where $BX \in \Gamma(\ker F_*)$ and $CX \in \Gamma(\mu)$.

Now we suppose that V is vertical and X is horizontal vector field. Using above relation and (16) we obtain

$$g_M(\phi V, CX) = 0.$$

From this last relation we have $g_N(F_*\phi V, F_*CX) = 0$ which implies that

$$TN = F_*(\phi(\ker F_*)) \oplus F_*(\mu). \tag{36}$$

Theorem 4.22. *Let $M(\phi, \xi, \eta, g_M)$ be a cosymplectic manifold of dimension $2m + 1$ and (N, g_N) is a Riemannian manifold of dimension n. Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion such that $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$.Then $m + 1 = n$.*

Proof. We assume that $U_1, ..., U_k$ be an orthonormal frame of $\Gamma(\ker F_*)$, where $k = 2m - n + 1$. Since $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$, $\phi U_1, ..., \phi U_k, \xi$ form an orthonormal frame of $\Gamma((\ker F_*)^\perp)$. So, by help of (21) we obtain $k = n-1$ which implies that $m + 1 = n$. □

Remark 4.23. *We note that Example 4.5., satisfies Theorem 4.22.*

From (15), (36) and (35) we obtain following Lemma.

Lemma 4.24. *Let F be an anti-invariant Riemannian submersion from a cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then we have*

$$\begin{aligned} BCX &= 0, \\ C^2X &= \phi^2 X - \phi BX, \\ C\phi V &= 0, \ C^3X + CX = 0, \\ B\phi V &= -V \end{aligned}$$

for any $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Using (17) one can easily obtain

$$\nabla_X Y = -\phi\nabla_X\phi Y + \eta(\nabla_X Y)\xi \tag{37}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Lemma 4.25. *Let F be an anti-invariant Riemannian submersion from a cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{A}_X\xi = 0, \tag{38}$$

$$\mathcal{T}_U\xi = 0, \tag{39}$$

$$g_M(CX, \phi U) = 0, \tag{40}$$

$$g_M(\nabla_Y CX, \phi U) = -g_M(CX, \phi\mathcal{A}_Y U) \tag{41}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.

Proof. By virtue of (8) and (18) we have (38). Using (6) and (18) we obtain (39). For $X \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$, by virtue of (16) and (35) we get

$$\begin{aligned} g_M(CX, \phi U) &= g_M(\phi X - BX, \phi U) \\ &= g_M(X, U) - \eta(X)\eta(U) + g_M(\phi BX, U). \end{aligned} \tag{42}$$

Since $\phi BX \in \Gamma((\ker F_*)^\perp)$ and $\xi \in \Gamma(\ker F_*)$, (42) implies (40). Now using (40) we get

$$g_M(\nabla_Y CX, \phi U) = -g_M(CX, \nabla_Y\phi U)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$. Then using (7) and (17) we have

$$g_M(\nabla_Y CX, \phi U) = -g_M(CX, \phi\mathcal{A}_Y U) - g_M(CX, \phi(\mathcal{V}\nabla_Y U)).$$

Since $\phi(\mathcal{V}\nabla_Y U) \in \Gamma((\ker F_*)^\perp)$, we obtain (41). □

We now study the integrability of the distribution $(\ker F_*)^\perp$ and then we investigate the geometry of leaves of $\ker F_*$ and $(\ker F_*)^\perp$.

Theorem 4.26. *Let F be an anti-invariant Riemannian submersion from a cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other:*

i) $(\ker F_*)^\perp$ *is integrable,*
ii) $g_N((\nabla F_*)(Y, BX), F_*\phi V) = g_N((\nabla F_*)(X, BX), F_*\phi V) + g_M(CY, \phi\mathcal{A}_X V) - g_M(CX, \phi\mathcal{A}_Y V),$
iii) $g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, \phi V) = g_M(CY, \phi\mathcal{A}_X V) - g_M(CX, \phi\mathcal{A}_Y V)$
for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Proof. Using (37), for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$ we get

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X Y, V) - g_M(\nabla_Y X, V) \\ &= g_M(\nabla_X\phi Y, \phi V) - g_M(\nabla_Y\phi X, \phi V). \end{aligned}$$

Then from (35) we have

$$\begin{aligned} g_M([X, Y], V) = {} & g_M(\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - g_M(\nabla_Y BX, \phi V) \\ & -g_M(\nabla_Y CX, \phi V). \end{aligned}$$

Using (2), (7) and (41) and if we take into account that F is a Riemannian submersion, we obtain

$$\begin{aligned} g_M([X,Y],V) &= g_N(F_*\nabla_X BY, F_*\phi V) - g_M(CY, \phi\mathcal{A}_X V) \\ &\quad - g_N(F_*\nabla_Y BX, F_*\phi V) + g_M(CX, \phi\mathcal{A}_Y V). \end{aligned}$$

Thus, from (12) we have

$$\begin{aligned} g_M([X,Y],V) &= g_N(-(\nabla F_*)(X,BY) + (\nabla F_*)(Y,BX), F_*\phi V) \\ &\quad + g_M(CX, \phi\mathcal{A}_Y V) - g_M(CY, \phi\mathcal{A}_X V) \end{aligned}$$

which proves (i) $\Leftrightarrow$ (ii).On the other hand using (12) we get

$$(\nabla F_*)(Y,BX) - (\nabla F_*)(X,BY) = -F_*(\nabla_Y BX - \nabla_X BY).$$

Then (7) implies that

$$(\nabla F_*)(Y,BX) - (\nabla F_*)(X,BY) = -F_*(\mathcal{A}_Y BX - \mathcal{A}_X BY).$$

From (2) $\mathcal{A}_Y BX - \mathcal{A}_X BY \in \Gamma((\ker F_*)^\perp)$, this shows that ($ii$) $\Leftrightarrow$ (iii). □

Remark 4.27. *We assume that* $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$. *Using (35) one can prove that* $CX = 0$ *for* $X \in \Gamma((\ker F_*)^\perp)$.

Corollary 4.28. *Let* $M(\phi, \xi, \eta, g_M)$ *be a cosymplectic manifold of dimension* $2m+1$ *and* (N, g_N) *is a Riemannian manifold of dimension* n. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant Riemannian submersion such that* $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$. *Then the following assertions are equivalent to each other:*
i) $(\ker F_*)^\perp$ *is integrable,*
ii) $(\nabla F_*)(X, \phi Y) = (\nabla F_*)(\phi X, Y)$, *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $X, Y \in \Gamma((\ker F_*)^\perp)$,
iii) $\mathcal{A}_X \phi Y = \mathcal{A}_Y \phi X$.

Theorem 4.29. *Let* $M(\phi, \xi, \eta, g_M)$ *be a cosymplectic of dimension* $2m+1$ *and* (N, g_N) *is a Riemannian manifold of dimension* n. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant Riemannian submersion. Then the following assertions are equivalent to each other:*

i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on* M,
ii) $g_M(\mathcal{A}_X BY, \phi V) = g_M(CY, \phi\mathcal{A}_X V)$,
iii) $g_N((\nabla F_*)(X, \phi Y), F_*\phi V) = -g_M(CY, \phi\mathcal{A}_X V)$
for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

Proof. From (16) and (17) we obtain

$$g_M(\nabla_X Y, V) = g_M(\nabla_X \phi Y, \phi V)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.Using (35)

$$g_M(\nabla_X Y, V) = g_M(\nabla_X BY + \nabla_X CY, \phi V)$$

From (7) and (26)

$$g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X BY + \mathcal{V}\nabla_X BY, \phi V) - g_M(CY, \phi\mathcal{A}_X V).$$

The last equation shows (i) $\Leftrightarrow$ (ii).
For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$,

$$\begin{aligned} g_M(\mathcal{A}_X BY, \phi V) &= g_M(CY, \phi\mathcal{A}_X V) \\ &\overset{(26)}{=} -g_M(\nabla_X CY, \phi V) \\ &\overset{(20)}{=} -g_M(\nabla_X \phi Y, \phi V) + g_M(\nabla_X BY, \phi V) \end{aligned} \tag{43}$$

Since differential F_* preserves the lenghts of horizontal vectors the relation (43) forms

$$g_M(\mathcal{A}_X BY, \phi V) = g_N(F_*\nabla_X BY, F_*\phi V) - g_M(\nabla_X \phi Y, \phi V) \tag{44}$$

Using, (17), (16), (12) and (13) in (44) respectively, we obtain

$$g_M(\mathcal{A}_X BY, \phi V) = g_N(-(\nabla F_*)(X, \phi Y), F_*\phi V)$$

which tells that (*ii*) ⇔ (*iii*). □

Corollary 4.30. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant Riemannian submersion such that* $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$*, where* $M(\phi, \xi, \eta, g_M)$ *is a cosymplectic manifold and* (N, g_N) *is a Riemannian manifold. Then the following assertions are equivalent to each other:*

i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on* M*,*

ii) $\mathcal{A}_X \phi Y = 0$*,*

iii) $(\nabla F_*)(X, \phi Y) = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$*.*

For the distribution $\ker F_*$, we have;

Theorem 4.31. *Let* F *be an anti-invariant Riemannian submersion from a cosymplectic manifold* $M(\phi, \xi, \eta, g_M)$ *to a Riemannian manifold* (N, g_N)*. Then the following assertions are equivalent to each other:*

i) $(\ker F_*)$ *defines a totally geodesic foliation on* M*,*

ii) $g_N((\nabla F_*)(V, \phi X), F_*\phi W) = 0$ *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V, W \in \Gamma(\ker F_*)$*,*

iii) $\mathcal{T}_V BX + \mathcal{A}_{CX} V \in \Gamma(\mu)$*.*

Proof. Since $g_M(W, X) = 0$ we have $g_M(\nabla_V W, X) = -g(W, \nabla_V X)$. From (16) and (17) we get $g_M(\nabla_V W, X) = -g_M(\phi W, H\nabla_V \phi X)$. Then Riemannian submersion F and (12) imply that

$$g_M(\nabla_V W, X) = g_N(F_*\phi W, (\nabla F_*)(V, \phi X))$$

which is (*i*) ⇔ (*ii*). By direct calculation, we derive

$$g_N((F_*\phi W, (\nabla F_*)(V, \phi X)) = -g_M(\phi W, \nabla_V \phi X).$$

Using (35) we have

$$g_N((F_*\phi W, (\nabla F_*)(V, \phi X)) = -g_M(\phi W, \nabla_V BX + \nabla_V CX).$$

Hence we get

$$g_N((F_*\phi W, (\nabla F_*)(V, \phi X)) = -g_M(\phi W, \nabla_V BX + [V, CX] + \nabla_{CX} V).$$

Since $[V, CX] \in \Gamma(\ker F_*)$, using (5) and (7), we obtain

$$g_N((F_*\phi W, (\nabla F_*)(V, \phi X)) = -g_M(\phi W, \mathcal{T}_V BX + \mathcal{A}_{CX} V).$$

This shows (*ii*) ⇔ (*iii*). □

Corollary 4.32. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant Riemannian submersion such that* $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$*, where* $M(\phi, \xi, \eta, g_M)$ *is a cosymplectic manifold and* (N, g_N) *is a Riemannian manifold. Then following assertions are equivalent to each other:*

i) $(\ker F_*)$ *defines a totally geodesic foliation on* M*,*

ii) $(\nabla F_*)(V, \phi X) = 0$*, for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V, W \in \Gamma(\ker F_*)$*,*

iii) $\mathcal{T}_V \phi W = 0$*.*

Theorem 4.33. *Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion such that $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$, where $M(\phi, \xi, \eta, g_M)$ is a cosymplectic manifold and (N, g_N) is a Riemannian manifold. Then F is a totally geodesic map if and only if*

$$\mathcal{T}_W \phi V = 0, \quad \forall\, W,\ V \in \Gamma(\ker F_*) \tag{45}$$

and

$$\mathcal{A}_X \phi W = 0, \quad \forall X \in \Gamma((\ker F_*)^\perp), \forall\, W \in \Gamma(\ker F_*). \tag{46}$$

Proof. First of all, we recall that the second fundamental form of a Riemannian submersion satisfies (13). For $W, V \in \Gamma(\ker F_*)$, by using (6), (12), (15) and (17), we get

$$(\nabla F_*)(W, V) = F_*(\phi \mathcal{T}_W \phi V). \tag{47}$$

On the other hand by using (12) and (17) we have

$$(\nabla F_*)(X, W) = F_*(\phi \nabla_X \phi W)$$

for $X \in \Gamma((\ker F_*)^\perp)$. Then from (8) and (15), we obtain

$$(\nabla F_*)(X, W) = F_*(\phi \mathcal{A}_X \phi W). \tag{48}$$

Since ϕ is non-singular, using (10) and (11) proof comes from (13), (47) and (48). □

Finally, we give a necessary and sufficient condition for an anti-invariant Riemannian submersion such that $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$ to be harmonic.

Theorem 4.34. *Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant Riemannian submersion such that $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$, where $M(\phi, \xi, \eta, g_M)$ is a cosymplectic manifold and (N, g_N) is a Riemannian manifold. Then F is harmonic if and only if Trace $\phi \mathcal{T}_V = 0$ for $V \in \Gamma(\ker F_*)$.*

Proof. From [6] we know that F is harmonic if and only if F has minimal fibres. Thus F is harmonic if and only if $\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = 0$, where k is dimension of $\ker F_*$. On the other hand, from (5), (6) and (17), we get

$$\mathcal{T}_V \phi W = \phi \mathcal{T}_V W \tag{49}$$

for any $W, V \in \Gamma(\ker F_*)$. Using (49), we get

$$\sum_{i=1}^{k} g_M(\mathcal{T}_{e_i} \phi e_i, V) = -\sum_{i=1}^{k} g_M(\mathcal{T}_{e_i} e_i, \phi V)$$

for any $V \in \Gamma(\ker F_*)$. (10) implies that

$$\sum_{i=1}^{k} g_M(\phi e_i, \mathcal{T}_{e_i} V) = \sum_{i=1}^{k} g_M(\mathcal{T}_{e_i} e_i, \phi V).$$

Then, using (3) we have

$$\sum_{i=1}^{k} g_M(\phi e_i, \mathcal{T}_{V e_i}) = \sum_{i=1}^{k} g_M(\mathcal{T}_{e_i} e_i, \phi V).$$

Hence, proof comes from (16). □

From Theorems 4.31., and 4.33., we have following theorem.

Theorem 4.35. *Let F be an anti-invariant Riemannian submersion from a cosymplectic manifold* $M(\phi, \xi, \eta, g_M)$ *to a Riemannian manifold* (N, g_N)*. Then M is a locally product manifold if and only if*

$$g_N((\nabla F_*)(X, \phi Y), F_*\phi V) = -g_M(CY, \phi \mathcal{A}_X V)$$

and

$$g_N((\nabla F_*)(V, \phi X), F_*\phi W) = 0$$

for $W, V \in \Gamma(\ker F_*)$, $X, Y \in \Gamma((\ker F_*)^\perp)$.

Using Corollary 4.30., and 4.32., we get following theorem.

Theorem 4.36. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant Riemannian submersion such that* $(\ker F_*)^\perp = \phi \ker F_* \oplus \{\xi\}$, *where* $M(\phi, \xi, \eta, g_M)$ *is a cosymplectic manifold and* (N, g_N) *is a Riemannian manifold. Then M is a locally product manifold if and only if* $\mathcal{A}_X \phi Y = 0$ *and* $\mathcal{T}_V \phi W = 0$ *for* $W, V \in \Gamma(\ker F_*)$, $X, Y \in \Gamma((\ker F_*)^\perp)$.

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