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Quadratic Quadrature Formula for Curves with Third Degree of Exactness

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Abstract. In this article, a quadrature formula of degree 2 is given that has degree of exactness 3 and order 5. The formula is valid for any planar curve given in parametric form unlike existing Gaussian quadrature formulas that are valid only for functions.

1. Introduction

Quadrature formulas are derived to calculate the area enclosed by a function. The quality of a given quadrature formula is usually measured by its degree of exactness (accuracy). Historically, quadrature is concerned with calculating the area. A square of the same area is constructed (squaring process); this gives the process the name of squaring or quadrature. The interpolation based quadrature formulas inherit their degree of exactness from the order of approximation of the interpolating polynomials. For example, the trapezoidal rule has degree of exactness of 1 and the Simpson rule has degree of exactness of 3. In general, approximating using the Lagrange interpolating polynomial of degree n and using it to form a quadrature formula gives the well-known $(n+1)$-point Newton-Cotes formula with degree of exactness of n, see [13].

The Gaussian quadrature formula uses the strong property of orthogonal polynomials to have degree of exactness of $2n-1$, see [5, 13].

In this paper, a different approach is given. The parametric curve is approximated using a polynomial parametric curve of order 4. Thereafter, this approximating parametric curve is used to get a quadrature formula which has the same degree of exactness like the Gaussian quadrature formula, but it is, moreover, valid for a wider class of curves, namely, the parametric curves.

This paper is organized as follows. In Section 2, some preliminaries are introduced. The quadratic approximation is given in Section 3. The quadrature formula that is valid for parametric curves is given in Section 4. In Section 5, the order of approximation of the proposed quadrature formula is shown. This paper ends with comparisons and conclusions in Section 6.

2. Preliminaries

For a real-valued function $f(x)$ of a real variable, defined on the finite interval $[a, b]$, we seek to compute the value of the integral, $\int_a^b f(x)dx$. Let $\mathcal{P}_m$ be the space of all polynomials of degree $\leq m$.

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Given a linear functional $L(f) = \int_a^b f(x)\,dx$, then a quadrature formula $Q_n(f)$ is a discrete linear functional that approximates the linear functional L, i. e. $Q_n(f) \approx L(f)$. If $Q_n(f) = L(f)$, $\forall f \in \mathcal{P}_m$, then Q_n has the degree of exactness m.

The error term for the $(n+1)$-point Newton-Cotes formula is as follows. For every function $f \in C^{(n+1)}[a,b]$, there exists ξ, $a < \xi < b$, that satisfies the equality

$$\int_a^b f(x)\,dx - \sum_{i=0}^{n} w_i\, f(x_i) = \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_0^n \prod_{i=0}^{n} (x-i)\,dx, \quad h = \frac{b-a}{n}.$$

It is clear that the degree of exactness is n.

Since the Gaussian quadrature formula is constructed to be exact for polynomials of degree $2n-1$, thus it is proper for functions that have polynomial behavior. The error estimate of the Gaussian quadrature method is given by the following formula. For every function $f \in C^{(2n)}[a,b]$, there exists ξ, $a < \xi < b$, such that

$$\int_a^b f(x)\,dx - \sum_{i=1}^{n} w_i\, f(x_i) = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)((2n!)^3)} f^{(2n)}(\xi).$$

Approximation methods for planar curves are stated in [7–9] that sufficiently improve the standard approximation rates via Taylor's method. These methods are based on the fact that the parametrization of a curve is not unique and can be suitably modified to improve the approximation order.

Given a regular smooth planar curve

$$C : t \to \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \; t \in \Re. \tag{1}$$

The curve C is approximated by the polynomial curve

$$P : t \to \begin{pmatrix} X_m(t) \\ Y_m(t) \end{pmatrix}, \; t \in \Re, \tag{2}$$

where $X_m(t)$ and $Y_m(t)$ are the Taylor (or Lagrange) polynomials of degree m. The polynomial curve P approximates the curve C with order $m+1$. An improvement over the standard order $m+1$ is possible because the parametrization of a curve is not unique. Without loss of generalization, we assume that $(f(0), g(0)) = (0,0)$, $(f'(0), g'(0)) = (1,0)$. In this case, for small t, the curve C can be parametrized in the form

$$C : t \to \begin{pmatrix} X_m(t) \\ \phi(X_m(t)) \end{pmatrix} \tag{3}$$

for a suitable $\phi(X_m(t))$. In this case, P approximates C with some order, say α, if and only if

$$\phi(X_m(t)) - Y_m(t) = O(t^\alpha). \tag{4}$$

Considering the approximation at both end points, then this is equivalent to

$$\begin{aligned} \frac{d^j}{dt^j}\{\phi(X_m(t)) - Y_m(t)\}|_{t=0} &= 0, \quad j = 0,1,\dots,\alpha_1 - 1, \\ \frac{d^j}{dt^j}\{\phi(X_m(t)) - Y_m(t)\}|_{t=1} &= 0, \quad j = 0,1,\dots,\alpha_2 - 1, \\ X_m(0) = 0, \quad X_m(1) &= 1, \quad \alpha_1 + \alpha_2 = \alpha. \end{aligned} \tag{5}$$

Normalizing P by choosing $X_m'(0) = 1$, then the polynomial approximation is determined by $2m$ free parameters. Comparing the number of parameters and the number of equations then we get the order of approximation of $2m$, see the details in [7].

3. Quadratic Case

We consider the quadratic piecewise approximation for planar curves, $\alpha_1 = \alpha_2 = 2$, which raises the standard approximation rate to 4 rather than 3. Take

$$X_2(t) = a_0 + a_1\,(t - t_0) + a_2\,(t - t_0)^2, \quad Y_2(t) = b_0 + b_1\,(t - t_0) + b_2\,(t - t_0)^2.$$

In this case, the above system becomes as follows:

$$X_2(t_0) = t_0,\ X_2(t_1) = t_1,\ Y_2(t_0) = \phi(t_0),\ Y_2(t_1) = \phi(t_1),$$
$$\phi'(t_0)X_2'(t_0) - Y_2'(t_0) = 0,\ \phi'(t_1)X_2'(t_1) - Y_2'(t_1) = 0.$$

Substituting for $X_2(t)$, $Y_2(t)$, $X_2'(t)$, $Y_2'(t)$ at $t = t_0$ and $t = t_1$, then the above system is simplified to:

$$a_0 = t_0, \quad b_0 = \phi(t_0), \quad a_1 + a_2\,h = 1, \quad b_1 + b_2 \quad h = \frac{v_h}{h},$$
$$b_1 = \phi'(t_0)a_1, \quad b_1 + 2b_2h = \phi'(t_1)(a_1 + 2a_2h),$$

where $h = t_1 - t_0$, $v_h = \phi(t_1) - \phi(t_0)$.

The solution to this system is

$$a_0 = t_0,\ b_0 = \phi(t_0),\ b_1 = \phi'(t_0)a_1, \quad a_2 = \frac{1 - a_1}{h}, \quad b_2 = \frac{1}{h}\left(\frac{v_h}{h} - \phi'(t_0)a_1\right),\ a_1 = 2\frac{\phi'(t_1) - \frac{v_h}{h}}{\phi'(t_1) - \phi'(t_0)}.$$

And in explicit forms:

$$a_0 = t_0, \quad a_1 = 2\frac{\phi'(t_1) - \frac{v_h}{h}}{\phi'(t_1) - \phi'(t_0)}, \quad a_2 = \frac{1}{h} - 2\frac{\phi'(t_1) - \frac{v_h}{h}}{h(\phi'(t_1) - \phi'(t_0))}, \tag{6}$$

$$b_0 = \phi(t_0), \quad b_1 = 2\phi'(t_0)\frac{\phi'(t_1) - \frac{v_h}{h}}{\phi'(t_1) - \phi'(t_0)}, \quad b_2 = \frac{v_h}{h^2} - 2\phi'(t_0)\frac{\phi'(t_1) - \frac{v_h}{h}}{h(\phi'(t_1) - \phi'(t_0))}. \tag{7}$$

Substituting these values in $X_2(t)$ and $Y_2(t)$ gives

$$X_2(t) = t_0 + 2\frac{\phi'(t_1) - \frac{v_h}{h}}{\phi'(t_1) - \phi'(t_0)}\,(t - t_0) + \left(\frac{1}{h} - 2\frac{\phi'(t_1) - \frac{v_h}{h}}{h(\phi'(t_1) - \phi'(t_0))}\right)(t - t_0)^2, \tag{8}$$

$$Y_2(t) = \phi(t_0) + 2\phi'(t_0)\frac{\phi'(t_1) - \frac{v_h}{h}}{\phi'(t_1) - \phi'(t_0)}\,(t - t_0) + \left(\frac{v_h}{h^2} - 2\phi'(t_0)\frac{\phi'(t_1) - \frac{v_h}{h}}{h(\phi'(t_1) - \phi'(t_0))}\right)(t - t_0)^2. \tag{9}$$

As $h \to 0$ then $t_1 \to t_0$, $t - t_0 \to h$ and $v_h/h \to \phi'(t_0)$. Let $t - t_0 = \beta h$, where $0 \le \beta \le 1$, then we have

$$\begin{aligned} X_2(t) &= t_0 + 2\,\beta\,h - \frac{1}{h}\,\beta^2h^2 \\ &= t_0 + (2\,\beta - \beta^2)h, \end{aligned} \tag{10}$$

$$\begin{aligned} Y_2(t) &= \phi(t_0) + 2\phi'(t_0)\,\beta\,h + \frac{1}{h}(\phi'(t_0) - 2\phi'(t_0))\,\beta^2h^2 \\ &= \phi(t_0) + \phi'(t_0)(2\,\beta - \beta^2)h. \end{aligned} \tag{11}$$

Let $h^* = (2\,\beta - \beta^2)h$. Since $0 \le \beta \le 1$, then $0 \le 2\,\beta - \beta^2 \le 1$, and thus $0 \le h^* \le h$ and $(X_2(t), Y_2(t)) \approx (t_0 + h^*, \phi(t_0) + h^*\phi'(t_0))$. If $t = t_1$ then $h^* = h$ and $(X_2(t_1), Y_2(t_1)) \approx (t_0 + h, \phi(t_0) + h\phi'(t_0))$. And if $t = t_0$ then $h^* = 0$ and $(X_2(t_0), Y_2(t_0)) \approx (t_0, \phi(t_0))$.

Example 3.1. We find the quadratic piecewise approximation of the circle when the number of arcs is 4, i.e. for the angle $\pi/4$. We consider the arc of the circle joining the points $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Applying the process by substituting in equations (8) and (9), we get the approximating polynomial given by:

$$X_2(t) = \sqrt{2}(t - \frac{1}{2}), \quad Y_2(t) = -\sqrt{2}((t - \frac{1}{2})^2 - \frac{3}{4}).$$

The curve and the quadratic approximation are plotted in Fig. 1.

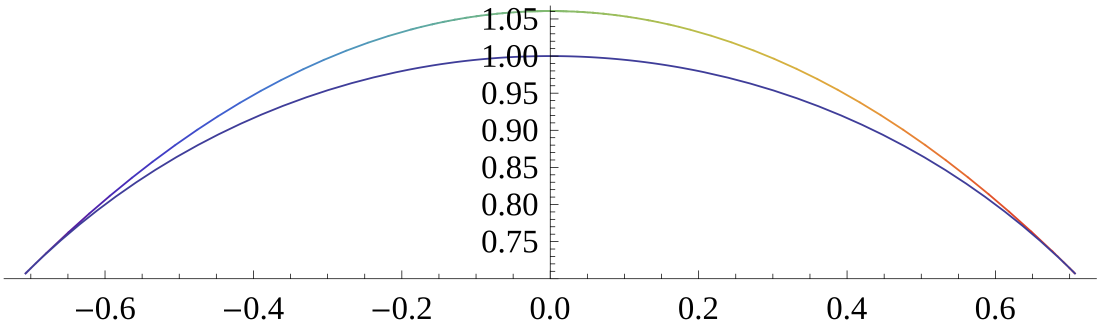

Figure 1: Arc of the circle and the quadratic approximation

4. Quadrature Formula

To find the area under the parametric curve C given by (1) for $a \le t \le b$, we proceed as follows: Since the area under $y = F(x)$, $a_1 \le x \le b_1$ is given by:

$$A_C = \int_{a_1}^{b_1} F(x)dx,$$

then we substitute $x = f(t)$ and substitute $a_1 = f(a)$ and $b_1 = f(b)$ and since $\frac{dx}{dt} = f'(t)$, thus

$$A_C = \int_{a_1}^{b_1} F(x)dx = \int_a^b F(f(t))f'(t)dt.$$

Since $y = F(x) = F(f(t)) = g(t)$, then we get

$$A_C = \int_a^b g(t)f'(t)dt. \tag{12}$$

Since P approximates C, thus, the area under the curve C is approximated by the area under P. Following a similar approach for the curve P, we get the area under the curve P by the formula

$$A_P = \int_0^1 Y_2(t)X_2'(t)dt. \tag{13}$$

Applying the formulas for $X_2(t)$ and $Y_2(t)$ in (8) and (9) we get the area under the curve P, $0 \le t \le 1$ by the formula:

$$\begin{aligned} A_P &= \int_0^1 Y_2(t)X_2'(t)dt \\ &= \sum_{i=1}^{2}\sum_{j=0}^{2} \frac{i}{i+j}\, a_i\, b_j \\ &= a_1b_0 + a_2b_0 + \frac{a_1b_1}{2} + \frac{2a_2b_1}{3} + \frac{a_1b_2}{3} + \frac{a_2b_2}{2}, \end{aligned} \tag{14}$$

where $a_i,\ b_i,\ i = 0, 1, 2$ are given by (6) and (7). Since P approximates C with order 4, $C(t) - P(t) = O(h^4)$, thus the value of the integral of P approximates the value of the integral of C with order 5, $\int_a^b C(t)\,dt - \int_0^1 P(t)\,dt = A_C - A_P = O(h^5)$. To verify this property numerically, the following approach in the next section is presented for the case of approximating the area enclosed by the circle.

5. Order of Approximation

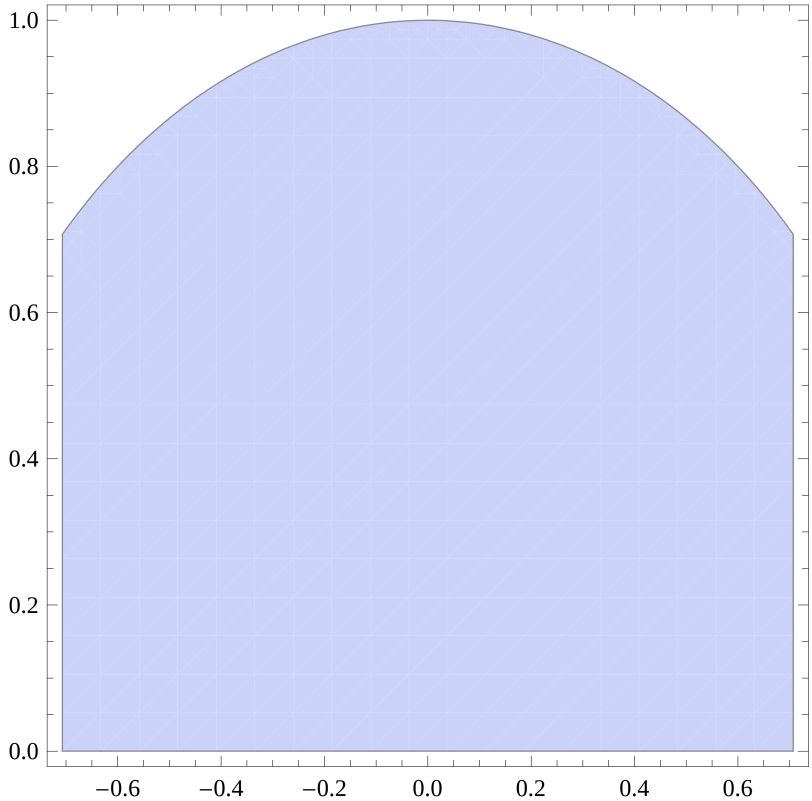

Figure 2: Area under the circle

We approximate consecutive arcs of the circle and find the area under these approximated arcs using the formula (14). To compute the order of approximation of the quadrature formula (14), we assume that the resulting error associated with calculating the area by dividing the circle into n parts has the form

$$e_n \approx c\, n^{\alpha},\ \text{for some } c, \alpha \in \Re.$$

Then for two consecutive errors $e_n \approx c\, n^{\alpha}$ and $e_m \approx c\, m^{\alpha}$, we get

$$\frac{e_n}{e_m} \approx \frac{c\, n^{\alpha}}{c\, m^{\alpha}} \approx (\frac{n}{m})^{\alpha}.$$

Take the natural logarithm (ln) for both sides to get:

$$\ln(\frac{e_n}{e_m}) \approx \alpha \ln(\frac{n}{m}).$$

This leads to the following formula for the order of approximation:

$$\alpha \approx \frac{\ln(e_n) - \ln(e_m)}{\ln(n) - \ln(m)}.$$

No. of points	Exact Area	Approximated Area	Error	Order
4	1.2853982	1.3333333	4.79×10^{-2}	
8	0.74625247	0.74754689	1.29×10^{-3}	-5.20
16	0.387691257	0.387730533	3.93×10^{-5}	-5.06
32	0.1957199314	0.1957211502	1.22×10^{-6}	-5.02
64	0.0980959553771	0.0980959933994	3.81×10^{-8}	-4.98
128	0.04907752976988	0.04907753095756	1.24×10^{-9}	-4.98

Table 1: The order of approximation

Table 1 presents the area under that portion of the upper semi-circle for the values of $-\cos(\frac{1}{2} - \frac{1}{2^n})\pi \leq x \leq \cos(\frac{1}{2} - \frac{1}{2^n})\pi$, $n = 2, 3, \ldots, 7$. The second column shows the exact values of the integral, while the third column shows the approximation using the method in formula (14). The fourth column shows the associated error. The fifth column shows that, as expected, the order of approximation is 5.

6. Comparison and Conclusions

Computing the quadratic Taylor and Lagrange methods shows that these methods can not compete with our method. So, we compare our method with the Gaussian quadrature method. Advantages of the method:

1. High degree of exactness as the Gaussian quadrature formula.
2. Gives area under any parametric curve, which can not be achieved by the Gaussian quadrature method.
3. The proposed quadrature formula is given in explicit form in terms of the values of the curve and its first derivatives at the end points.

As further research, we propose to follow the following proposals:

1. Finding a cubic quadrature formula that has degree of exactness of five, see [11].
2. Building a spline oriented method that divides a complex curve into subcurves and applying the method on each part with different kinds of smoothness conditions, see [1].
3. Constructing composite quadrature methods for the quadratic and cubic cases and compare their numerical performances with the composite Newton-Cotes methods, see [12].

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