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On the Index of Product Systems of Hilbert Modules

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Abstract. In this note we prove that the set of all uniformly continuous units on a product system over a C^* algebra $\mathcal{B}$ can be endowed with a structure of left-right $\mathcal{B}$ - $\mathcal{B}$ Hilbert module after identifying similar units by the suitable equivalence relation. We use this construction to define the index of the initial product system, and prove that it is a generalization of earlier defined indices by Arveson (in the case $\mathcal{B} = \mathbb{C}$) and Skeide (in the case of spatial product system). We prove that such defined index is a covariant functor from the category of continuous product systems to the category of $\mathcal{B}$ bimodules. We also prove that the index is subadditive with respect to the outer tensor product of product systems, and prove additional properties of the index of product systems that can be embedded into a spatial one.

1. Introduction

Product systems over $\mathbb{C}$ have been studied during last several decades in connection with E_0-semigroups acting on a type I factor. Although the main problem of classification of all non isomorphic product systems is still open, this theory is well developed. The reader is referred to the book [2] and references therein. In the present century there are some significant results that generalize this theory to product systems over some C^*-algebra $\mathcal{B}$, either in connection with E_0 semigroups (see [13], [3], [15]) or in connection with quantum probability dynamics (see [7], [4], [14]).

There are many difficulties in generalizing the notion of index of a product system introduced in [1] to this more general concept. Up to our knowledge there is one attempt in this direction done in [16], using redefining the notion of tensor product of two product systems in order to retain nice behaviour of index with respect to tensor product.

The main point of this note is to find the natural generalization of the index of product systems from Arveson's $\mathbb{C}$-case to a more general C^*-algebra case. To this purpose we consider the quotient set $\mathcal{U}/\sim$ of all uniformly continuous units on the given product system E by a suitable equivalence relation, and prove that $\mathcal{U}/\sim$ carries a natural structure of a two-sided $\mathcal{B}-\mathcal{B}$ module.

Throughout the whole paper $\mathcal{B}$ will denote a unital C^*-algebra and 1 will denote its unit. Also, we shall use $\otimes$ for tensor product, either algebraic or other, although $\odot$ is also in common use.

The rest of the introduction is devoted to basic definitions.

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Definition 1.1. *a) A product system over C^*-algebra $\mathcal{B}$ is a family $(E_t)_{t\geq 0}$ of Hilbert $\mathcal{B}-\mathcal{B}$ modules, with $E_0 \cong \mathcal{B}$, and a family of (unitary) isomorphisms*

$$\varphi_{t,s} : E_t \otimes E_s \to E_{t+s},$$

where $\otimes$ stands for the so-called inner tensor product obtained by identifications $ub\otimes v \sim u\otimes bv$, $u\otimes vb \sim (u\otimes v)b$, $bu\otimes v \sim b(u\otimes v)$, $(u \in E_t, v \in E_s, b \in \mathcal{B})$ and then completing in the inner product $\langle u\otimes v, u_1\otimes v_1\rangle = \langle v, \langle u, u_1\rangle v_1\rangle$;

b) Unit on E is a family $u_t \in E_t$, $t \geq 0$, so that $u_0 = 1$ and $\varphi_{t,s}(u_t \otimes u_s) = u_{t+s}$, which we shall abbreviate to $u_t \otimes u_s = u_{t+s}$. A unit u_t is unital if $\langle u_t, u_t\rangle = 1$. It is central if for all $b \in \mathcal{B}$ and all $t \geq 0$ there holds $bu_t = u_t b$;

The previous definition does not include any technical condition, such as measurability, or continuity. It occurs that it is sometimes more convenient to pose the continuity condition directly on units, although there is a definition of a continuous product system which we shall use in Section 3.

Definition 1.2. *Two units u_t and v_t give rise to the family of mappings $\mathcal{K}_t^{u,v} : \mathcal{B} \to \mathcal{B}$, given by*

$$\mathcal{K}_t^{u,v}(b) = \langle u_t, bv_t\rangle .$$

All $\mathcal{K}_t^{u,v}$ are bounded $\mathbb{C}$-linear operators on $\mathcal{B}$, and this family forms a semigroup. We say that the set of units S is continuous if the corresponding semigroup $(K_t^{\xi,\eta})_{\xi,\eta\in S}$ (with respect to Schur multiplying) is uniformly continuous. For a single unit u_t we say that it is uniformly continuous, or briefly just continuous, if the set $\{u\}$ is continuous, that is, the corresponding family $\mathcal{K}_t^{u,u}$ is continuous in the norm of the space $L(\mathcal{B})$.

Given a (uniformly) continuous set of units $\mathcal{U}$, we can form, as it was shown in [4] a uniformly continuous completely positive definite semigroup (*CPD*-semigroup further on) $\mathcal{K} = (\mathcal{K}_t)_{t\in\mathbb{R}_+}$

Denote by $\mathcal{L} = \frac{d}{dt}\mathcal{K}\mid_{t=0}$ the generator of CPD-semigroup $\mathcal{K}$. It is well known [4] that $\mathcal{L}$ is conditionally completely positive definite, that is, for all finite n-tuple $x_1, \dots, x_n \in \mathcal{U}$ and for all $a_j, b_j \in \mathcal{B}$ we have

$$\sum_{j=1}^{n} a_j b_j = 0 \Longrightarrow \sum_{i,j=1}^{n} b_i^* \mathcal{L}^{x_i,x_j}(a_i^* a_j) b_j \geq 0. \tag{1}$$

It holds, also, $\mathcal{L}^{y,x}(b) = \mathcal{L}^{x,y}(b^*)^*$.

Also, $\mathcal{K}$ is uniquely determined by $\mathcal{L}$. More precisely, $\mathcal{K}$ can be recovered from $\mathcal{L}$ by $\mathcal{K} = e^{t\mathcal{L}}$ using Schur product, i.e.

$$\mathcal{K}_t^{x,y}(b) = \langle x_t, by_t\rangle = (\exp t\mathcal{L}^{x,y})(b). \tag{2}$$

Remark 1.3. *We should tell the difference between the continuous set of units and the set of continuous units. In the second case only $\mathcal{K}_t^{\xi,\xi}$ should be uniformly continuous for $\xi \in S$, whereas in the first case all $\mathcal{K}_t^{\xi,\eta}$ should be uniformly continuous.*

In Section 2 we list and prove auxiliary statements that are necessary for the proofs of main results. In Section 3 we define the notion of the index of a given product system and prove its functoriality from the category of a continuous product systems to the category of left-right $\mathcal{B}$ - $\mathcal{B}$ Hilbert modules. Section 4 is devoted to the outer tensor product of product systems and to the behaviour of the index with respect to it. In Section 5 we discuss how the existence of a central unit, either in the product system E or in some of its extensions, affects the index. All examples are left for the last Section 6, as well as concluding remarks.

The proofs in this note requires a technique specific for Hilbert C^*-modules reduced to a few initial statements. Nevertheless, we refer the reader to books [11] and [8] for elaborate approach to this topic.

2. Preliminary Results

In [10], Liebscher and Skeide introduce an interesting way to obtain new units in a given product system. The results are stated in Lemma 3.1, Proposition 3.3 and Lemma 3.4 of the mentioned paper. We briefly quote them as

Proposition 2.1. *a) Suppose that a continuous set S of units generates a product system E. Let $t \mapsto y_t \in E_t$ be a mapping (not necessarily unit), so that for all $b \in \mathcal{B}$ and some $K, K_\xi \in L(\mathcal{B})$ ($\xi \in S$) we have*

$$\langle y_t, b y_t\rangle = b + tK(b) + O(t^2),$$

$$\langle y_t, b\xi_t\rangle = b + tK_\xi(b) + O(t^2).$$

Then there is a product system $F \supseteq E$ and a unit ζ so that $S \cup \{\zeta\}$ is continuous and

$$\mathcal{L}^{\zeta,\zeta} = K \text{ and } \mathcal{L}^{\zeta,\xi} = K_\xi.$$

b) Also, the following three conditions are mutually equivalent:

1. *$\zeta \in E$;*
2. *ζ can be obtained as the norm limit of the sequence $(y_{t/n})^{\otimes n}$;*
3. $\lim\limits_{n\to\infty} \langle \zeta_t, (y_{t/n})^{\otimes n}\rangle = \langle \zeta_t, \zeta_t\rangle$.

Remark 2.2. *In [10] a more general limit over the filter of all partitions of segment $[0, t]$ instead of $\lim_{n\to\infty}(y_{t/n})^{\otimes n}$ is considered. However, we do not need such a general context.*

These results are used, in the same paper, to construct units starting with mappings

$$t \mapsto \sum_{j=1}^{n} \varkappa_j x_t^j, \qquad \varkappa_j \in \mathbb{C}, \quad \sum_{j=1}^{n} \varkappa_j = 1$$

and

$$t \mapsto x_t e^{\beta t}, \tag{3}$$

where $x, x^1, \dots, x^n \in S$, $\beta \in \mathcal{B}$, proving, also, that the resulting units, denoted by $\varkappa_1 x^1 \boxplus \cdots \boxplus \varkappa_n x^n$ and x^β, respectively, belong to S. (Obviously the same unit x^β is obtained if we start with mapping $t \mapsto e^{\beta t} x_t$ instead of (3), since both of them have the same generators $\mathcal{L}^{x,y}(b) + \beta^* b$, $\mathcal{L}^{x,x}(b) + \beta^* b + b\beta$.) It was, also, noted that the product $\boxplus$ is associative unless the expression makes sense. In fact, it holds

$$\varkappa_1 x^1 \boxplus \varkappa_2 x^2 \boxplus \varkappa_3 x^3 = (\varkappa_1 + \varkappa_2)\left(\frac{\varkappa_1}{\varkappa_1 + \varkappa_2} x^1 \boxplus \frac{\varkappa_2}{\varkappa_1 + \varkappa_2} x^2\right) \boxplus \varkappa_3 x^3,$$

provided that $\varkappa_1 + \varkappa_2 \neq 0$, and a similar equality

$$\varkappa_1 x^1 \boxplus \varkappa_2 x^2 \boxplus \varkappa_3 x^3 = \varkappa_1 x^1 \boxplus (\varkappa_2 + \varkappa_3)\left(\frac{\varkappa_2}{\varkappa_2 + \varkappa_3} x^2 \boxplus \frac{\varkappa_3}{\varkappa_2 + \varkappa_3} x^3\right)$$

provided that $\varkappa_2 + \varkappa_3 \neq 0$.

The kernels of x^β are given by

$$\begin{aligned} \mathcal{L}^{x^\beta,x^\beta} &= \mathcal{L}^{x,x} + \beta^* \mathrm{id}_{\mathcal{B}} + \mathrm{id}_{\mathcal{B}}\beta, \\ \mathcal{L}^{x^\beta,\xi} &= \mathcal{L}^{x,\xi} + \beta^* \mathrm{id}_{\mathcal{B}}. \end{aligned} \tag{4}$$

For our purpose, however, it is useful to substitute the complex numbers $\varkappa_j$ by elements of $\mathcal{B}$. In other words we have

Proposition 2.3. *Suppose that a continuous set S of units generates a product system E. Let $x^j \in S$, let let $\varkappa_j \in \mathcal{B}$, $j = 1, \dots, n$ and let $\sum \varkappa_j = 1$. Then the functions*

$$t \mapsto \sum_{j=1}^{n} \varkappa_j x_t^j \quad \text{and } t \mapsto \sum_{j=1}^{n} x_t^j \varkappa_j$$

satisfy all assumptions of Proposition 2.1, and the resulting units belong to S. We shall denote them by

$$\varkappa_1 x^1 \boxplus \dots \boxplus \varkappa_n x^n \quad \text{and } x^1 \varkappa_1 \boxplus \dots \boxplus x^n \varkappa_n.$$

Proof. We get the assertion as a special case of the third example in [10, Section 4.2], putting $\xi^{k+1} = \xi^0$, $a_{k+1} = 1$, $b_{k+1} = -1$, and either $a_j = 1$ or $b_j = 1$ ($j = 1, 2, \dots, k$).

The proof can also be obtained directly by writing down the kernels. In the case $n = 2$ the mappings

$$t \mapsto x_t^1 \varkappa_1 + x_t^2 \varkappa_2 \quad \text{and} \quad t \mapsto \varkappa_1 x_t^1 + \varkappa_2 x_t^2$$

lead to the kernels

$$K = \varkappa_1^* \mathcal{L}^{x^1,x^1} \varkappa_1 + \varkappa_1^* \mathcal{L}^{x^1,x^2} \varkappa_2 + \varkappa_2^* \mathcal{L}^{x^2,x^1} \varkappa_1 + \varkappa_2^* \mathcal{L}^{x^2,x^2} \varkappa_2,$$

$$K_\xi = \varkappa_1^* \mathcal{L}^{x^1,\xi} + \varkappa_2^* \mathcal{L}^{x^2,\xi}$$

and

$$K = \mathcal{L}^{x^1,x^1} L_{\varkappa_1^*} R_{\varkappa_1} + \mathcal{L}^{x^1,x^2} L_{\varkappa_1^*} R_{\varkappa_2} + \mathcal{L}^{x^2,x^1} L_{\varkappa_2^*} R_{\varkappa_1} + \mathcal{L}^{x^2,x^2} L_{\varkappa_2^*} R_{\varkappa_2},$$

$$K_\xi = \mathcal{L}^{x^1,\xi} L_{\varkappa_1^*} + \mathcal{L}^{x^2,\xi} L_{\varkappa_2^*},$$

where $L_a, R_a : \mathcal{B} \to \mathcal{B}$ are the left and right multiplication operators for $a \in \mathcal{B}$, and these kernels generate required units. □

We, also, need the following Lemmata.

Lemma 2.4. *Let $\mathcal{U}$ be the set of all uniformly continuous units on a given product system E, and let $x, y \in \mathcal{U}$. If for all $\xi \in \mathcal{U}$ there holds $\mathcal{L}^{x,\xi} = \mathcal{L}^{y,\xi}$ then $x = y$.*

Proof. From (2) we obtain

$$\langle x_t, b\xi_t \rangle = \langle y_t, b\xi_t \rangle.$$

For $b = 1$, $\xi = x$, it becomes $\langle x_t, x_t \rangle = \langle y_t, x_t \rangle$, and for $b = 1$, $\xi = y$ it becomes $\langle x_t, y_t \rangle = \langle y_t, y_t \rangle$. Combining the last two relations we find $\langle x_t - y_t, x_t - y_t \rangle = 0$. □

Remark 2.5. *The assumption "for all $\xi \in \mathcal{U}$" is superfluous in the previous Lemma. We only use the equality for $\xi = x$ and $\xi = y$.*

Lemma 2.6. *Let x be a continuous unit on a product system E. Then $x^{-\beta/2}$ is a unital unit, where $\beta = \mathcal{L}^{x,x}(1)$. Moreover, if x is central, then $x^{-\beta/2}$ is central, as well.*

Proof. Since it always holds

$$\mathcal{L}^{y,x}(b) = (\mathcal{L}^{x,y}(b^*))^* \tag{5}$$

we get $\beta = \beta^*$. Further, by (4) we obtain $\mathcal{L}^{x^{-\beta/2},x^{-\beta/2}}(1) = \mathcal{L}^{x,x}(1) - \beta^*/2 - \beta/2 = 0$, and therefore

$$\left\langle x_t^{-\beta/2}, x_t^{-\beta/2} \right\rangle = \left(\exp t\mathcal{L}^{x^{-\beta/2},x^{-\beta/2}}\right)(1) = 1.$$

If, in addition, x is central, then

$$be^{t\beta} = b\langle x_t, x_t \rangle = \langle x_t b^*, x_t \rangle = \langle x_t, x_t b \rangle = e^{t\beta} b,$$

which implies that $b\beta = \beta b$ for all $b \in \mathcal{B}$. As it is easy to see, $x_t^{-\beta/2} = x_t e^{-t\beta/2}$ and we conclude that $x^{-\beta/2}$ is, also, central. □

3. Definition of Index

Let E be a product system. We define the index as a quotient of a certain set of continuous units on E by a suitable inner product. Thus, the index is defined rather as operand on a set of continuous units then on a product system. However, choosing a reference unit ω, there is a maximal continuous set of units $\mathcal{U}_\omega$ that contains ω (in the product system E) - see next Proposition. Therefore we refer the index as $\operatorname{ind}(E,\omega)$ and show that it is independent of the choice of ω in the same continuous set of units.

Proposition 3.1. *Let $\mathcal{U}$ denote the set of all continuous units on a product system E. We define the relation $\simeq$ on $\mathcal{U}$ by*

$$x \simeq y \Leftrightarrow \{x,y\} \text{ is a continuous set.}$$

This relation is an equivalence relation.

Proof. This relation is obviously reflexive and symmetric. We have only to prove that it is transitive, i.e. that $\{\xi,\eta\}$ and $\{\eta,\zeta\}$ are continuous sets implies that $\{\xi,\zeta\}$ is also a continuous set.

It suffices to prove the uniform continuity of the mapping $b \mapsto \langle \xi_t, b\zeta_t\rangle$, at $t=0$. We begin considering the difference $\xi_t-\eta_t$. Choosing $b=1$ we have

$$\langle \xi_t-\eta_t, \xi_t-\eta_t\rangle = \mathcal{K}^{\xi,\xi}(1) - \mathcal{K}_t^{\xi,\eta}(1) - \mathcal{K}_t^{\eta,\xi}(1) + \mathcal{K}_t^{\eta,\eta}(1) \to 1-1-1+1=0.$$

We also have

$$\langle \xi_t, b\zeta_t\rangle = \langle \xi_t-\eta_t, b\zeta_t\rangle + \langle \eta_t, b\zeta_t\rangle.$$

The second summand is uniformly continuous in b, by continuity of $\{\eta,\zeta\}$ and, consequently, tends to b (as $t\to 0+$), whereas for the first summand the following estimate holds

$$\|\langle \xi_t-\eta_t, b\zeta_t\rangle\| \le \|\xi_t-\eta_t\|\,\|b\zeta_t\| \le C\|\xi_t-\eta_t\| \to 0.$$

(See, also the proof of [4, Lemma 4.4.11].) $\square$

Thus, the set $\mathcal{U}$ can be decomposed into mutually disjoint collection of maximal continuous sets of units.

Let E be a product system over a unital C^*-algebra $\mathcal{B}$ with at least one continuous unit. (In view of [14, Definition 4.4] this means that E is non type *III* product system.) Further, let ω be an arbitrary continuous unit in E and let $\mathcal{U}=\mathcal{U}_\omega$ be the set of all uniformly continuous units that are equivalent to ω.

We define the addition and multiplication by $b\in\mathcal{B}$ on $\mathcal{U}_\omega$ by

$$x+y = x\boxplus y\boxplus -\omega,\quad b\cdot x = bx\boxplus(1-b)\omega,\quad x\cdot b = xb\boxplus\omega(1-b). \tag{6}$$

The kernels of $x+y$, $x\cdot a$, $a\cdot x$ are

$$\begin{aligned}\mathcal{L}^{x+y,x+y} &= \mathcal{L}^{x,x}+\mathcal{L}^{x,y}-\mathcal{L}^{x,\omega}+\mathcal{L}^{y,x}+\mathcal{L}^{y,y}-\mathcal{L}^{y,\omega}-\mathcal{L}^{\omega,x}-\mathcal{L}^{\omega,y}+\mathcal{L}^{\omega,\omega},\\ \mathcal{L}^{x+y,\xi} &= \mathcal{L}^{x,\xi}+\mathcal{L}^{y,\xi}-\mathcal{L}^{\omega,\xi},\end{aligned} \tag{7}$$

$$\begin{aligned}\mathcal{L}^{x\cdot a,x\cdot a} &= a^*\mathcal{L}^{x,x}a+(1-a)^*\mathcal{L}^{\omega,x}a+a^*\mathcal{L}^{x,\omega}(1-a)+(1-a)^*\mathcal{L}^{\omega,\omega}(1-a),\\ \mathcal{L}^{x\cdot a,\xi} &= a^*\mathcal{L}^{x,\xi}+(1-a)^*\mathcal{L}^{\omega,\xi},\ \xi\in\mathcal{U},\end{aligned} \tag{8}$$

$$\begin{aligned}\mathcal{L}^{a\cdot x,a\cdot x} &= \mathcal{L}^{x,x}L_{a^*}R_a+\mathcal{L}^{\omega,x}L_{1-a^*}R_a+\mathcal{L}^{x,\omega}L_{a^*}R_{1-a}+\mathcal{L}^{\omega,\omega}L_{1-a^*}R_{1-a},\\ \mathcal{L}^{a\cdot x,\xi} &= \mathcal{L}^{x,\xi}L_{a^*}+\mathcal{L}^{\omega,\xi}L_{1-a^*},\ \xi\in\mathcal{U}.\end{aligned} \tag{9}$$

We, also, define an equivalence relation $\approx$ by: $x\approx y$ if and only if $x=y^\beta$ for some $\beta\in\mathcal{B}$.

Theorem 3.2. *a) The set $\mathcal{U}$ with respect to operations defined by (6) is a left-right $\mathcal{B}-\mathcal{B}$ module.*
b) The relation $\approx$ is an equivalence relation and it is compatible with all algebraic operations in $\mathcal{U}$, i.e.

$$x \approx y \Longrightarrow x \cdot b \approx y \cdot b,\ b \cdot x \approx b \cdot y,$$

$$x^1 \approx x^2, y^1 \approx y^2 \Longrightarrow x^1 + y^1 \approx x^2 + y^2;$$

Proof. a) The associativity follows from the associativity of $\boxplus$. In more details, both $(x+y)+z$ and $x+(y+z)$ are equal to $x \boxplus y \boxplus z \boxplus (-2\omega)$.

The neutral element is ω and the inverse is $2\omega \boxplus (-x)$ which can be easily checked.

Commutativity is obvious.

The other axioms of left-right $\mathcal{B}-\mathcal{B}$ module $(x \cdot a) \cdot b = x \cdot (ab)$, $a \cdot (b \cdot x) = (ab) \cdot x$, $a \cdot (x+y) = a \cdot x + a \cdot y$, $(x+y) \cdot a = x \cdot a + y \cdot a$ and $1 \cdot x = x \cdot 1 = x$ can be easily checked by comparing the kernels.

b) Reflexivity follows choosing $\beta = 0$. If $x = y^\beta$ then $\mathcal{L}^{x,\xi} = \mathcal{L}^{y,\xi} + \beta^*\mathrm{id}_{\mathcal{B}}$, and hence

$$\mathcal{L}^{y,\xi} = \mathcal{L}^{x,\xi} - \beta^*\mathrm{id}_{\mathcal{B}} = \mathcal{L}^{x^{-\beta},\xi},$$

from which and from Lemma 2.4 the symmetry follows.

Transitivity. If $x = y^\beta$ and $y = z^\alpha$ then

$$\mathcal{L}^{x,\xi} = \mathcal{L}^{y,\xi} + \beta^*\mathrm{id}_{\mathcal{B}} = \mathcal{L}^{z,\xi} + (\alpha+\beta)^*\mathrm{id}_{\mathcal{B}} = \mathcal{L}^{z^{\alpha+\beta},\xi}$$

for any $\xi \in \mathcal{U}$. From this and from Lemma 2.4 we conclude that $x = z^{\alpha+\beta}$.

Let us now prove that the result of addition and multiplication by $b \in \mathcal{B}$ does not depend on the choice of β. Indeed, if $x, y \in \mathcal{U}$ and $x_1 = x^\beta$, $y_1 = y^\alpha$, for some $\alpha, \beta \in \mathcal{B}$. Then, by (7) and (4)

$$\begin{aligned}\mathcal{L}^{x_1+y_1,\xi} &= \mathcal{L}^{x_1,\xi} + \mathcal{L}^{y_1,\xi} - \mathcal{L}^{\omega,\xi} = \mathcal{L}^{x,\xi} + \beta^*\,\mathrm{id}_{\mathcal{B}} + \mathcal{L}^{y_1,\xi} + \alpha^*\,\mathrm{id}_{\mathcal{B}} - \mathcal{L}^{\omega,\xi} = \\ &= \mathcal{L}^{x+y,\xi} + (\alpha+\beta)^*\,\mathrm{id}_{\mathcal{B}} = \mathcal{L}^{(x+y)^{\alpha+\beta},\xi}\end{aligned}$$

for any $\xi \in \mathcal{U}$. It follows that, again using Lemma 2.4

$$x_1 + y_1 = (x+y)^{\alpha+\beta}.$$

Further, let $x_1 = x^\beta$ and let $a \in \mathcal{B}$. Then by (8)

$$\begin{aligned}\mathcal{L}^{x_1\cdot a,\xi} &= a^*\mathcal{L}^{x_1,\xi} + (1-a)^*\mathcal{L}^{\omega,\xi} = a^*(\mathcal{L}^{x,\xi} + \beta^*\,\mathrm{id}_{\mathcal{B}}) + (1-a)^*\mathcal{L}^{\omega,\xi} = \\ &= \mathcal{L}^{x\cdot a,\xi} + a^*\beta^*\,\mathrm{id}_{\mathcal{B}} = \mathcal{L}^{(x\cdot a)^{\beta a},\xi},\end{aligned}$$

for any $\xi \in \mathcal{U}$. It follows, once again using Lemma 2.4, that

$$x_1 \cdot a = (x \cdot a)^{\beta a}.$$

A similar argument shows that $a \cdot x_1 = (a \cdot x)^{a\beta}$. □

We can immediately form the quotient module $\mathcal{U}/\approx$. However, it might not be the accurate choice, taking into account possible choices of the inner product. Thus, we are looking for the most suitable choice of a $\mathcal{B}$ valued inner product on $\mathcal{U}$. For a while, we shall consider a family of candidates. Namely, for every positive element $b \in \mathcal{B}$ there is a map $\langle\ ,\ \rangle_b : \mathcal{U} \times \mathcal{U} \longrightarrow \mathcal{B}$ given by

$$\langle x, y\rangle_b = (\mathcal{L}^{x,y} - \mathcal{L}^{x,\omega} - \mathcal{L}^{\omega,y} + \mathcal{L}^{\omega,\omega})(b), \tag{10}$$

where ω is the same as in (6).

Any of these mappings is $\mathcal{B}$-valued semi-inner product (in the sense that it can be degenerate, i.e. $\langle x, x\rangle_b = 0$ need not imply $x = 0$). Nevertheless, it satisfies all other customary properties.

Proposition 3.3. *The pairing (10) satisfies the following properties:*

(a). *For all* $x, y, z \in \mathcal{U}$, *and* $\alpha, \beta \in \mathbb{C}$ $\langle x, \alpha y + \beta z\rangle_b = \alpha\langle x, y\rangle_b + \beta\langle x, z\rangle_b$;
(b). *For all* $x, y \in \mathcal{U}$, $a \in \mathcal{B}$ $\langle x, y \cdot a\rangle_b = \langle x, y\rangle_b a$;
(c). *For all* $x, y \in \mathcal{U}$ $\langle x, y\rangle_b = \langle y, x\rangle_b^*$;
(d). *For all* $x \in \mathcal{U}$ $\langle x, x\rangle_b \geq 0$;
(e). *If* $x \approx x'$ *and* $y \approx y'$ *then* $\langle x, y\rangle_b = \langle x', y'\rangle_b$;
(f). *For all* $x, y \in \mathcal{U}$, $0 \leq a \in \mathcal{B}$ $\langle x, a \cdot y\rangle_1 = \langle x, y\rangle_a$;
(g). *If* $0 \leq b (\in \mathcal{B}) \leq 1$ *then for all* $x \in \mathcal{U}$ *we have* $\langle x, x\rangle_b \leq \langle x, x\rangle_1$.
(h). *There holds* $\langle x - y, x - y\rangle_1 = \lim_{t\to 0+} \dfrac{\langle x_t - y_t, x_t - y_t\rangle}{t}$.

Proof. (a)-(c) is easy to check. (d) follows, since $\mathcal{L}$ is conditionally CPD, precisely we can put $n = 2$, $x^1 = x$, $x^2 = \omega$, $a_1 = a_2 = \sqrt{b}$, $b_1 = 1$, $b_2 = -1$ in (1). (e) follows from the cancellation of terms $\beta^* b$ and $b\gamma$ in expanded form of $\langle x', y'\rangle$, where $x' = x^\beta$ and $y' = y^\gamma$.

To conclude (f), expand $\langle x, a \cdot y\rangle$ and use (9).

(g) - Since $\mathcal{L}$ is conditionally completely positive definite, we get $\mathcal{L}^{x,x}(1-b) - \mathcal{L}^{x,\omega}(1-b) - \mathcal{L}^{\omega,x}(1-b) + \mathcal{L}^{\omega,\omega}(1-b) \geq 0$. It follows that $\langle x, x\rangle_b \leq \langle x, x\rangle_1$.

(h) follows from (10) and the definition of $\mathcal{L}$ after a few cancellations. □

Our choice for the inner product will be $\langle\cdot,\cdot\rangle_1$, which we shall abbreviate to $\langle\cdot,\cdot\rangle$ further on, i.e. if we omit the index b we shall assume $b = 1$. From the properties (b) – (d) of the previous Proposition we can derive the Cauchy Schwartz inequality (see [8, Proposition 1.1] or [11, Proposition 1.2.4])

$$\langle x, y\rangle \langle x, y\rangle^* \leq \langle x, x\rangle \, \| \langle y, y\rangle \|. \tag{11}$$

It follows that the set $N = \{x \in \mathcal{U} \mid \langle x, x\rangle = 0\}$ is equal to $\{x \in \mathcal{U} \mid \forall y \in \mathcal{U},\ \langle x, y\rangle = 0\}$ and that it contains $\{\omega^\beta \mid \beta \in \mathcal{B}\}$ (property (e)). From this and from (a) – (c) and (f) we conclude that N is a submodule of $\mathcal{U}$, and $\mathcal{U}/N$ is a pre-Hilbert left right $\mathcal{B}$ - $\mathcal{B}$ module.

Definition 3.4. *Let E be a product system, and let ω be a continuous unit on E. The index of a pair (E, ω) is the completion of pre-Hilbert left-right module $\mathcal{U}/\sim$, where $\mathcal{U} = \mathcal{U}_\omega$ is the maximal continuous set of units containing ω, and $\sim$ is the equivalence relation defined by $x \sim y$ if and only if $x - y \in N$. Naturally, the index will be denoted by* $\mathrm{ind}(E, \omega)$.

Remark 3.5. *If E can be embedded into a spatial product system, as we shall see in the next section, the completion is unnecessary.*

Remark 3.6. *If $\{\omega, \omega'\}$ is a continuous set, then $\mathrm{ind}(E, \omega) \cong \mathrm{ind}(E, \omega')$. Indeed, then $\mathcal{U}_\omega = \mathcal{U}_{\omega'}$ and the isometric isomorphism is given by translation $x \mapsto x \boxplus -\omega \boxplus \omega'$*

The following definition of a *continuous* product system [14, Section 7] will allow us to speak of the index of E without highlighting the unit ω.

Definition 3.7. *Continuous product system is a product system $(E_t)_{t\geq 0}$, together with a family of isometric embeddings $i_t : E_t \to E$ into a unital Hilbert bimodule, which satisfies*

1. *For every $y_s \in E_s$ there is a continuous section $(x_t) \in CS_i(E)$ so that $y_s = x_s$;*
2. *For every pair $x, y \in CS_i(E)$ of continuous sections the function $(s, t) \mapsto i_{s+t}(x_s \otimes y_t)$ is continuous;*

where the set of continuous sections (with respect to i) is

$$CS_i(E) = \{x = (x_t)_{t\geq 0} \mid x_t \in E_t,\ t \mapsto i_t x_t \text{ is continuous}\}.$$

By [14, Theorems 7.5 and 7.7] (see also [5, Theorems 2.4 and 2.5]) there is at most one continuous structure on E that makes a given continuous unit ω a continuous section. Further, given a continuous unit $\omega \in CS_i(E)$, the set $\mathcal{U}_\omega$ coincides with the set of all continuous units that belong to $CS_i(E)$. Indeed, if a continuous unit x belongs to $CS_i(E)$ then $\omega \simeq x$ by [14, Theorem 7.7]. Conversely, if $\omega \simeq x$ and x is a continuous unit then

$$\|x_{t+\varepsilon} - x_t\| \le \|x_t \otimes x_\varepsilon - x_t \otimes \omega_\varepsilon\| + \|x_t \otimes \omega_\varepsilon - x_t\| \le \|x_t\|(\|x_\varepsilon - \omega_\varepsilon\| + \|\omega_\varepsilon - 1\|) \to 0,$$

as $\varepsilon \to 0+$, because the first summand tends to zero by $\omega \simeq x$, whereas the second summand tends to zero, by $\omega \in CS_i(E)$. Left continuity follows from right, since $\|x_{t-\varepsilon} - x_t\| \le \|x_{t-\varepsilon}\| \|1 - x_\varepsilon\|$

Thus, for continuous product systems, we shall not underlain the unit ω, i.e. we shall write ind(E).

The class of all continuous product systems is a category, if morphisms are defined as follows.

Definition 3.8. *The mapping $\theta : E \to F$ between two continuous product systems (with embeddings i and j respectively) is a morphism if:*

1. *$\theta|_{E_t}$ is a bounded adjointable $\mathcal{B}-\mathcal{B}$ linear mappings $\theta_t : E_t \to F_t$ fulfilling $\theta_{t+s} = \theta_t \otimes \theta_s$ and $\theta_0 = \mathrm{id}_{\mathcal{B}}$;*
2. *Both θ and θ^* preserve continuous structure, i.e. if $(x_t)_{t\ge0} \in CS_i(E)$ is a continuous section then $(\theta(x_t))_{t\ge0} \in CS_j(F)$, and if $(y_t)_{t\ge0} \in CS_j(F)$ then $(\theta^*(y_t))_{t\ge0} \in CS_i(E)$;*
3. *$\limsup\limits_{t\to0+} \|\theta_t\| < +\infty$.*

Remark 3.9. *In [16, Section 2] morphisms are defined as mappings that satisfy only condition (1) (in previous Definition). This definition is, however, pure algebraic, and we cannot say anything about continuous structure, without additional assumptions.*

Proposition 3.10. *The index is a covariant functor from the category of continuous product systems over $\mathcal{B}$ to the category of all left-right $\mathcal{B}-\mathcal{B}$ modules.*

Proof. Let $\theta : E \to F$ be a morphism. For a reference unit in $\mathcal{U}_F$ choose $\omega' = \theta(\omega)$. For an arbitrary unit $x = (x_t)$ on E, $\theta(x) = (\theta_t(x_t))$ is a unit on F since $\theta_{t+s}(x_{t+s}) = \theta_{t+s}(x_t \otimes x_s) = \theta_t(x_t) \otimes \theta_s(x_s)$ and $\theta_0(x_0) = 1$. Further, if x is continuous, then $x \in CS_i(E)$, implying $\theta(x) \in CS_j(F)$, that is $\theta(x)$ is continuous.

Using Lemma 2.4 and noting that $\mathcal{L}^{\theta(x+y),\xi} = \mathcal{L}^{\theta(x)+\theta(y),\xi}$, it follows that $\theta(x+y) = \theta(x)+\theta(y)$ for $x, y \in \mathcal{U}_E$. Similarly, $\theta(x \cdot a) = \theta(x) \cdot a$ and $\theta(a \cdot x) = a \cdot \theta(x)$, $a \in \mathcal{B}$.

Hence, the mapping $\mathcal{U}_E \ni x \mapsto \theta(x) \in \mathcal{U}_F$ is an algebraic homomorphism.

Let $\theta^* : F \to E$ denote the morphism whose fibers are $\theta_t^* : F_t \to E_t$, and let $x = (x_t)$ be a unit in E. Then $(\theta_t^*\theta_t x_t)$ is also a unit. Let $y \sim y_1$ in E, and denote $\psi_t = \theta_t^*\theta_t$. Then, for all $x \in \mathcal{U}_E$, using Proposition 3.3 ((h)), we obtain $\langle \theta x, \theta y\rangle = \langle \psi x - \psi\omega, y\rangle$ and hence $\langle \theta x, \theta y - \theta y_1\rangle = \langle \psi x - \psi\omega, y - y_1\rangle = 0$, implying $\theta y \sim \theta y_1$. Thus, we obtain a well defined homomorphism $\mathcal{U}_E/\sim\ni [x] \mapsto \mathrm{ind}(\theta)([x]) = [\theta x] \in \mathcal{U}_F/\sim$.

Let us prove that ind(θ) is an adjointable mapping. For any $y \in \mathcal{U}_F$, $\theta^* y \in \mathcal{U}_E$. Then we have

$$\begin{aligned}\langle \mathrm{ind}(\theta)x, y\rangle =&\mathcal{L}^{\theta x,y}(1) - \mathcal{L}^{\theta x,\omega'}(1) - \mathcal{L}^{\omega',y}(1) + \mathcal{L}^{\omega',\omega'}(1) = \\ =&\mathcal{L}^{x,\theta^* y}(1) - \mathcal{L}^{x,\theta^*\omega'}(1) - \mathcal{L}^{\omega,\theta^* y}(1) + \mathcal{L}^{\omega,\theta^*\omega'}(1) = \langle x, \theta^* y - \theta^*\omega'\rangle\,.\end{aligned}$$

This shows that the adjoint of ind(θ) is the mapping $(\mathrm{ind}(\theta))^*(y) = \theta^* y - \theta^*\omega'$ (the composition of ind(θ^*) and translation $x \mapsto x - \theta^*\omega'$).

Finally, let us prove that ind(θ) is bounded, and, therefore, that it can be extended to ind(E). Using Proposition 3.3 ((h)) and [8, Proposition1.2] (or [11, Corollary 2.1.6]) we obtain

$$\langle \theta x, \theta x\rangle = \lim_{t\to0+} \frac{\langle \theta_t x_t - \theta_t\omega_t, \theta_t x_t - \theta_t\omega_t\rangle}{t} \le \limsup_{t\to0+} \|\theta_t\|^2 \frac{\langle x_t - \omega_t, x_t - \omega_t\rangle}{t} \le (\limsup_{t\to0+} \|\theta_t\|^2)\,\langle x, x\rangle\,.$$

Hence ind$(E) \in B^{a,bil}(\mathrm{ind}(E); \mathrm{ind}(F))$ is a morphism in the category of all left-right $\mathcal{B}-\mathcal{B}$ modules over $\mathcal{B}$.

It can be easily seen that $\mathrm{ind}(\mathrm{id}_E) = \mathrm{id}_{\mathrm{ind}(E)}$ and $\mathrm{ind}(\psi\theta) = \mathrm{ind}(\psi)\mathrm{ind}(\theta)$ for all morphisms θ between product systems E and F and ψ between product systems F and G. □

Remark 3.11. *The induced mapping* $\operatorname{ind}(\theta)$ *preserves the relation* $\approx$*. Indeed, if* $x' = x^\beta$, $\beta \in \mathcal{B}$*, then* $\mathcal{L}^{\theta(x'),\xi} = \mathcal{L}^{\theta(x)^\beta,\xi}$ *for* $\xi \in \mathcal{U}_F$ *and Lemma 2.4 implies that* $\theta(x') = \theta(x)^\beta$.

Remark 3.12. *If the condition (3) is suppressed, we only can obtain that* $\operatorname{ind}(\theta)$ *is densely defined adjointable (possibly unbounded) operator from* $\operatorname{ind}(E)$ *to* $\operatorname{ind}(F)$.

Corollary 3.13. *Suppose that E and F are algebraically isomorphic product systems (there is a unitary morphism* $\theta : E \to F$*). Then* $\operatorname{ind}(E, \omega) = \operatorname{ind}(F, \theta(\omega))$.

Proof. Fibers of $\theta : E \to F$ are unitary operators, so their norm is equal to 1 and the condition (3) in Definition 3.8 is fulfilled. Further, $\langle \theta_t x_t, b\theta_t y_t \rangle = \langle x_t, b y_t \rangle$, from which we conclude that θ converts continuous units into continuous, as well as $\langle \theta x, \theta x \rangle = \langle x, x \rangle$. Therefore, in this case $\operatorname{ind}(\theta)$ is a unitary operator, implying $\operatorname{ind}(E) \cong \operatorname{ind}(F)$, or more precisely $\operatorname{ind}(E, \omega) = \operatorname{ind}(F, \theta(\omega))$. $\square$

4. Subadditivity of the index

Given two product systems, E over a unital C^*-algebra $\mathcal{A}$ and F over $\mathcal{B}$, we can consider its (outer) tensor product $E \otimes F$ as a product system over $\mathcal{A} \otimes \mathcal{B}$, taking pointwise outer tensor product $E_t \otimes F_t$ as a Hilbert module over $\mathcal{A} \otimes \mathcal{B}$. (Here $\mathcal{A} \otimes \mathcal{B}$ denotes the spatial tensor product of C^*-algebras.) This is the direct generalization of a tensor product within the category of Arveson product system. On the other hand, it appears as a product system generated by the action of E_0 semigroup $\alpha \otimes \beta$ on $\mathcal{M} \otimes \mathcal{N}$, where α and β are E_0 semigroups on type II_1 factors $\mathcal{M}$ and $\mathcal{N}$ (see [3]).

It is easy to see that units x_t on E, and y_t on F give rise to the unit $x_t \otimes y_t$ on $E \otimes F$. The corresponding semigroup is (evaluated on elementary tensors)

$$\langle x_t \otimes x'_t, (a \otimes b)(y_t \otimes y'_t)\rangle = \langle x_t, ay_t\rangle \otimes \langle x'_t, by'_t\rangle,$$

and its continuity is obvious. Thus, we have the mapping $\mathcal{U}_E \times \mathcal{U}_F \to \mathcal{U}_{E\otimes F}$. If ω and ω' are reference units in $\operatorname{ind} E$ and $\operatorname{ind} F$, then it is natural to choose $\omega \otimes \omega'$ to be the reference unit in $E \otimes F$.

First, we list some basic properties of $x \otimes y$.

Proposition 4.1. *Let* 1 *and* $1'$ *denote the identity elements in* $\mathcal{A}$ *and* $\mathcal{B}$. *Then for all* $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x, y \in \mathcal{U}_E$ *and* $x', y' \in \mathcal{U}_F$ *there holds:*

(a). $\mathcal{L}^{x\otimes x', y\otimes y'}(a \otimes b) = a \otimes \mathcal{L}^{x',y'}(b) + \mathcal{L}^{x,y}(a) \otimes b$ *- Leibnitz rule;*

(b). $\langle x \otimes x', y \otimes y'\rangle = 1 \otimes \langle x', y'\rangle + \langle x, y\rangle \otimes 1'$, *where the inner products are those in* $\mathcal{U}_{E\otimes F}$, $\mathcal{U}_E$ *and* $\mathcal{U}_F$, *respectively;*

(c). $(x \otimes \omega') \cdot (\alpha \otimes 1') = (x \cdot \alpha) \otimes \omega'$, $(\omega \otimes y) \cdot (1 \otimes \beta) = \omega \otimes (y \cdot \beta)$, *where* $\cdot$ *denotes the multiplying in modules* $\mathcal{U}_{E\otimes F}$, $\mathcal{U}_E$ *and* $\mathcal{U}_F$, *respectively;*

(d). $x \otimes y = x \otimes \omega' + \omega \otimes y$, *where the addition is that in module* $\mathcal{U}_{E\otimes F}$;

(e). $(x \otimes \omega') \cdot (1 \otimes \beta) = (1 \otimes \beta) \cdot (x \otimes \omega')$ *and* $(\omega \otimes y) \cdot (\alpha \otimes 1') = (\alpha \otimes 1') \cdot (\omega \otimes y)$;

(f). $\langle x \otimes \omega', \omega \otimes y\rangle = 0$.

Proof. (a) Straightforward calculation;

(b) Follows from (a) and the definition of the inner product;

(c) Using part (a), (8) and (9), after straightforward, but unpleasant calculations we conclude that all kernels:

$$\mathcal{L}^{(x\otimes\omega')\cdot(\alpha\otimes 1'),(x\otimes\omega')\cdot(\alpha\otimes 1')}(a \otimes b) \qquad \mathcal{L}^{(x\otimes\omega')\cdot(\alpha\otimes 1'),x\cdot\alpha\otimes\omega'}(a \otimes b)$$

$$\mathcal{L}^{x\cdot\alpha\otimes\omega',(x\otimes\omega')\cdot(\alpha\otimes 1')}(a \otimes b) \qquad \mathcal{L}^{x\cdot\alpha\otimes\omega',x\cdot\alpha\otimes\omega'}(a \otimes b)$$

are equal to

$$a \otimes \mathcal{L}^{\omega',\omega'}(b) + (\alpha^*\mathcal{L}^{x,x}(a)\alpha + \alpha^*\mathcal{L}^{x,\omega}(a)(1-\alpha) + (1-\alpha^*)\mathcal{L}^{\omega,x}(a)\alpha + (1-\alpha^*)\mathcal{L}^{\omega,\omega}(a)(1-\alpha)) \otimes b.$$

By this, Lemma 2.4 and Remark 2.5 we conclude the first equality. The second follows similarly.

(d) After a few steps we get

$$\mathcal{L}^{x\otimes\omega'+\omega\otimes y,x\otimes\omega'+\omega\otimes y}(a\otimes b) = \mathcal{L}^{x\otimes\omega'+\omega\otimes y,x\otimes y}(a\otimes b) = \mathcal{L}^{x\otimes y,x\otimes\omega'+\omega\otimes y}(a\otimes b) = \mathcal{L}^{x\otimes y,x\otimes y}(a\otimes b) = a\otimes\mathcal{L}^{y,y}(b)+\mathcal{L}^{x,x}(a)\otimes b;$$

(e) Once again, using part (a), (8) and (9) we conclude that all kernels

$$\mathcal{L}^{(x\otimes\omega')\cdot(1\otimes\beta),(x\otimes\omega')\cdot(1\otimes\beta)}(a \otimes b) \qquad \mathcal{L}^{(x\otimes\omega')\cdot(1\otimes\beta),(1\otimes\beta)\cdot(x\otimes\omega')}(a \otimes b)$$

$$\mathcal{L}^{(1\otimes\beta)\cdot(x\otimes\omega'),(x\otimes\omega')\cdot(1\otimes\beta)}(a \otimes b) \qquad \mathcal{L}^{(1\otimes\beta)\cdot(x\otimes\omega'),(1\otimes\beta)\cdot(x\otimes\omega')}(a \otimes b)$$

are equal to

$$a \otimes \mathcal{L}^{\omega',\omega'}(b) + \mathcal{L}^{x,x}(a) \otimes \beta^*b\beta + \mathcal{L}^{\omega,x}(a) \otimes (1'-\beta^*)b\beta + \mathcal{L}^{x,\omega}(a) \otimes \beta^*b(1'-\beta) + \mathcal{L}^{\omega,\omega}(a) \otimes (1'-\beta^*)b(1-\beta);$$

(f) Follows easily from (b). $\square$

Remark 4.2. *Note that, in general,* $(x \otimes y) \cdot (\alpha \otimes \beta) \neq (x \cdot \alpha) \otimes (y \cdot \beta)$*, so that* $\operatorname{ind}(E \otimes F)$ *cannot be considered as a tensor product of* $\operatorname{ind} E$ *and* $\operatorname{ind} F$*.*

Proposition 4.3. *The mapping* $T : (\operatorname{ind}(E) \otimes \mathcal{B}) \oplus (\mathcal{A} \otimes \operatorname{ind}(F)) \to \operatorname{ind}(E \otimes F)$ *defined on the elementary tensors from the dense subset* $((\mathcal{U}_E/\sim) \otimes \mathcal{B}) \oplus (\mathcal{A} \otimes (\mathcal{U}_F/\sim))$ *by*

$$T([x] \otimes \beta, \alpha \otimes [y]) = [(x \otimes \omega') \cdot (1 \otimes \beta) + (\alpha \otimes 1') \cdot (\omega \otimes y)]$$

is a module homomorphism and isometric embedding.

Proof. First, taking into account Proposition 4.1 (parts (f) and (b)), for $z = (x \otimes \omega') \cdot (1 \otimes \beta) + (\alpha \otimes 1') \cdot (\omega \otimes y)$ we obtain

$$\langle z, z\rangle = \langle x, x\rangle \otimes \beta^*\beta + \alpha^*\alpha \otimes \langle y, y\rangle = \big\langle (x \otimes \beta, \alpha \otimes y), (x \otimes \beta, \alpha \otimes y)\big\rangle . \tag{12}$$

Hence, we get that T is well defined. Indeed, if $x \sim x_1$ then $(x \otimes \omega') \cdot (1 \otimes \beta) \sim (x_1 \otimes \omega') \cdot (1 \otimes \beta)$, since their difference multiplied by itself is equal to zero. Similarly for $y \sim y_1$.

Additivity is obvious.

For the right multiplication, using Proposition 4.1 (parts (c) and (e)), we get

$$\begin{aligned} T(([x] \otimes \beta, \alpha \otimes [y]) \cdot (a \otimes b)) =& T([x \cdot a] \otimes \beta b, \alpha a \otimes [y \cdot b]) = \\ =& [(x \otimes \omega') \cdot (a \otimes \beta b) + (\omega \otimes y) \cdot (\alpha a \otimes b)] = \\ =& T([x] \otimes \beta, \alpha \otimes [y]) \cdot (a \otimes b), \end{aligned}$$

and similarly for the left multiplication.

Finally, from (12) it follows that T is an isometry, and hence embedding. □

Remark 4.4. *In Arveson case, i.e. in the case* $\mathcal{A} = \mathcal{B} = \mathbb{C}$*, the above embedding is actually an isomorphism, due to [2, Theorem 3.7.2 and Corollary 3.7.3] which asserts that any unit* w *in* $E \otimes F$ *has the form* $w = u \otimes v$ *for some units* u *in* E*, and* v *in* F*. However, almost every substantial step in the proof of these statements fails in a general situation. Therefore, it should find either entirely different proof, or a suitable counterexample.*

5. (Sub)spatial Product Systems

In this section we prove that the index of a spatial or a subspatial product system can be described more precisely. In more details, the relations $\sim$ and $\approx$ coincide, $\mathcal{U}$ can be recovered from $\mathrm{ind}(E)$ as $\mathcal{U} \cong \mathcal{B} \oplus \mathrm{ind}(E)$, and finally, completion in the Definition 3.4 is not necessary. Some of these properties can be obtained using the fact that any spatial product system contains a subsystem isomorphic to a time ordered Fock module [16, Theorem 6.3]. However, the proofs presented here are independent of this characterization, and henceforth they don't use Kolmogorov decomposition of completely positive definite kernels.

We begin with the definition of the spatial [16, Section 2] and subspatial product system.

Definition 5.1. *The spatial product system is a product system that contains a central unital unit. The system is subspatial if it can be embedded into a spatial one.*

Remark 5.2. *Recall that unit ω is central if and only if $b\omega_t = \omega_t b$ for all $b \in \mathcal{B}$ and all $t \geq 0$. Such a unit might not exist (see [4, Example 4.2.4]). However, its nice behaviour allows to obtain plenty of interesting results.*

In view of Lemma 2.6, it is enough to assume that E admits a central continuous unit, instead of assuming that it admits a central unital unit.

Note, also, that subspatial system might not be spatial (see [5, Section 3]). The converse is trivially satisfied.

Throughout this section, the reference unit ω is always assumed to be central and we shall assume that it is specified, even though it is not emphasized.

The following Lemma establishes the most important property of central units. Although it is very simple and seen in many papers, we give its proof for the convenience of the reader.

Lemma 5.3. *If a unit ω is central then for all $b \in \mathcal{B}$ and all $x \in \mathcal{U}$*

$$\mathcal{L}^{x,\omega}(b) = \mathcal{L}^{x,\omega}(1)b. \tag{13}$$

Then, also $\mathcal{L}^{\omega,x}(b) = b\mathcal{L}^{\omega,x}(1)$.

Proof. If ω is central, we have

$$\mathcal{L}^{x,\omega}(b) = \lim_{t\to 0+} \frac{\langle x_t, b\omega_t\rangle - b}{t} = \lim_{t\to 0+} \frac{\langle x_t, \omega_t\rangle - 1}{t} b = \mathcal{L}^{x,\omega}(1)b.$$

□

The next Proposition allows us to translate the statements proved for spatial product systems to subspatial.

Proposition 5.4. *Let E be a subspatial product system embedded into a spatial system $\hat{E}$ with a central unit $\hat{\omega}$, and let ω be an arbitrary unit on E. Then the mapping*

$$\Phi : \mathcal{U}_E \to \{x - \omega \mid x \in \mathcal{U}_E\} \subseteq \mathcal{U}_{\hat{E}}, \quad \Phi(x) = x - \omega,$$

is an embedding, where the substraction is that in $\mathcal{U}_{\hat{E}}$, i.e. $\Phi(x) = x - \omega = x \boxplus \hat{\omega} \boxplus (-\omega)$.

In other words, $\mathcal{U}_E$ is an affine subspace of $\mathcal{U}_{\hat{E}}$.

Proof. Indeed,

$$\Phi(x + y) = \Phi(x \boxplus y \boxplus (-\omega)) = x \boxplus y \boxplus (-\omega) \boxplus \hat{\omega} \boxplus (-\omega) =$$
$$= (x \boxplus \hat{\omega} \boxplus (-\omega)) \boxplus (y \boxplus \hat{\omega} \boxplus (-\omega)) \boxplus (-\hat{\omega}) = \Phi(x) + \Phi(y)$$

and also

$$\Phi(x \cdot a) = \Phi(xa \boxplus \omega(1 - a)) = (xa \boxplus \omega(1 - a)) \boxplus \hat{\omega} \boxplus (-\omega) = (x \boxplus \hat{\omega} \boxplus (-\omega))a \boxplus \hat{\omega}(1 - a) = \Phi(x) \cdot a$$

and similarly for $\Phi(a \cdot x)$. Finally, we easily find that $\langle \Phi(x), \Phi(y)\rangle = \langle x, y\rangle$. □

The following Proposition establishes that the relations $\approx$ and $\sim$ coincide.

Proposition 5.5. *Let E be a subspatial product system. Then the equivalence relation $\sim$ from Definition 3.4 is characterized as follows:*

$$x \sim y \Longleftrightarrow x = y^{\beta}, \quad \text{for some } \beta \in \mathcal{B}.$$

Proof. Let us, first, assume that E is spatial, and that ω is its central unit. Since both relations $\approx$ and $\sim$ are compatible with algebraic operations and $x + \omega^{\beta} = x^{\beta}$, it suffices to prove that $\langle x, x\rangle = 0$ implies $x \approx \omega$, i.e. $x^{\beta} = \omega$ for some $\beta \in \mathcal{B}$.

Let $\langle x, x\rangle_1 = 0$, let $b \in \mathcal{B}$, $b \geq 0$ and let denote $\tilde{b} = b/\|b\|$. The element $\tilde{b}$ is positive and $1 - \tilde{b} \geq 0$. By Proposition 3.3 ((g)) we have

$$\langle x, x\rangle_{\tilde{b}} \leq \langle x, x\rangle\,, \tag{14}$$

and hence $\langle x, x\rangle_{\tilde{b}} = 0$.

¿From Cauchy Schwartz inequality (11), we have $\langle x, y\rangle_{\tilde{b}} = 0$ for all $y \in \mathcal{U}$. Let $\beta = \mathcal{L}^{\omega,\omega}(1) - \mathcal{L}^{\omega,x}(1) \in \mathcal{B}$. Then $\mathcal{L}^{x^{\beta},y}(\tilde{b}) = \mathcal{L}^{x,y}(\tilde{b}) + \beta^{*}\tilde{b} = \mathcal{L}^{\omega,y}(\tilde{b})$ and hence $\mathcal{L}^{x^{\beta},y}(b) = \mathcal{L}^{\omega,y}(b)$. Since every element in $\mathcal{B}$ is a linear combination of at most four positive elements, we get $\mathcal{L}^{x^{\beta},y} = \mathcal{L}^{\omega,y}$. Using Lemma 2.4, we conclude $x^{\beta} = \omega$.

Let, now, E be a subspatial product system. Then it can be embedded in a spatial system $\hat{E}$ that contains a central unit $\hat{\omega}$. If $x \sim y$, then, obviously, $x - \omega \sim y - \omega$, and by previous part, $x - \omega = (y - \omega)^{\beta}$ for some $\beta \in \mathcal{B}$, which is equal to $y^{\beta} - \omega$ by Theorem 3.2. Hence $x = y^{\beta}$. □

Theorem 5.6. *If E is subspatial product system, then $\mathcal{U}/_{\sim}$ is a Hilbert left-right $\mathcal{B}$ - $\mathcal{B}$-module.*

Proof. We, only, have to prove that $\mathcal{U}/\sim$ is norm complete. First, assume that E is spatial.

Let $([x^n])$ be a Cauchy sequence in $\mathcal{U}/_{\sim}$, that is: for all $0 < \varepsilon \leq 1$ there is $n_0 \in \mathbb{N}$ such that

$$\|\langle x^n - x^m, x^n - x^m\rangle_1\| = \|[x^n] - [x^m]\|^2 < \varepsilon^2 \quad \text{for } m, n \geq n_0. \tag{15}$$

We would like to show that $([x^n])$ is convergent. This sequence is, of course, bounded. First, we prove that the sequence $\langle x^n, x^n\rangle_{\tilde{b}}$ is also a Cauchy sequence, where $\tilde{b} = b/\|b\|$, and $b \geq 0$ is arbitrary. For $m, n \geq n_0$, we have by (14),

$$\begin{aligned}\|\langle x^n, x^n\rangle_{\tilde{b}} - \langle x^m, x^m\rangle_{\tilde{b}}\| &\leq \|\langle x^n - x^m, x^n - x^m\rangle_{\tilde{b}}\| + \|\langle x^n - x^m, x^m\rangle_{\tilde{b}}\| + \|\langle x^m, x^n - x^m\rangle_{\tilde{b}}\| \leq \\ &\leq \|\langle x^n - x^m, x^n - x^m\rangle_1\| + 2\sqrt{\|\langle x^n - x^m, x^n - x^m\rangle_1\|}\sqrt{\|\langle x^m, x^m\rangle_1\|} < \varepsilon^2 + 2\varepsilon\sqrt{\|\langle x^m, x^m\rangle_1\|} < \varepsilon \text{ const.}\end{aligned} \tag{16}$$

The unit x^n is an arbitrary representative of the class $[x^n]$, and now we are going to pick the most suitable one. Let $\beta_n = -\mathcal{L}^{\omega,x^n}(1) \in \mathcal{B}$, and let x^{β_n} denote the unit $(x^n)^{\beta_n}$. By (13) we have $\mathcal{L}^{\xi,\omega}(b) = \mathcal{L}^{\xi,\omega}(1)b$. This ensures that

$$\mathcal{L}^{\omega,x^{\beta_n}}(\tilde{b}) = (\mathcal{L}^{x^{\beta_n},\omega}(\tilde{b}))^{*} = (\mathcal{L}^{x^n,\omega}(1) + (\beta^n)^{*})\tilde{b} = 0,$$

for $n \in \mathbb{N}$. Now we obtain

$$\mathcal{L}^{x^{\beta_n},x^{\beta_n}}(\tilde{b}) - \mathcal{L}^{x^{\beta_m},x^{\beta_m}}(\tilde{b}) = \langle x^{\beta_n}, x^{\beta_n}\rangle_{\tilde{b}} - \langle x^{\beta_m}, x^{\beta_m}\rangle_{\tilde{b}} = \langle x^n, x^n\rangle_{\tilde{b}} - \langle x^m, x^m\rangle_{\tilde{b}}.$$

It follows, by (16),

$$\|(\mathcal{L}^{x^{\beta_n},x^{\beta_n}} - \mathcal{L}^{x^{\beta_m},x^{\beta_m}})(b)\| < \varepsilon \text{ const}\|b\|, \tag{17}$$

for any $b \geq 0$. Since every element of $\mathcal{B}$ is a linear combination of at most four positive elements, we conclude that $\mathcal{L}^{x^{\beta_n},x^{\beta_n}}$ is a Cauchy sequence in $L(\mathcal{B})$, multiplying the constant in (17) by 4, if necessary. Hence it converges.

For every $y \in \mathcal{U}$, we have

$$\langle y, x^n - x^m\rangle_{\tilde{b}} \langle x^n - x^m, y\rangle_{\tilde{b}} \leq \|\langle x^n - x^m, x^n - x^m\rangle_{\tilde{b}}\| \langle y, y\rangle_{\tilde{b}}.$$

By (14) and (15), $\|\langle x^n - x^m, y\rangle_{\tilde{b}}\| < \varepsilon\sqrt{\|\langle y, y\rangle_1\|}$, implying

$$\|(\mathcal{L}^{x^{\beta_n}, y} - \mathcal{L}^{x^{\beta_m}, y})(b)\| < \varepsilon\sqrt{\|\langle y, y\rangle_1\|}\|b\|, \tag{18}$$

for all $\mathcal{B} \ni b \geq 0$. As above, we conclude that $\mathcal{L}^{x^{\beta_n}, y}$ is a Cauchy sequence in $L(\mathcal{B})$, and hence convergent. Moreover, it satisfies the Cauchy condition uniformly with respect to y, $\|\langle y, y\rangle_1\| \leq 1$.

Therefore, we proved that there are $K, K_y \in L(\mathcal{B})$ so that

$$\lim_{n\to+\infty} \|\mathcal{L}^{x^{\beta_n}, x^{\beta_n}} - K\| = 0, \tag{19}$$

$$\lim_{n\to+\infty} \|\mathcal{L}^{x^{\beta_n}, y} - K_y\| = 0. \tag{20}$$

Since $\|\mathcal{L}^{x^{\beta_n}, x^{\beta_n}}\|$, $\|\mathcal{L}^{x^{\beta_n}, y}\| \leq$ const, for $n \in \mathbb{N}$ and $\|\langle y, y\rangle_1\| \leq 1$, the series

$$\sum_{m=0}^{+\infty} \frac{t^m(\mathcal{L}^{x^{\beta_n}, x^{\beta_n}})^m}{m!} \quad \text{and} \quad \sum_{m=0}^{+\infty} \frac{t^m(\mathcal{L}^{x^{\beta_n}, y})^m}{m!}$$

uniformly converge with respect to $n \in \mathbb{N}$, which by Lebesgue dominant convergence theorem implies

$$\lim_{n\to+\infty} \langle x_t^{\beta_n}, \bullet x_t^{\beta_n}\rangle = \lim_{n\to+\infty} e^{t\mathcal{L}^{x^{\beta_n}, x^{\beta_n}}} = \lim_{n\to+\infty} \sum_{m=0}^{+\infty} \frac{t^m(\mathcal{L}^{x^{\beta_n}, x^{\beta_n}})^m}{m!} = e^{tK},$$

$$\lim_{n\to+\infty} \langle x_t^{\beta_n}, \bullet y_t\rangle = \lim_{n\to+\infty} e^{t\mathcal{L}^{x^{\beta_n}, y}} = \lim_{n\to+\infty} \sum_{m=0}^{+\infty} \frac{t^m(\mathcal{L}^{x^{\beta_n}, y})^m}{m!} = e^{tK_y}.$$

So,

$$\lim_{n\to+\infty} \langle x_t^{\beta_n}, \bullet x_t^{\beta_n}\rangle = \mathrm{id}_{\mathcal{B}} + tK + O(t^2), \tag{21}$$

$$\lim_{n\to+\infty} \langle x_t^{\beta_n}, \bullet y_t\rangle = \mathrm{id}_{\mathcal{B}} + tK_y + O(t^2). \tag{22}$$

Thus, we found the kernels of the desired limit of our Cauchy sequence. We can immediately apply Proposition 2.1 to bring up the unit u_t with kernels K, K_y. However, it is disputable whether or not, this unit satisfies one of conditions of Proposition 2.1, and therefore, whether or not it belongs to $\mathcal{U}$. So, we need to find another way to obtain u_t.

Let $\varepsilon > 0$. Since limits $\lim\limits_{n\to+\infty} \langle x_t^{\beta_n}, x_t^{\beta_n}\rangle$ and $\lim\limits_{n\to+\infty} \langle x_t^{\beta_n}, y_t\rangle$ exist in $\mathcal{B}$, uniformly in y, $\|\langle y, y\rangle_1\| \leq 1$, there are $n_1, n_2 \in \mathbb{N}$ so that

$$\|\langle x_t^{\beta_n}, x_t^{\beta_n}\rangle - \langle x_t^{\beta_{n_1}}, x_t^{\beta_{n_1}}\rangle\| < \frac{\varepsilon}{2} \text{ for } n \geq n_1, \tag{23}$$

$$\|\langle x_t^{\beta_n}, y_t\rangle - \langle x_t^{\beta_{n_2}}, y_t\rangle\| < \frac{\varepsilon}{2} \text{ for } n \geq n_2. \tag{24}$$

Let $n_0 = \max\{n_1, n_2\}$ and $m, n \geq n_0$. We have, by (23) and (24)

$$\begin{aligned}
\|x_t^{\beta_n} - x_t^{\beta_m}\|_{E_t}^2 =& \|\langle x_t^{\beta_n} - x_t^{\beta_m}, x_t^{\beta_n} - x_t^{\beta_m}\rangle\|_{\mathcal{B}} = \\
=& \|\langle x_t^{\beta_n}, x_t^{\beta_n}\rangle - \langle x_t^{\beta_m}, x_t^{\beta_m}\rangle + \langle x_t^{\beta_m}, x_t^{\beta_m} - x_t^{\beta_n}\rangle + \langle x_t^{\beta_m} - x_t^{\beta_n}, x_t^{\beta_m}\rangle\| \leq \\
\leq& \|\langle x_t^{\beta_n}, x_t^{\beta_n}\rangle - \langle x_t^{\beta_{n_0}}, x_t^{\beta_{n_0}}\rangle\| + \|\langle x_t^{\beta_m}, x_t^{\beta_m}\rangle - \langle x_t^{\beta_{n_0}}, x_t^{\beta_{n_0}}\rangle\| + \\
&+ 2\|\langle x_t^{\beta_m}, x_t^{\beta_m}\rangle - \langle x_t^{\beta_{n_0}}, x_t^{\beta_{n_0}}\rangle\| + 2\|\langle x_t^{\beta_{n_0}}, x_t^{\beta_{n_0}}\rangle - \langle x_t^{\beta_{n_0}}, x_t^{\beta_n}\rangle\| + \\
&+ 2\|\langle x_t^{\beta_m}, x_t^{\beta_n}\rangle - \langle x_t^{\beta_{n_0}}, x_t^{\beta_n}\rangle\| < 8\varepsilon,
\end{aligned}$$

It follows that $(x_t^{\beta_n})$ is convergent in Hilbert $\mathcal{B}-\mathcal{B}$ module E_t. Its limit we denote by

$$\lim_{n\to+\infty} x_t^{\beta_n} = u_t \in E_t. \tag{25}$$

By (21) and (22), we get

$$\langle u_t, \bullet u_t\rangle = \mathrm{id}_{\mathcal{B}} + tK + O(t^2),$$

$$\langle u_t, \bullet y_t\rangle = \mathrm{id}_{\mathcal{B}} + tK_y + O(t^2).$$

To conclude that u_t is unit, we only need to apply limit (as $n\to\infty$) to relation

$$x_t^{\beta_n}\otimes x_s^{\beta_n} = x_{s+t}^{\beta_n}.$$

From (19) and (20) we find that

$$\lim_{n\to+\infty} \|\langle x^{\beta_n}-u, x^{\beta_n}-u\rangle_1\| = 0,$$

i.e.

$$\lim_{n\to+\infty}[x^n] = [u],$$

in $\mathcal{U}/_\sim$. Therefore $\mathcal{U}/_\sim$ is a Hilbert $\mathcal{B}$-module.

If E is only subspatial, we can embed it into a spatial system $\hat{E}$ with central unit ω. We can apply the previous case. The only question is whether the limit unit belongs to $E\le\hat{E}$. However it immediately follows from (25). □

Proposition 5.7. *If E is a subspatial product system, then $\mathcal{U}$ is (algebraically) isomorphic to* $\mathrm{ind}(E)\oplus\mathcal{B}$ *as right $\mathcal{B}$ module. If E is, in addition, spatial, then $\mathcal{U}$ is isomorphic to* $\mathrm{ind}(E)\oplus\mathcal{B}$ *as left-right $\mathcal{B}-\mathcal{B}$ module.*

Proof. We can assume that ω is unital, since the index does not depend on ω.

By the previous Theorem and Proposition, we have a short exact sequence of Hilbert modules.

$$0\to\mathcal{B}\overset{i}{\hookrightarrow}\mathcal{U}\overset{\pi}{\to}\mathrm{ind}(E)\to 0$$

where $i(\beta)=\omega^\beta$, and π is the canonical projections. We shall show that this sequence splits, constructing the homomorphism $j:\mathcal{U}\to\mathcal{B}$ by

$$j(x) = \mathcal{L}^{x,\omega}(1)^* = \mathcal{L}^{\omega,x}(1).$$

This mapping satisfies

$$j(x+y) = (\mathcal{L}^{x,\omega}(1)+\mathcal{L}^{y,\omega}(1)-\mathcal{L}^{\omega,\omega}(1))^* = j(x)+j(y),$$

$$j(x\cdot a) = (a^*\mathcal{L}^{x,\omega}(1)+(1-a^*)\mathcal{L}^{\omega,\omega}(1))^* = j(x)a,$$

since ω is unital, implying $\mathcal{L}^{\omega,\omega}(1)=0$.

If, in addition, ω is central, then

$$j(a\cdot x) = (\mathcal{L}^{x,\omega}(a^*)+\mathcal{L}^{\omega,\omega}(1-a^*))^* = (\mathcal{L}^{x,\omega}(1)a^*)^* = aj(x),$$

since for central unital unit ω there holds $\mathcal{L}^{x,\omega}(a)=\mathcal{L}^{x,\omega}(1)a$ (Lemma 5.3).

To finish the proof, note that $j\circ i(\beta) = j(\omega^\beta)=\beta$. □

The following Corollary was proved in [4, Theorem 3.5.2] under additional assumption that $\mathcal{B}$ is a von Neumann algebra and in full generality in [14, Theorem 5.2]. Here, we give an easy proof that does not use Kolmogorov decomposition of completely positive definite kernels.

Corollary 5.8. *Let x be a continuous unit on a product system E over $\mathcal{B}$. If E can be embedded in some spatial product system, then the generator of CPD semigroup $\langle x_t, bx_t\rangle$ has the Christensen-Evans form, that is*

$$\mathcal{L}^{x,x}(b) = \langle \zeta_x, b\zeta_x\rangle + \beta_x^* b + b\beta_x,$$

where ζ_x is element of some Hilbert left-right $\mathcal{B} - \mathcal{B}$ module, and $\beta_x \in \mathcal{B}$.

Moreover, the generator of $\langle x_t, by_t\rangle$ has the form

$$\mathcal{L}^{x,y}(b) = \left\langle \zeta_x, b\zeta_y \right\rangle + \beta_x^* b + b\beta_y.$$

Proof. Let $E \leq \hat{E}$ and let ω be a central unital unit in $\hat{E}$. A straightforward calculation gives

$$\langle x, b \cdot y\rangle = \mathcal{L}^{x,y}(b) - \mathcal{L}^{x,\omega}(b) - \mathcal{L}^{\omega,y}(b) + \mathcal{L}^{\omega,\omega}(b).$$

Using the fact that ω is central and unital, we get

$$\mathcal{L}^{x,\omega}(b) = \mathcal{L}^{x,\omega}(1)b = j(x)^* b, \qquad \mathcal{L}^{\omega,y}(b) = b\mathcal{L}^{\omega,y}(1) = bj(y) \qquad \text{and} \qquad \mathcal{L}^{\omega,\omega}(b) = 0,$$

where j is the mapping from Proposition 5.7. Thus we obtain

$$\mathcal{L}^{x,y}(b) = \langle x, b \cdot y\rangle + j(x)^* b + bj(y) = \langle [x], b[y]\rangle + j(x)^* b + bj(y),$$

which finishes the proof. $\square$

6. Examples, Remarks

Following two Examples demonstrate that $\operatorname{ind}(E)$ defined in this note is a generalization of the notion of the index defined by Arveson in the case $\mathcal{B} = \mathbb{C}$ in [1], and by Skeide in the case when E is a spatial product system [16].

Example 6.1. *Let E be an Arveson product system, i.e. a product system over $\mathbb{C}$. Then $\mathcal{U}/_\sim$ is isomorphic to a vector space of dimension* $\operatorname{ind}(E)$. *Indeed, as any Arveson product system contains a unique maximal type I subsystem of the same index (namely, the system generated by its units), we may assume that E is generated by a continuous set of units. By [2, Theorem 6.7.1] and [2, Proposition 3.1.5], E is isomorphic to the concrete product system of the CCR flow of rank $n = \operatorname{ind}(E)$, acting on $\mathcal{B}(e^{L^2((0,\infty),K)})$ where K is a Hilbert space of dimension n. By [2, Theorem 2.6.4], $\mathcal{U}/_\sim = \{[U^\zeta],\ \zeta \in K\}$, where U^ζ are units defined by $U_t^\zeta(\exp(f)) = \exp(\chi_{(0,t)} \otimes \zeta + S_t f)$, $t \geq 0$, $f \in L^2((0,\infty);K)$, and (S_t) is the shift semigroup of index n that acts on $L^2((0,\infty);K)$ by way of*

$$S_t f(x) = \begin{cases} f(x-t), & x > t \\ 0, & 0 < x \leq t. \end{cases}$$

Taking the unit U^0 for ω we see that $\left\langle [U^\zeta], [U^\eta] \right\rangle = \left\langle U^\zeta, U^\eta \right\rangle = \langle \zeta, \eta \rangle$. Noting that

$$\left\langle [U^{\zeta+\eta}] - [U^\zeta + U^\eta], [U^{\zeta+\eta}] - [U^\zeta + U^\eta] \right\rangle = 0$$

and

$$\left\langle [U^{a\zeta}] - [aU^\zeta], [U^{a\zeta}] - [aU^\zeta] \right\rangle = 0,$$

for $a \in \mathbb{C}$, we obtain that $[U^{\zeta+\eta}] = [U^\zeta + U^\eta]$ and $[U^{a\zeta}] = [aU^\zeta]$. Hence, $K \ni \zeta \mapsto [U^\zeta] \in \mathcal{U}/_\sim$ is an isomorphism.

Since the dimension of vector space completely determines it (up to isomorphism), it allows us to consider the $\mathcal{B}$-module $\operatorname{ind}(E)$ *as a suitable generalization of the index.*

Example 6.2. *Let $\mathbb{I}\Gamma(F)$ be the time ordered Fock module where F is a two sided Hilbert module over $\mathcal{B}$. In [9, Theorems 3 and 6] it was proved that all continuous units in $\mathbb{I}\Gamma(F)$ can be parameterized by the set $F \times \mathcal{B}$. Let the unit that corresponds to pair (ζ, β) denote by $u(\zeta, \beta)$. The corresponding kernels are given by (see [4, formula (3.5.2)])*

$$\mathfrak{L}^{u(\zeta,\beta),u(\zeta',\beta')}(b) = \langle \zeta, b\zeta' \rangle + \beta^* b + b\beta' \tag{26}$$

Comparing the kernels, we can conclude that the mapping $F \times \mathcal{B} \ni (\zeta, \beta) \to u(\zeta, \beta) \in \mathcal{U}_{\mathbb{I}\Gamma(F)}$ is an (algebraic) isomorphism of modules, if we choose $\omega = u(0,0)$. Also, it is easy to see that $u(\zeta,\beta)^\gamma = u(\zeta, \beta + \gamma)$, so that $\mathcal{U}_{\mathbb{I}\Gamma(F)}/\sim = \{[u(\zeta, 0)] \mid \zeta \in F\}$ and therefore $\operatorname{ind}(\mathbb{I}\Gamma(F)) \cong F$, *in algebraic sense.*

Further, from (26) we easily get

$$\langle u(\zeta, \beta), u(\zeta', \beta') \rangle = \langle \zeta, \zeta' \rangle .$$

Thus $\operatorname{ind}(\mathbb{I}\Gamma(F))$ *is isomorphic to F as Hilbert left-right module. Therefore, our definition of the index generalizes that of Skeide [16].*

Remark 6.3. *It would be interesting to compute the index of the subspatial system exhibited in [5, Section 3], that is not spatial.*

Next Example is the Example of a product system without any central unit.

Example 6.4. *In [4, Example 4.2.4], there is an example of a product system that does not contain any central unit. In more details, let $\mathcal{B} = K(G) + \mathbb{C}\mathrm{id}_G$ be the unitization of compact operators on an infinite-dimensional Hilbert space G and let $h \in B(G)$ be a self-adjoint operator. The Hilbert $\mathcal{B} - \mathcal{B}$ modules $\mathcal{B}_t$ defined to coincide with $\mathcal{B}$ as right Hilbert modules and with left multiplication $b \cdot x_t = e^{ith} b e^{-ith} x_t$ form a product system $(\mathcal{B}_t)_{t \geq 0}$ with identification*

$x_s \otimes y_t = e^{ith} x_s e^{-ith} y_t$. *Such product system does not admit a central unit if and only if $h \notin \mathcal{B}$, and it is generated by the single unit $1_t \equiv 1$ and hence it is of type I, as it was shown in [4, Example 4.2.4].*

Let ξ_t be an arbitrary continuous unit and let $\xi'_t = e^{-ith}\xi_t$. Obviously, ξ'_t is a uniformly continuous family. We have

$$\xi'_{s+t} = e^{-i(s+t)h}\xi_s \otimes \xi_t = e^{-i(s+t)h} e^{ith}\xi_s e^{-ith}\xi_t = \xi'_s \xi'_t.$$

It follows that $\xi'_t = e^{tB_\xi}$ for some bounded operator B_ξ on G. Set $A_\xi = B_\xi + ih$ and we obtain that any continuous unit on $(\mathcal{B}_t)_{t\ge 0}$ has a form

$$\xi_t = e^{ith} e^{t(A_\xi - ih)},$$

for some A_ξ. Moreover, we find that

$$A_\xi = \lim_{t\to 0+}\left[\frac{e^{ith}-1}{t} e^{t(A_\xi - ih)} + \frac{e^{t(A_\xi - ih)}-1}{t}\right] = \lim_{t\to 0+}\frac{e^{ith}e^{t(A_\xi - ih)}-1}{t} \in \mathcal{B},$$

because the last fraction belongs to $\mathcal{B}$, and the limit converges uniformly, since exponentials are analytic functions.

Pick the unit ω choosing $A_\omega = 0$. Then $\omega_t = 1$. Noting that $\mathcal{L}^{\xi,\eta} = \mathrm{id}_{\mathcal{B}}(A_\eta - ih) + (A^*_\xi + ih)\mathrm{id}_{\mathcal{B}}$, we see that $\langle [\xi], [\eta]\rangle = \langle \xi, \eta\rangle = 0$ for every $[\xi], [\eta] \in \mathcal{U}/_\sim$. Hence, $\mathcal{U}/_\sim = \{0\}$.

Remark 6.5. *This example shows that product systems, even of type I cannot be classified by its index. Namely, for $h \in \mathcal{B}$ it has a central unit, and for $h \notin \mathcal{B}$ it does not. Therefore, such product systems are not isomorphic in spite of the fact that they have the same index.*

In [5, Theorem 4.8] it was shown that the product system from previous Example is not subspatial, by proving that the kernel of the unit 1 has no Christensen-Evans form. This is, up to our knowledge, the only example of nonsubspatial product system. However for this system, Propositions 5.5, 5.7 and Theorem 5.6 remain valid. This example is trivial, (despite twisted left action of $\mathcal{B}$) in the sense that all E_t are isomorphic (as right modules) to the algebra $\mathcal{B}$ itself. On the other hand, the index is constructed to "measure" how many dimensions the continuous units can generate (in a certain sense) after taking a quotient by $\mathcal{B}$.

It is, therefore, natural to ask if there are any product systems (of course that are not subspatial) for which Propositions 5.5, 5.7 and Theorem 5.6 fail.

Remark 6.6. *The other problem that arises from this example is what is actually, the trivial product system. Following Skeide, it must be this example with $h = 0$ (as well as any example where $E_t \cong \mathcal{B}$ - the algebra itself, and left and right multiplication are canonical). However, then we have a problem how to define short exact sequences in the category of product systems. Namely, injective morphisms have trivial kernel, i.e. isomorphic to $\{0\}$ at each fiber. Such product system has no continuous units, since it must be equal to 1 at time $t = 0$. Thus, we are forced to consider injective morphisms modulo trivial systems. In this case, the previous Example cannot be seen from short exact sequences. Therefore, in the absence of a suitable definition we cannot speak about exact functoriality of the index.*

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