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Computing the (k-)monopoly Number of Direct Product of Graphs

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Abstract. Let $G = (V, E)$ be a simple graph without isolated vertices and minimum degree $\delta(G)$, and let $k \in \{1 - \lceil \delta(G)/2 \rceil, \ldots, \lfloor \delta(G)/2 \rfloor\}$ be an integer. Given a set $M \subset V$, a vertex v of G is said to be k-controlled by M if $\delta_M(v) \geq \frac{\delta(v)}{2} + k$ where $\delta_M(v)$ represents the quantity of neighbors v has in M and $\delta(v)$ the degree of v. The set M is called a k-monopoly if it k-controls every vertex v of G. The minimum cardinality of any k-monopoly is the k-monopoly number of G. In this article we study the k-monopoly number of direct product graphs. Specifically we obtain tight lower and upper bounds for the k-monopoly number of direct product graphs in terms of the k-monopoly numbers of its factors. Moreover, we compute the exact value for the k-monopoly number of several families of direct product graphs.

1. Introduction and Preliminaries

Throughout this article we consider simple graphs $G = (V, E)$. Given a set $S \subset V$ and a vertex $v \in V$, we denote by $\delta_S(v)$ the number of neighbors v has in S. If $S = V$, then $\delta_V(v)$ is the degree of v and we just write $\delta(v)$. The *minimum degree* of G is denoted by $\delta(G)$ and the *maximum degree* by $\Delta(G)$. Given an integer $k \in \left\{1 - \left\lceil \frac{\delta(G)}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta(G)}{2} \right\rfloor\right\}$ and a set M, a vertex v of G is said to be *k-controlled* by M if $\delta_M(v) \geq \frac{\delta(v)}{2} + k$. The set M is called a *k-monopoly* if it k-controls every vertex v of G. The minimum cardinality of any k-monopoly is the *k-monopoly number* and it is denoted by $\mathcal{M}_k(G)$. A monopoly of cardinality $\mathcal{M}_k(G)$ is called a $\mathcal{M}_k(G)$-set. In particular notice that for a graph with a leaf (vertex of degree one), there exist only 0-monopolies and the neighbor of every leaf is in each $\mathcal{M}_0(G)$-set. Monopolies in graphs were defined first in [10] and they were generalized to k-monopolies recently in [9]. Other studies about monopolies in graphs and some of its applications can be found in [2, 7, 11, 12, 16].

If $\overline{M}$ represents the *complement* of the set M, then we can use the following equivalent definition for a k-monopoly in G. A set of vertices M is a k-monopoly in G if and only if for every vertex v of G, $\delta_M(v) \geq \delta_{\overline{M}}(v) + 2k$ (we call this expression *the k-monopoly condition* for v) and we say that M is a k-monopoly in G if and only if every v of G satisfies the k-monopoly condition for M.

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Monopolies in graphs have a quite long range of applications in several problems related to overcoming failures, since they frequently have some common approaches around the notion of majorities, for instance to consensus problems [1], diagnosis problems [13] or voting systems [3], among other applications and references. Moreover, monopolies in graphs are closely related to different parameters in graphs. According to several connections which exist between monopolies, global alliances and signed domination in graphs (see [9]), it is known that the complexity of computing the k-monopoly number of a graph is an NP-complete problem for $k \geq 0$, while this is an open problem for $k < 0$. In this sense, it is desirable to reduce the problem of computing the k-monopoly number of some graph classes to other ones in which could be easier to compute the value of such a parameter. That is the case of product graphs. In this article we obtain several relationships between the k-monopoly number of product graphs and the k-monopoly number of its factors.

We use for a graph G standard notations, $N_G(g)$ for the *open neighborhood* $\{g' : gg' \in E(G)\}$ and $N_G[g]$ for the *closed neighborhood* $N_G(g) \cup \{g\}$. Let $S \subset V(G)$. Neighborhoods over S form a *subpartition* of a graph G if $N_G(u) \cap N_G(v) = \emptyset$ for every different $u, v \in S$. A set S forms a *maximum subpartition* if neighborhoods centered in S form a subpartition, where $V(G) - \bigcup_{v\in S} N_G(v)$ has the minimum cardinality among all possible sets $S' \subset V(G)$ which neighborhoods centered in S' form a subpartition. In the extreme case, where $\bigcup_{v\in S} N_G(v) = V(G)$, we call G an *efficient open domination graph* and the set S an *efficient open dominating set* of G. Efficient open domination graphs were first studied in [4]. The work was continued in [5], where all efficient open domination trees have been inductively described. Also, there was proved that the problem of deciding whether G is an efficient open domination graph or not is NP-complete. Recently in [14] the efficient open dominating sets of Cayley graphs were considered and a discussion with respect to product graphs can be found in [8]. Clearly not all graphs are efficient open domination graphs.

One lower bound of this work is based on the following observation.

Observation 1.1. *Let G be a graph of minimum degree δ. If $S \subset V(G)$ forms a subpartition of G, then for any $k \in \left\{1 - \left\lceil \frac{\delta}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta}{2} \right\rfloor\right\}$*

$$\mathcal{M}_k(G) \geq k|S| + \sum_{v\in S} \left\lceil \frac{\delta(v)}{2} \right\rceil.$$

The Observation clearly holds, since S forms a subpartition. That is, the condition $\delta_M(v) \geq \frac{\delta(v)}{2} + k$ is fulfilled for all vertices of S, but not necessarily for all vertices of G. Hence the lower bound follows. Clearly we can expect the best results for maximum subpartitions and, particularly, in efficient open domination graphs. The following consequence is useful to obtain some other results (in particular for regular graphs).

Corollary 1.2. *Let G be a graph of minimum degree δ. If $S \subset V(G)$ forms a subpartition of G, then for any $k \in \left\{1 - \left\lceil \frac{\delta}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta}{2} \right\rfloor\right\}$*

$$\mathcal{M}_k(G) \geq |S| \left(\left\lceil \frac{\delta}{2} \right\rceil + k \right).$$

The following general result will also be useful to obtain a lower bound on the monopoly number of direct product graphs.

Proposition 1.3. *Let G be a graph order n, minimum degree δ and maximum degree Δ. Then, for any $k \in \left\{1 - \left\lceil \frac{\delta}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta}{2} \right\rfloor\right\}$*

$$\mathcal{M}_k(G) \geq \left\lceil \frac{n}{\Delta} \left(\left\lceil \frac{\delta}{2} \right\rceil + k \right) \right\rceil.$$

Proof. Let S be a $\mathcal{M}_k(G)$-set and let v be a vertex of G. Notice that v appears at most Δ times in the neighborhoods of vertices $u \neq v$. Also, we have that $\delta_S(v) \geq \lceil \delta/2 \rceil + k$. Thus,

$$|S| \geq \frac{1}{\Delta} \sum_{v\in V(G)} \delta_S(v) \geq \frac{1}{\Delta} \sum_{v\in V(G)} \left(\left\lceil \frac{\delta}{2} \right\rceil + k \right) = \frac{n}{\Delta} \left(\left\lceil \frac{\delta}{2} \right\rceil + k \right),$$

and the result follows. □

Various graph products have been investigated in the last few decades and a rich theory involving the structure and recognition of classes of these graphs has emerged, cf. the new book [6]. The most studied graph products are the Cartesian product, the strong product, the direct product, and the lexicographic product which are also called *standard products*. The other standard approach to graph products is to deduce properties of a product with respect to its factors. The latest aproach is followed also in this work.

The *direct product* $G \times H$ of graphs G and H is a graph with $V(G \times H) = V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \times H$ whenever $gg' \in E(G)$ and $hh' \in E(H)$. For a fix $h \in V(H)$ we call $G^h = \{(g, h) \in G \times H : g \in V(G)\}$ a *G-layer* in $G \times H$. H-layers ${}^gH$ for a fix $g \in V(G)$ are defined symmetrically. Notice that in direct product of graphs without loops, the subgraphs induced by both a G-layer or an H-layer represent a graph without edges on $|V(G)|$ or $|V(H)|$ vertices, respectively. The map $p_G : V(G \times H) \to V(G)$ defined by $p_G((g, h)) = g$ is called a *projection map onto* G. Similarly, we define p_H the *projection map onto* H. Projections are defined as maps between vertices, but many times it is more comfortably to see them as maps between graphs. In this case we observe the subgraphs induced by $A \subseteq V(G \times H)$ and $p_X(A)$ for $X \in \{G, H\}$.

The open neighborhoods of vertices in direct product are nicely connected to open neighborhoods of projections to the factors. Namely, $N_{G\times H}(g, h) = N_G(g) \times N_H(h)$ for every vertex $(g, h) \in V(G) \times V(H)$. The direct product is not always a connected graph even if both factors are connected. Indeed, $G \times H$ is a connected graph if and only if at least one of the graphs G or H is non bipartite. Moreover, if both G and H are bipartite, $G \times H$ has exactly two components (see [6, 15]) where the vertices (g, h) and (g, h') with $hh' \in E(H)$ are in different components, as are (g, h) and (g', h) with $gg' \in E(G)$. In particular, a graph $G \times H$ has an isolated vertex if and only if there exists an isolated vertex in G or in H. Thus, we should omit graphs with isolated vertices.

2. The Upper Bounds

In this section we prove two upper bounds for $\mathcal{M}_k(G \times H)$ with respect to the properties of factors G and H.

Theorem 2.1. *Let G, H be two graphs without isolated vertices and let $\ell = \min\{\delta(G), \delta(H)\}$. For any $k \in \{1 - \left\lceil \frac{\ell}{2} \right\rceil, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor\}$ we have*

$$\mathcal{M}_k(G \times H) \le \min\{\mathcal{M}_k(G)|V(H)|, |V(G)|\mathcal{M}_k(H)\}.$$

Proof. Let (g, h) be an arbitrary vertex of $G \times H$ and let M_G and M_H be a $\mathcal{M}_k(G)$-set and a $\mathcal{M}_k(H)$-set, respectively. We can split $N_{G\times H}(g, h) = (A \times C) \cup (B \times C) \cup (A \times D) \cup (B \times D)$ where $A = N_G(g) \cap M_G$, $B = N_G(g) \cap \overline{M_G}$, $C = N_H(h) \cap M_H$ and $D = N_H(h) \cap \overline{M_H}$. Clearly $|A \times C| = \delta_{M_H}(h)\delta_{M_G}(g)$, $|B \times C| = \delta_{\overline{M_G}}(g)\delta_{M_H}(h)$, $|A \times D| = \delta_{M_G}(g)\delta_{\overline{M_H}}(h)$ and $|B \times D| = \delta_{\overline{M_G}}(g)\delta_{\overline{M_H}}(h)$.

Notice that $\delta_{M_H}(h) \ge \delta_{\overline{M_H}}(h) + 2k$ since M_H is a $\mathcal{M}_k(H)$-set. Hence, it follows that $\delta_{M_H}(h) \ge \delta_{\overline{M_H}}(h) + \frac{2k}{\delta_{M_G}(g)+\delta_{\overline{M_G}}(g)}$ while there are no isolated vertices in G (note that the equality can be attained only if $k = 0$ and $\delta_{M_H}(h) = \delta_{\overline{M_H}}(h)$). Thus we obtain

$$\delta_{M_H}(h)(\delta_{M_G}(g) + \delta_{\overline{M_G}}(g)) \ge \delta_{\overline{M_H}}(h)(\delta_{M_G}(g) + \delta_{\overline{M_G}}(g)) + 2k$$

and finally

$$\delta_{M_H}(h)\delta_{M_G}(g) + \delta_{M_H}(h)\delta_{\overline{M_G}}(g) \ge \delta_{\overline{M_H}}(h)\delta_{M_G}(g) + \delta_{\overline{M_H}}(h)\delta_{\overline{M_G}}(g) + 2k.$$

If we use the language of sets, we get

$$|A \times C| + |B \times C| \ge |A \times D| + |B \times D| + 2k$$

and for $M = V(G) \times M_H$ this means $\delta_M(g, h) \ge \delta_{\overline{M}}(g, h) + 2k$. Thus $\mathcal{M}_k(G \times H) \le |V(G)|\mathcal{M}_k(H)$. By commutativity of direct product and symmetry of operations we also derive $\delta_{M'}(g, h) \ge \delta_{\overline{M'}}(g, h) + 2k$ for $M' = M_G \times V(H)$. Therefore, $\mathcal{M}_k(G \times H) \le \mathcal{M}_k(G)|V(H)|$ which ends the proof. □

From the proof above we immediately see that the upper bound from Theorem 2.1 behaves better for k close to 0 and it is the best when $k = 0$. Also, the bound is not defined for all k-monopolies where $k \in \left\{1 - \left\lceil \frac{\delta(G\times H)}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta(G\times H)}{2} \right\rfloor\right\}$. For instance $\mathcal{M}_{\left\lfloor \frac{\delta(G\times H)}{2} \right\rfloor}(G)$-sets and $\mathcal{M}_{\left\lfloor \frac{\delta(G\times H)}{2} \right\rfloor}(H)$-sets do not exist whenever $\delta(G) > 1$ and $\delta(H) > 1$.

Next we present another upper bound, which sometimes behaves better than the bound of Theorem 2.1.

Theorem 2.2. *For any graphs G and H of order r and t, respectively,*

$$\mathcal{M}_{2k^2}(G \times H) \leq rt + 2\mathcal{M}_k(G)\mathcal{M}_k(H) - r\mathcal{M}_k(H) - t\mathcal{M}_k(G).$$

Proof. Let S_G and S_H be a $\mathcal{M}_k(G)$-set and a $\mathcal{M}_k(H)$-set, respectively. We shall prove that $S = (S_G\times S_H)\cup(\overline{S_G}\times\overline{S_H})$ is a $(2k^2)$-monopoly in $G \times H$. It is clear that S is a dominating set in $G \times H$. Let (u, v) be a vertex of $G \times H$. Thus,

$$\begin{aligned}
\delta_S(u, v) &= \delta_{S_G\times S_H}(u, v) + \delta_{\overline{S_G}\times\overline{S_H}}(u, v)\\
&= \delta_{S_G}(u)\delta_{S_H}(v) + \delta_{\overline{S_G}}(u)\delta_{\overline{S_H}}(v)\\
&= \delta_{\overline{S_G}}(u)\delta_{S_H}(v) + \delta_{S_G}(u)\delta_{\overline{S_H}}(v) - \delta_{\overline{S_G}}(u)\delta_{S_H}(v) - \delta_{S_G}(u)\delta_{\overline{S_H}}(v) +\\
&\quad + \delta_{S_G}(u)\delta_{S_H}(v) + \delta_{\overline{S_G}}(u)\delta_{\overline{S_H}}(v)\\
&= \delta_{\overline{S_G}\times S_H}(u, v) + \delta_{S_G\times\overline{S_H}}(u, v) + (\delta_{S_G}(u) - \delta_{\overline{S_G}}(u))(\delta_{S_H}(v) - \delta_{\overline{S_H}}(v))\\
&\geq \delta_{\overline{S}}(u, v) + (\delta_{\overline{S_G}}(u) + 2k - \delta_{\overline{S_G}}(u))(\delta_{\overline{S_H}}(v) + 2k - \delta_{\overline{S_H}}(v))\\
&= \delta_{\overline{S}}(u, v) + 4k^2.
\end{aligned}$$

Therefore, S is $(2k^2)$-monopoly in $G \times H$ and the result follows. □

3. The Lower Bounds

Recently, in [8], all efficient open domination graphs among direct products have been characterized as those for which both factors are efficient open domination graphs. We can not generalized this result to maximum subpartitions completely, but in the part that is actually needed.

Proposition 3.1. *Let G and H be graphs. If S_G and S_H form maximum subpartitions of G and H, respectively, then $S = S_G \times S_H$ forms a maximum subpartition of $G \times H$.*

Proof. Let S_G and S_H be maximum subpartitions of G and H, respectively, and let $S = S_G \times S_H$. Since $N_{G\times H}(g, h) = N_G(g) \times N_H(h)$ holds, we have $N_{G\times H}(g, h) \cap N_{G\times H}(g', h') = \emptyset$ for every different vertices (g, h) and (g', h') from S. If S is not a maximum partition of $G \times H$, then there exists $(g, h) \notin S$ with $N_{G\times H}(g, h) \cap N_{G\times H}(g', h') = \emptyset$ for every $(g', h') \in S$. Since S_G and S_H are maximum partitions of G and H, respectively, there exists $u \in S_G$ and $v \in S_H$, such that $N_G(g) \cap N_G(u) \neq \emptyset$ and $N_H(h) \cap N_H(v) \neq \emptyset$. If $w \in N_G(g) \cap N_G(u)$ and $z \in N_H(h) \cap N_H(v)$, then $(w, z) \in N_{G\times H}(g, h) \cap N_{G\times H}(u, v)$, which is a contradiction. Hence S is a maximum partition of $G \times H$. □

The following theorem follows directly from the Observation 1.1, Proposition 3.1 and the fact that the degree in the direct product is a product of degrees in factors.

Theorem 3.2. *Let G and H be graphs without isolated vertices and let $\ell = \delta(G)\delta(H)$. If S_G and S_H form maximum subpartitions of G and H, respectively, then for every $k \in \left\{1 - \left\lceil \frac{\ell}{2} \right\rceil, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor\right\}$,*

$$\mathcal{M}_k(G \times H) \geq k|S_G||S_H| + \sum_{(g,h)\in S_G\times S_H} \left\lceil \frac{\delta_G(g)\delta_H(h)}{2} \right\rceil.$$

An interesting consequence of the theorem above is the next one, which sometimes gives better results than the bound of Theorem 3.2.

Corollary 3.3. *Let G and H be two graphs without isolated vertices and let $\ell = \delta(G)\delta(H)$. If S_G and S_H form maximum subpartitions of G and H, respectively, then for every $k \in \left\{1 - \left\lceil \frac{\ell}{2} \right\rceil, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor\right\}$,*

$$\mathcal{M}_k(G \times H) \geq |S_G||S_H| \left(\left\lceil \frac{\delta(G)\delta(H)}{2} \right\rceil + k \right).$$

Another lower bound for the k-monopoly number of direct product graph is obtained as a consequence of Proposition 1.3.

Corollary 3.4. *Let G and H be two graphs without isolated vertices of order n and m, respectively. For any $k \in \left\{1 - \left\lceil \frac{\delta(G)\delta(H)}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta(G)\delta(H)}{2} \right\rfloor\right\}$ we have*

$$\mathcal{M}_k(G \times H) \geq \left\lceil \frac{mn}{\Delta(G)\Delta(H)} \left(\left\lceil \frac{\delta(G)\delta(H)}{2} \right\rceil + k \right) \right\rceil.$$

4. Exact Values

The upper and lower bounds presented in the two sections above are tight as we will see in this next section. The following propositions ([9]) about the monopoly number of some families of graphs are useful to prove our results.

Proposition 4.1. [9] *For every complete graph K_n and every $k \in \left\{1 - \left\lceil \frac{\delta(G)}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta(G)}{2} \right\rfloor\right\}$,*

$$\mathcal{M}_k(K_n) = \left\lceil \frac{n + 2k + 1}{2} \right\rceil.$$

Proposition 4.2. [9] *For every integer $n \geq 3$,*

$$\mathcal{M}_0(C_n) = \mathcal{M}_0(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \text{ mod } 4, \\ \frac{n+2}{2} & \text{if } n \equiv 2 \text{ mod } 4, \\ \frac{n+1}{2} & \text{if } n \equiv 1 \text{ mod } 4 \text{ or } n \equiv 3 \text{ mod } 4. \end{cases}$$

Proposition 4.3. [9] *For every complete bipartite graph $K_{r,t}$ and every $k \in \left\{1 - \left\lceil \frac{\delta(K_{r,t})}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta(K_{r,t})}{2} \right\rfloor\right\}$,*

$$\mathcal{M}_k(K_{r,t}) = \left\lceil \frac{r + 2k}{2} \right\rceil + \left\lceil \frac{t + 2k}{2} \right\rceil.$$

Next we present several formulas for the 0-monopoly number of some direct product graphs. The following four results follow from Theorem 2.2 (or 2.1), Corollary 3.4 and Propositions 4.1, 4.2 and 4.3. We present the proof of the first one and we omit the other ones due to its similarity.

Proposition 4.4. *If r and t are odd integers greater than 2, then*

$$\mathcal{M}_0(K_r \times K_t) = \left\lceil \frac{rt}{2} \right\rceil.$$

Proof. If r, t are odd integers, then $r-1, t-1$ are even numbers. Now, since $K_r \times K_t$ is a regular graph of degree $(r-1)(t-1)$, Corollary 3.4 leads to the following.

$$\mathcal{M}_0(K_r \times K_t) \geq \left\lceil \frac{rt}{(r-1)(t-1)} \left\lceil \frac{(r-1)(t-1)}{2} \right\rceil \right\rceil = \left\lceil \frac{rt}{2} \right\rceil.$$

On the other hand, since $\mathcal{M}_0(K_n) = \left\lceil \frac{n+1}{2} \right\rceil$ (Lemma 4.1) and $r+1, t+1$ are even numbers, by Theorem 2.2 we have that

$$\mathcal{M}_0(K_r \times K_t) \leq rt + 2\frac{r+1}{2}\frac{t+1}{2} - r\frac{t+1}{2} - t\frac{r+1}{2} = \frac{rt+1}{2} = \left\lceil \frac{rt}{2} \right\rceil.$$

□

Proposition 4.5. *If $r \geq 2$ is any integer and $t \equiv 0 \pmod 4$, then*

$$\mathcal{M}_0(K_r \times C_t) = \frac{rt}{2}.$$

Proposition 4.6. *If r and t are positive even integers, then*

$$\mathcal{M}_0(K_{r,r} \times K_{t,t}) = 2rt.$$

Proposition 4.7. *If r is a positive even integer and t is a positive odd integer, then*

$$\mathcal{M}_0(K_{r,r} \times K_t) = rt.$$

From [4] we know that a path P_n is an efficient open domination graph if and only if $n \not\equiv 1 \pmod 4$ and a cycle C_n is an efficient open domination graph if and only if $n \equiv 0 \pmod 4$. The following observations are useful to prove our results.

Observation 4.8.

(i) *If $n \equiv 0 \pmod 4$, then every vertex belonging to an efficient open dominating set in P_n has degree two.*
(ii) *If $n \equiv 2 \pmod 4$, then all but two vertices belonging to an efficient open dominating set in P_n have degree two. The other two vertices have degree one.*
(iii) *If $n \equiv 3 \pmod 4$, then all but one vertex belonging to an efficient open dominating set in P_n have degree two. The other vertex has degree one.*

Observation 4.9. *Let G be a path or a cycle. If G is an efficient open domination graph with an efficient open dominating set S, then $\mathcal{M}_0(G) = |S|$.*

Proof. Since G has maximum degree two, $\mathcal{M}_0(G) \leq |S|$ follows from the fact that every vertex of G has one neighbor in S. On the other hand, $\mathcal{M}_0(G) \geq |S|$ follows from Observation 1.1. □

Theorem 4.10. *If G and H are two efficient open domination graphs being paths or cycles of order r and t, respectively, then*

$$\mathcal{M}_0(G \times H) = \begin{cases} \frac{rt}{2}, & \text{if } r \equiv 0 \pmod 4 \text{ or } t \equiv 0 \pmod 4, \\ \frac{rt+4}{2}, & \text{if } r, t \equiv 2 \pmod 4, \\ \frac{rt+2}{2}, & \text{if } r \equiv 2 \pmod 4 \text{ and } t \equiv 3 \pmod 4, \\ \frac{rt+1}{2}, & \text{if } r, t \equiv 3 \pmod 4. \end{cases}$$

Proof. Let S_G and S_H be efficient open dominating sets of G and H, respectively. From Theorem 3.2 we have that $\mathcal{M}_0(G\times H) \geq \sum_{(g,h)\in S_G\times S_H} \left\lceil \frac{\delta_G(g)\delta_H(h)}{2} \right\rceil$. From Observation 4.9 we have that $|S_G\times S_H| = \mathcal{M}_0(G)\mathcal{M}_0(H)$. Also, by Observation 4.8, S_G and S_H can have at most two vertices of degree one in G or H, respectively. Thus, to obtain a lower bounds for $\mathcal{M}_0(G\times H)$ we only need to count the number of vertices of S_G and S_H having degree one or two. We consider the following cases.

Case 1: $r, t \equiv 0 \pmod 4$. By Observation 4.8 (when G or H is a path), every vertex of S_G and S_H has degree two. Thus, we have

$$\mathcal{M}_0(G\times H) \geq \sum_{(g,h)\in S_G\times S_H} \left\lceil \frac{\delta_G(g)\delta_H(h)}{2} \right\rceil = 2\mathcal{M}_0(G)\mathcal{M}_0(H).$$

Since $r, t \equiv 0 \pmod 4$, Proposition 4.2 leads to $\mathcal{M}_0(G) = r/2$ and $\mathcal{M}_0(H) = t/2$. Thus, from the expression above we obtain that $\mathcal{M}_0(G\times H) \geq \frac{rt}{2}$. By Theorem 2.1 we obtain that $\mathcal{M}_0(G\times H) \leq \min\{r\mathcal{M}_0(H), t\mathcal{M}_0(G)\} = \frac{rt}{2}$.

Case 2: $r \equiv 0 \pmod 4$ and $t \equiv 2 \pmod 4$. Hence H must be a path, every vertex of S_G has degree two and, by Observation 4.8, all but two vertices of S_H, say x, y, have degree two and x and y have degree one. Thus,

$$\begin{aligned}
\mathcal{M}_0(G\times H) &\geq \sum_{(g,h)\in S_G\times S_H} \left\lceil \frac{\delta_G(g)\delta_H(h)}{2} \right\rceil \\
&= \sum_{(g,h)\in(S_G\times S_H)-(S_G\times\{x,y\})} 2 + \sum_{(g,h)\in S_G\times\{x,y\}} 1 \\
&= 2\mathcal{M}_0(G)(\mathcal{M}_0(H)-2) + 2\mathcal{M}_0(G) \\
&= 2\mathcal{M}_0(G)(\mathcal{M}_0(H)-1).
\end{aligned}$$

Now, from Lemma 4.2 we have that $\mathcal{M}_0(G) = r/2$ and $\mathcal{M}_0(H) = \frac{t+2}{2}$, which lead to $\mathcal{M}_0(G\times H) \geq \frac{rt}{2}$. Again, by using the Theorem 2.1 we obtain that $\mathcal{M}_0(G\times H) \leq \min\{r\mathcal{M}_0(H), t\mathcal{M}_0(G)\} = \frac{rt}{2}$.

Case 3: $r \equiv 0 \pmod 4$ and $t \equiv 3 \pmod 4$. Hence H must be a path, every vertex of S_G has degree two and, by Observation 4.8, all but one vertex of S_H, say w, have degree two and w has degree one. Thus,

$$\begin{aligned}
\mathcal{M}_0(G\times H) &\geq \sum_{(g,h)\in S_G\times S_H} \left\lceil \frac{\delta_G(g)\delta_H(h)}{2} \right\rceil \\
&= \sum_{(g,h)\in(S_G\times S_H)-(S_G\times\{w\})} 2 + \sum_{(g,h)\in S_G\times\{w\}} 1 \\
&= 2\mathcal{M}_0(G)(\mathcal{M}_0(H)-1) + \mathcal{M}_0(G) \\
&= 2\mathcal{M}_0(G)\mathcal{M}_0(H) - \mathcal{M}_0(G).
\end{aligned}$$

Now, from Lemma 4.2 we have that $\mathcal{M}_0(G) = r/2$ and $\mathcal{M}_0(H) = \frac{t+1}{2}$, which lead to $\mathcal{M}_0(G\times H) \geq \frac{rt}{2}$. Again, by using the Theorem 2.1 we obtain that $\mathcal{M}_0(G\times H) \leq \min\{r\mathcal{M}_0(H), t\mathcal{M}_0(G)\} = \frac{rt}{2}$.

Case 4: $r, t \equiv 2 \pmod 4$. Hence G and H must be paths. So, by Observation 4.8, all but two vertices of S_G, say x, y, have degree two and x and y have degree one. By the same reason all but two vertices of S_H, say u, v, have degree two and u and v have degree one. Thus, if $B_1 = (S_G\times S_H) - (\{x,y\}\times S_H) - (S_G\times\{u,v\})$,

$B_2 = (S_G \times \{u, v\}) - (\{x, y\} \times \{u, v\})$, $B_3 = (\{x, y\} \times S_H) - (\{x, y\} \times \{u, v\})$ and $B_4 = \{x, y\} \times \{u, v\}$, then

$$\begin{aligned}
\mathcal{M}_0(G \times H) &\geq \sum_{(g,h)\in S_G\times S_H} \left\lceil \frac{\delta_G(g)\delta_H(h)}{2} \right\rceil \\
&= \sum_{(g,h)\in B_1} 2 + \sum_{(g,h)\in B_2} 1 + \sum_{(g,h)\in B_3} 1 + \sum_{(g,h)\in B_4} \left\lceil \frac{1}{2} \right\rceil \\
&= 2(\mathcal{M}_0(G) - 2)(\mathcal{M}_0(H) - 2) + 2(\mathcal{M}_0(G) - 2) + 2(\mathcal{M}_0(H) - 2) + 4\left\lceil \frac{1}{2} \right\rceil \\
&= 2\mathcal{M}_0(G)\mathcal{M}_0(H) - 2\mathcal{M}_0(G) - 2\mathcal{M}_0(H) + 4.
\end{aligned}$$

Now, from Lemma 4.2 we have that $\mathcal{M}_0(G) = \frac{r+2}{2}$ and $\mathcal{M}_0(H) = \frac{t+2}{2}$, which lead to $\mathcal{M}_0(G \times H) \geq \frac{rt+4}{2}$.

On the other hand, from Theorem 2.2 we have that

$$\mathcal{M}_0(G \times H) \leq rt + 2\frac{r+2}{2}\frac{t+2}{2} - r\frac{t+2}{2} - t\frac{r+2}{2} = \frac{rt+4}{2},$$

which completes the proof of this case.

Case 5: $r \equiv 2 \pmod 4$ and $t \equiv 3 \pmod 4$. Hence G and H must be paths. So, by Observation 4.8, all but two vertices of S_G, say a, b, have degree two and a amd b have degree one. Similar all but one vertex of S_H, say c, have degree two and c have degree one. Thus, if $A_1 = (S_G \times S_H) - (\{a, b\} \times S_H) - (S_G \times \{c\})$, $A_2 = (S_G \times \{c\}) - (\{a, b\} \times \{c\})$, $A_3 = (\{a, b\} \times S_H) - (\{a, b\} \times \{c\})$ and $A_4 = \{a, b\} \times \{c\}$, then

$$\begin{aligned}
\mathcal{M}_0(G \times H) &\geq \sum_{(g,h)\in S_G\times S_H} \left\lceil \frac{\delta_G(g)\delta_H(h)}{2} \right\rceil \\
&= \sum_{(g,h)\in A_1} 2 + \sum_{(g,h)\in A_2} 1 + \sum_{(g,h)\in A_3} 1 + \sum_{(g,h)\in A_4} \left\lceil \frac{1}{2} \right\rceil \\
&= 2(\mathcal{M}_0(G) - 2)(\mathcal{M}_0(H) - 1) + (\mathcal{M}_0(G) - 2) + 2(\mathcal{M}_0(H) - 1) + 2\left\lceil \frac{1}{2} \right\rceil \\
&= 2\mathcal{M}_0(G)\mathcal{M}_0(H) - \mathcal{M}_0(G) - 2\mathcal{M}_0(H) + 2.
\end{aligned}$$

Now, from Lemma 4.2 we have that $\mathcal{M}_0(G) = \frac{r+2}{2}$ and $\mathcal{M}_0(H) = \frac{t+1}{2}$, which lead to $\mathcal{M}_0(G \times H) \geq \frac{rt+2}{2}$.

Similarly to Case 4, from Theorem 2.2 we have that

$$\mathcal{M}_0(G \times H) \leq rt + 2\frac{r+2}{2}\frac{t+1}{2} - r\frac{t+1}{2} - t\frac{r+2}{2} = \frac{rt+2}{2},$$

which completes the proof of this case.

Case 6: $r, t \equiv 3 \pmod 4$, then G and H must be paths. Hence, by Observation 4.8, all but one vertex of S_G, say a, have degree two and a has degree one. Also all but one vertex of S_H, say c, have degree two and c has degree one. Thus, if $A_1 = (S_G \times S_H) - (\{a\} \times S_H) - (S_G \times \{c\})$, $A_2 = (S_G \times \{c\}) - (\{a\} \times \{c\})$, $A_3 = (\{a\} \times S_H) - (\{a\} \times \{c\})$ and $A_4 = \{a\} \times \{c\}$, then

$$\begin{aligned}
\mathcal{M}_0(G \times H) &\geq \sum_{(g,h)\in S_G\times S_H} \left\lceil \frac{\delta_G(g)\delta_H(h)}{2} \right\rceil \\
&= \sum_{(g,h)\in A_1} 2 + \sum_{(g,h)\in A_2} 1 + \sum_{(g,h)\in A_3} 1 + \sum_{(g,h)\in A_4} \left\lceil \frac{1}{2} \right\rceil \\
&= 2(\mathcal{M}_0(G) - 1)(\mathcal{M}_0(H) - 1) + (\mathcal{M}_0(G) - 1) + (\mathcal{M}_0(H) - 1) + \left\lceil \frac{1}{2} \right\rceil \\
&= 2\mathcal{M}_0(G)\mathcal{M}_0(H) - \mathcal{M}_0(G) - \mathcal{M}_0(H) + 1.
\end{aligned}$$

Now, from Lemma 4.2 we have that $\mathcal{M}_0(G) = \frac{r+1}{2}$ and $\mathcal{M}_0(H) = \frac{t+1}{2}$, which lead to $\mathcal{M}_0(G \times H) \geq \frac{rt+1}{2}$.

Finally, like in the Cases 4 and 5, from Theorem 2.2 we have that

$$\mathcal{M}_0(G \times H) \leq rt + 2\frac{r+1}{2}\frac{t+1}{2} - r\frac{t+1}{2} - t\frac{r+1}{2} = \frac{rt+1}{2},$$

which completes the proof. □

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