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Lévy Processes Time-Changed by the First-Exit Time of the Inverse Gaussian Subordinator

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Abstract. This paper deals with a characterization of the first-exit time of the inverse Gaussian subordinator in terms of natural exponential family. This leads us to characterize, by means its variance function, the class of Lévy processes time-changed by the first-exit time of the inverse Gaussian subordinator.

1. Introduction

The inverse Gaussian process plays an important role in the data analysis and statistical modeling since it represents the first-exit time of the Brownian motion. In the last decades, several authors have focused on this process and have used it in many areas of applications (see for instance [17], [18], [20] and [21]). Moreover, the first-exit time of this process has drawn considerable attention of researchers since it has been widely used as a time-change process. The essence of the time-changed Lévy processes is closely linked to the concept of stochastic volatility modeling for asset prices (see [4], [7], [12] and [13]). It has arisen naturally in diverse fields such as finance, insurance, process control and survival analysis (see for example [8] and [15]).

In the present work, we focus on the study of these mixed models in terms of natural exponential family (NEF). Several works combine the notion of the NEFs and the variance-mean mixture processes. Within this framework, [6] have characterized the exponential families of the Markov processes using an additive functional model. Besides, [11, 12] have investigated on the characterization of the class of normal tempered stable distributions, which can be interpreted as a Brownian motion time-changed by a stable subordinator. They have also established a characterization of a multivariate Lévy process based on the notion of cut in natural exponential family. Moreover, [2] have introduced the normal stable Tweedie distribution and they have determined its variance function. The importance of the variance function comes from, on the one hand, it is expressed in terms of the mean vector, on the other hand, it characterizes the NEF. From statistical point of view, many papers have used the variance functions in order to estimate the parameters of exponential dispersion models, which are related to NEFs additively and reproductively. We may refer to [22], where variance function is used in order to give some asymptotic properties of the estimator for a finite mixture of exponential dispersion models and have applied the estimation results in the image segmentation. Furthermore, [5] have investigated the variance function of the Tweedie model for the modeling of the signal path loss prediction. In fact, they have studied asymptotic normality and the

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confidence interval of the predicted signal path loss and they have illustrated their approach by considering an experimental study.

In this paper, we explicit the variance function of the NEF generated by the first-exit time of the inverse Gaussian subordinator. Furthermore, we investigate this variance function in order to characterize the NEF generated by the distribution of a Lévy process time-changed by the first-exit time of the inverse Gaussian subordinator. From this, we compute the variance functions of the NEFs of the inverse Gaussian and Poisson processes time-changed by the first-exit time of the inverse Gaussian subordinator. Note that the Poisson process time-changed by the first-exit time of the inverse Gaussian subordiantor was introduced and studied by [7] and [13]. However, the inverse Gaussian process time-changed by the first-exit time of the inverse Gaussian subordinator is lacking in the literature.

The paper is organized as follows: After recalling some essential definitions and generalities indispensable to the present work, we characterize, in Section 3, the NEF generated by the first-exit time of the inverse Gaussian subordinator by means of its variance function. In Section 4, we compute the variance function of the NEF generated by the distribution of a Lévy process time-changed by first-exit time of the inverse Gaussian subordinator.

2. Preliminaries

In this section, we recall some basic definitions and results. Our notations are the ones used by [9].

2.1. *Natural exponential families on $\mathbb{R}^d$*

Let μ be a positive random measure on $\mathbb{R}^d$. The set of probability distributions

$$F = F(\mu) = \{P(\theta,\mu)(dx) = e^{\langle\theta,x\rangle-\ln(L_\mu(\theta))}\mu(dx);\ \theta\in\Theta(\mu)\}$$

is called the NEF generated by μ, where $\langle\theta,x\rangle$ is the ordinary scalar product on $\mathbb{R}^d$, $L_\mu(\theta)=\displaystyle\int_{\mathbb{R}^d} e^{\langle\theta,x\rangle}\mu(dx)$ is the Laplace transform of μ and $\Theta(\mu)$ is the interior of the convex set $D(\mu)=\{\theta\in\mathbb{R}^d;\ L_\mu(\theta)<\infty\}$. We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of positive measures μ such that $\Theta(\mu)$ is non-empty and μ is not concentrated on an affine hyperplane of $\mathbb{R}^d$. For each $\mu\in\mathcal{M}(\mathbb{R}^d)$ and $\theta\in\Theta(\mu)$, we define the cumulant function of μ by $k_\mu(\theta)=\ln\left(L_\mu(\theta)\right)$. The map $\Theta(\mu)\longrightarrow M_{F(\mu)};\ \theta\longmapsto m=k'_\mu(\theta)=\displaystyle\int_{\mathbb{R}^d} xP(\theta,\mu)(dx)$ defines a diffeomorphisme between $\Theta(\mu)$ and its image $M_{F(\mu)}$, called the domain of the means of $F(\mu)$. We denote its inverse by ψ_μ. The second derivative k''_μ represents the covariance operator of $P(\theta,\mu)$. It is given by $k''_\mu(\theta)=\int_{\mathbb{R}^d} x\otimes xP(\theta,\mu)(dx)-k'_\mu(\theta)\otimes k'_\mu(\theta)$, where $x\otimes x(u,v)=\langle x,u\rangle\langle x,v\rangle$. The variance function of $F(\mu)$ is defined on $M_{F(\mu)}$ by

$$m\longmapsto V_{F(\mu)}(m)=k''_\mu(\psi_\mu(m))=\left(\psi'_\mu(m)\right)^{-1}. \tag{1}$$

The importance of the variance function comes from the fact that it is a function of the means and it characterizes the family F within the class of all NEFs. In fact, if F_1 and F_2 are two natural exponential families such that the variance functions $V_{F_1}(m)$ and $V_{F_2}(m)$ coincide on a nonempty open set of the intersection of the means domains $M_{F_1}\cap M_{F_2}$, then $F_1=F_2$. In other words, the knowledge of the NEF is given by the knowledge of its variance function (for more details about NEFs, the reader can see [9]).

2.2. *Lévy process time-changed by the first-exit time of the inverse Gaussian subordinator*

Let $X(t)$ be an inverse Gaussian subordinator with distribution ν_t. It is well known that this subordinator has an $\frac{1}{2}$–stable distribution. According to [16], p. 94, its Laplace transform is equal to $L_{\nu_t}(\theta)=e^{-t\sqrt{-\theta}}$, for all $\theta<0$. The process $Y(t)=\inf\{s>0;\ X(s)>t\}$ represents the first-exit time of the inverse Gaussian subordiantor $X(t)$ and is widely used as a time-change subordinator (see [4], [7] and

[13]). We denote by ρ_t its distribution . Recently, [14] have shown that the Laplace transform of $Y(t)$ is equal to

$$L_{\rho_t}(\theta) = \frac{1}{\theta} k_{\nu_1}(\theta) e^{tk_{\nu_1}(\theta)} = \frac{1}{\sqrt{-\theta}} e^{-t\sqrt{-\theta}} \text{ for all } \theta < 0. \tag{2}$$

Consider now a Lévy process $G(t)$ independent of the subordinator $Y(t)$. The process $Z(t) = G(Y(t))$ represents a Lévy process time-changed by the first-exit time of the inverse Gaussian subordinator. Recently, many works have studied this kind of processes. In fact, [19] have estimated the parameters of the fractional Brownian motion time-changed by the first-exit time of the inverse Gaussian subordinator. Besides, [7] have studied the diffusion equation of the Poisson processes time-changed by the stable and the inverse stable subordinators. Furthermore, [3] have determined some properties of the geometric Brownian motion time-changed by the inverse tempered stable subordinator. Note that the first-exit time of the inverse Gaussian subordinator is an element of the class of inverse stable subordinators. In what follows, we denote by Q_t the distribution of the Lévy process $G(t)$, μ_t the bivariate distribution $\mu_t(dy, dz) = \rho_t(dy)Q_y(dz)$ and $\eta_t(dz) = \int_0^{+\infty} \mu_t(dy, dz)$ the distribution of $Z(t)$.

The Laplace transform of μ_t is, for all $(\theta_1, \theta_2) \in \{(\theta_1, \theta_2) \in \mathbb{R} \times \Theta(Q_1)\ ;\ \theta_1 + k_{Q_1}(\theta_2) < 0\}$, given by

$$L_{\mu_t}(\theta_1, \theta_2) = \int_0^{+\infty} e^{\theta_1 y} L_{Q_y}(\theta_2)\rho_t(dy) = \int_0^{+\infty} e^{y(\theta_1 + k_{Q_1}(\theta_2))}\rho_t(dy) = L_{\rho_t}\left(\theta_1 + k_{Q_1}(\theta_2)\right). \tag{3}$$

In addition, the Laplace transform of η_t is, for all $\theta_2 \in \{\theta_2 \in \Theta(Q_1)\ ;\ k_{Q_1}(\theta_2) < 0\}$, equal to

$$L_{\eta_t}(\theta_2) = \int_0^{+\infty} L_{Q_y}(\theta_2)\rho_t(dy) = L_{\rho_t}(k_{Q_1}(\theta_2)) = \frac{1}{\sqrt{-k_{Q_1}(\theta_2)}} e^{-t\sqrt{-k_{Q_1}(\theta_2)}} = L_{\mu_t}(0, \theta_2). \tag{4}$$

3. Variance function of the first-exit time of the inverse Gaussian subordinator

In this section, we characterize the NEF generated by the first-exit time of the inverse Gaussian subordinator by means of its variance function. More precisely, we have

Theorem 3.1. *For all $m > 0$,*

$$V_{F(\rho_1)}(m) = \frac{4m^3\sqrt{1+8m}}{(2m+1)\sqrt{1+8m} + 6m + 1}. \tag{5}$$

Proof. According to (2), we have

$$k'_{\rho_1}(\theta) = -\frac{1}{2\theta} + \frac{1}{2\sqrt{-\theta}} = m, \text{ for all } \theta < 0. \tag{6}$$

This implies that $M_{F(\rho_1)} = (0, \infty)$. Setting $z = \sqrt{-\theta}$, the equation (6) becomes $z - 2mz^2 + 1 = 0$. Its unique positive solution is given by $z = \frac{1+\sqrt{1+8m}}{4m}$. It follows that $\theta = \psi_{\rho_1}(m) = -\left(\frac{1+\sqrt{1+8m}}{4m}\right)^2$. Using (1), we get the result. □

4. Variance function of a Lévy process time-changed by the first-exit time of the inverse Gaussian subordinator

In this section, we characterize $F(\mu_1)$ and $F(\eta_1)$ by means of their variance functions. For this purpose, we first give the corresponding second derivative.

Proposition 4.1.
1. For all $\theta = (\theta_1, \theta_2) \in \{(\theta_1, \theta_2) \in \mathbb{R} \times \Theta(Q_1)\ ;\ \theta_1 + k_{Q_1}(\theta_2) < 0\}$,

$$k''_{\mu_1}(\theta_1, \theta_2) = \frac{2 + \sqrt{-(\theta_1 + k_{Q_1}(\theta_2))}}{4(\theta_1 + k_{Q_1}(\theta_2))^2} \begin{pmatrix} 1 & k'_{Q_1}(\theta_2) \\ k'_{Q_1}(\theta_2) & \left(k'_{Q_1}(\theta_2)\right)^2 \end{pmatrix} - \frac{1 + \sqrt{-(\theta_1 + k_{Q_1}(\theta_2))}}{2(\theta_1 + k_{Q_1}(\theta_2))} \begin{pmatrix} 0 & 0 \\ 0 & k''_{Q_1}(\theta_2) \end{pmatrix}. \quad (7)$$

2. For all $\theta_2 \in \{\theta_2 \in \Theta(Q_1)\ ;\ k_{Q_1}(\theta_2) < 0\}$, $k''_{\eta_1}(\theta_2) = \dfrac{\left(2 + \sqrt{-k_{Q_1}(\theta_2)}\right)\left(k'_{Q_1}(\theta_2)\right)^2}{4\left(k_{Q_1}(\theta_2)\right)^2} - \dfrac{\left(1 + \sqrt{-k_{Q_1}(\theta_2)}\right)k''_{Q_1}(\theta_2)}{2k_{Q_1}(\theta_2)}$.

Proof. 1. Using (3), we obtain

$$k'_{\mu_1}(\theta_1, \theta_2) = \left(k'_{\rho_1}(\theta_1 + k_{Q_1}(\theta_2)), k'_{Q_1}(\theta_2)k'_{\rho_1}(\theta_1 + k_{Q_1}(\theta_2))\right) = (m_1, m_2). \quad (8)$$

It follows that

$$\frac{\partial^2 k_{\mu_1}(\theta_1, \theta_2)}{\partial \theta_1^2} = k''_{\rho_1}(\theta_1 + k_{Q_1}(\theta_2)),\ \frac{\partial^2 k_{\mu_1}(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2} = k'_{Q_1}(\theta_2)k''_{\rho_1}(\theta_1 + k_{Q_1}(\theta_2)) \quad (9)$$

and

$$\frac{\partial^2 k_{\mu_1}(\theta_1, \theta_2)}{\partial \theta_2^2} = k''_{Q_1}(\theta_2)k'_{\rho_1}(\theta_1 + k_{Q_1}(\theta_2)) + \left(k'_{Q_1}(\theta_2)\right)^2 k''_{\rho_1}(\theta_1 + k_{Q_1}(\theta_2)). \quad (10)$$

Moreover, differentiating (2) wit respect to θ gives

$$k'_{\rho_1}(\theta) = -\frac{1 + \sqrt{-\theta}}{2\theta} \text{ and } k''_{\rho_1}(\theta) = \frac{2 + \sqrt{-\theta}}{4\theta^2}. \quad (11)$$

Inserting this in (9) and (10), we obtain the announced result.
2. According to (4), we have

$$k''_{\eta_1}(\theta_2) = k''_{\rho_1}(k_{Q_1}(\theta_2))\left(k'_{Q_1}(\theta_2)\right)^2 + k'_{\rho_1}(k_{Q_1}(\theta_2))k''_{Q_1}(\theta_2). \quad (12)$$

Inserting (11) in (12), we get the desired result. □

Next, we give the variance function of the NEF $F(\mu_1)$.

Theorem 4.2. *For all* $(m_1, m_2) \in (0, \infty) \times M_{F(Q_1)}$,

$$V_{F(\mu_1)}(m_1, m_2) = \begin{pmatrix} \frac{4m_1^3\sqrt{1+8m_1}}{(2m_1+1)\sqrt{1+8m_1}+6m_1+1} & \frac{4m_1^2 m_2\sqrt{1+8m_1}}{(2m_1+1)\sqrt{1+8m_1}+6m_1+1} \\ \frac{4m_1^2 m_2\sqrt{1+8m_1}}{(2m_1+1)\sqrt{1+8m_1}+6m_1+1} & \frac{4m_1 m_2^2\sqrt{1+8m_1}}{(2m_1+1)\sqrt{1+8m_1}+6m_1+1} + m_1 V_{F(Q_1)}\left(\frac{m_2}{m_1}\right) \end{pmatrix}.$$

Proof. According to (8), we have $\theta_2 = \psi_{Q_1}\left(\frac{m_2}{m_1}\right)$ and $\theta_1 = \psi_{\rho_1}(m_1) - k_{Q_1}\left(\psi_{Q_1}\left(\frac{m_2}{m_1}\right)\right)$. Inserting this in (7), we deduce that

$$\begin{aligned} V_{F(\mu_1)}(m_1, m_2) &= k''_{\mu_1}\left(\psi_{\rho_1}(m_1) - k_{Q_1}\left(\psi_{Q_1}\left(\frac{m_2}{m_1}\right)\right), \psi_{Q_1}\left(\frac{m_2}{m_1}\right)\right) \\ &= \begin{pmatrix} V_{F(\rho_1)}(m_1) & \frac{V_{F(\rho_1)}(m_1)}{m_1}m_2 \\ \frac{V_{F(\rho_1)}(m_1)}{m_1}m_2 & \frac{V_{F(\rho_1)}(m_1)}{m_1^2}m_2^2 + m_1 V_{F(Q_1)}\left(\frac{m_2}{m_1}\right) \end{pmatrix}. \end{aligned} \quad (13)$$

Together with (5), this achieves the proof. □

Next, we characterize the NEF generated by a Lévy process time-changed by the first-exit time of the inverse Gaussian subordinator by means its variance function. For this purpose, we need first to recall some results about the notion of cuts in NEF introduced by [1]. Let $p : (0,\infty)\times\mathbb{R} \longrightarrow (0,\infty);\ (y,z) \longmapsto y$ be a canonical projection on $(0,\infty)$. We say that a NEF $F = F(\mu_1)$ has a cut on $(0,\infty)$, if $p(F)$ is also a NEF on $(0,\infty)$.

Lemma 4.3. *([1])*
Let $M_{F(\rho_1)} = p(M_F)$ and $p(F) = \{p(\nu); \nu \in F\}$. The following statements are equivalent:
(1) The family $p(F)$ is a NEF on $(0,\infty)$ (F has a cut on $(0,\infty)$).
(2) For all $m_1 \in M_{F(\rho_1)}$, the marginal variance function $p(V_F(m_1,m_2)|_{(0,\infty)\times\{0\}})$ is constant while m_2 runs over $m_1(M_F) = \{m_2 \in \mathbb{R}^{d-1}; (m_1,m_2) \in M_F\}$ (the marginal variance function $p(V_F(m_1,m_2)|_{(0,\infty)\times\{0\}})$ depends only on m_1).
(3) For all $\theta_2 \in \Theta(\eta_1)$, $y \longmapsto k_{Q_y}(\theta_2)$ is an affine function on $(0,\infty)$ (i.e. There exist maps $a : \Theta(\eta_1) \longrightarrow \mathbb{R}$ and $b : \Theta(\eta_1) \longrightarrow (0,\infty)$ such that for any $y \in (0,\infty)$, $k_{Q_y}(\theta_2) = ya(\theta_2) + b(\theta_2)$)
(4) There exist analytic maps $\kappa : \Theta(\eta_1) \longrightarrow \mathcal{L}(\mathbb{R},(0,\infty))$ and $\sigma : \Theta(\mu_2) \longrightarrow \mathbb{R}$ such that $m_2 = m_1\kappa(\theta_2) + \sigma(\theta_2)$, for all $(m_1,m_2) \in M_F$ with $(\theta_1,\theta_2) = \psi_{\mu_1}(m_1,m_2)$ (the functions κ and σ are equal to $\kappa = a'$ and $\sigma = b'$).

Theorem 4.4. *For all $m_2 \in M_{F(\eta_1)} \setminus \{0\}$,*

$$V_{F(\eta_1)}(m_2) = \left(k'_{Q_1}\left(\psi_{\eta_1}(m_2)\right)\right)^2 V_{F(\rho_1)}\left(\frac{m_2}{k'_{Q_1}\left(\psi_{\eta_1}(m_2)\right)}\right) + \frac{m_2 V_{F(Q_1)}\left(k'_{Q_1}\left(\psi_{\mu_1}(m_2)\right)\right)}{k'_{Q_1}\left(\psi_{\eta_1}(m_2)\right)}. \tag{14}$$

Before embarking in the proof of this theorem, we need the following lemma.

Lemma 4.5. *For all $m_2 \in M_{F(\eta_1)} \setminus \{0\}$,*

$$m_1 = \frac{m_2}{k'_{Q_1}\left(\psi_{\eta_1}(m_2)\right)}. \tag{15}$$

Proof. According to (13), we deduce that $p(V_F(m_1,m_2)|_{(0,\infty)\times\{0\}}) = V_{F(\rho_1)}(m_1)$ depends only on m_1. It follows that the NEF $F = F(\mu_1)$ has a cut on $(0,\infty)$. Using Lemma 4.3, we deduce that there exist a and b such that $k_{Q_y}(\theta_2) = ya(\theta_2) + b(\theta_2)$. Since, for all $y > 0$, Q_y is a convolution semigroup, then $a(\theta_2) = k_{Q_1}(\theta_2)$ and $b(\theta_2) = 0$. Therefore

$$m_2 = m_1 k'_{Q_1}(\theta_2) = m_1 k'_{Q_1}(\psi_{\eta_1}(m_2)).$$

Hence, we get the result. □

Now, we are in a better position to prove Theorem 4.4.
Proof of Theorem 4.4. Using (3), we deduce that,

$$\frac{\partial k_{\mu_1}(\theta_1,\theta_2)}{\partial\theta_1}\Big|_{\theta_1=0} = k'_{\rho_1}(k_{Q_1}(\theta_2)) = m_1(0,\theta_2) \text{ and } k'_{\eta_1}(\theta_2) = \frac{\partial k_{\mu_1}(\theta_1,\theta_2)}{\partial\theta_2}\Big|_{\theta_1=0} = k'_{Q_1}(\theta_2).k'_{\rho_1}(k_{Q_1}(\theta_2)) = m_2(0,\theta_2).$$

Setting $m_1(0,\theta_2) = m_1$ and $m_2(0,\theta_2) = m_2$. According to (7) and (12), we obtain

$$k''_{\mu_1}(0,\theta_2) = \begin{pmatrix} k''_{\rho_1}\left(k_{Q_1}(\theta_2)\right) & k''_{\rho_1}\left(k_{Q_1}(\theta_2)\right)k'_{Q_1}(\theta_2) \\ k''_{\rho_1}\left(k_{Q_1}(\theta_2)\right)k'_{Q_1}(\theta_2) & k''_{\eta_1}(\theta_2) \end{pmatrix}.$$

Together, with (13), this implies that, for all $(m_1(0,\theta_2), m_2(0,\theta_2)) = (m_1,m_2) \in M_{F(\mu_1)}$,

$$V_{F(\mu_1)}(m_1,m_2) = \begin{pmatrix} V_{F(\rho_1)}(m_1) & \frac{V_{F(\rho_1)}(m_1)}{m_1}m_2 \\ \frac{V_{F(\rho_1)}(m_1)}{m_1}m_2 & V_{F(\eta_1)}(m_2) \end{pmatrix} = \begin{pmatrix} V_{F(\rho_1)}(m_1) & \frac{V_{F(\rho_1)}(m_1)}{m_1}m_2 \\ \frac{V_{F(\rho_1)}(m_1)}{m_1}m_2 & \frac{V_{F(\rho_1)}(m_1)}{m_1^2}m_2^2 + m_1 V_{F(Q_1)}\left(\frac{m_2}{m_1}\right) \end{pmatrix}. \tag{16}$$

Inserting (15) in (16), we get the announced result. □

In the next corollary, we give the variance functions of the NEFs of the inverse Gaussian and Poisson processes time-changed by the first-exit time of the inverse Gaussian subordinator.

Corollary 4.6. *1. The variance function of the NEF generated the Poisson process time-changed by the first-exit time of the inverse Gaussian subordinator is given by*

$$V_{F(\eta_1)}(m_2) = \left(1-(f(m_2))^2\right)^2 V_{F(\rho_1)}\left(\frac{m_2}{1-(f(m_2))^2}\right) + m_2,\ \textit{for all}\ m_2 > 0,$$

where $f(m_2) = \dfrac{a(m_2)}{b(m_2)} - \dfrac{2m_2}{3} + b(m_2) - \dfrac{1}{3}$, $b(m_2) = \left(\left[\left(\frac{m_2}{3} + \frac{(2m_2+1)^3}{27} - \frac{1}{3}\right)^2 - (a(m_2))^3\right]^{\frac{1}{2}} - \frac{m_2}{3} - \frac{(2m_2+1)^3}{27} + \frac{1}{3}\right)^{\frac{1}{3}}$ *and* $a(m_2) = \dfrac{(2m_2+1)^2}{9} + \dfrac{1}{3}$.

2. The variance function of the NEF generated the inverse Gaussian process time-changed by the first-exit time of the inverse Gaussian subordinator is given by

$$V_{F(\eta_1)}(m_2) = \frac{1}{4(g(m_2))^4} V_{F(\rho_1)}\left(2m_2(g(m_2))^2\right) + \frac{m_2}{2(g(m_2))^4},\ \textit{for all}\ m_2 > 0,$$

where $g(m_2) = \dfrac{\left[\sqrt{6}\left(\frac{27}{16m_2^2} + 3\sqrt{3}u(m_2)\right)^{\frac{3}{2}} - 36\sqrt{3}m_2(l(m_2))^{\frac{2}{3}}w(m_2) + 12\sqrt{3}w(m_2)\right]^{\frac{1}{2}}}{12\sqrt{m_2}(l(m_2))^{\frac{1}{6}}(9(l(m_2))^{\frac{2}{3}} - \frac{3}{m_2})^{\frac{1}{4}}} + \dfrac{\sqrt{3}w(m_2)}{6(l(m_2))^{\frac{1}{6}}}$, $u(m_2) = \sqrt{\dfrac{4}{m_2^3} + \dfrac{27}{256m_2^4}}$, $l(m_2) = \dfrac{1}{32m_2^2} + \dfrac{\sqrt{3}u(m_2)}{18}$ *and* $w(m_2) = \sqrt{3(l(m_2))^{\frac{2}{3}} - \dfrac{1}{m_2}}$.

Proof. 1. According to [10], $k'_{Q_1}(\theta_2) = e^{\theta_2}$ and $V_{F(Q_1)}(m_2) = m_2$. This together with (14) imply that the variance function of the NEF generated the Poisson process time-changed by the first-exit time of the inverse Gaussian subordinator is given by

$$V_{F(\eta_1)}(m_2) = \exp\left(2\psi_{\eta_1}(m_2)\right) V_{F(\rho_1)}\left(\frac{m_2}{\exp\left(\psi_{\eta_1}(m_2)\right)}\right) + m_2,\ \text{for all}\ m_2 > 0. \tag{17}$$

Furthermore, using (4), we get, for all $\theta_2 < 0$,

$$k_{\eta_1}(\theta_2) = -\frac{1}{2}\ln(1-e^{\theta_2}) - \sqrt{1-e^{\theta_2}}\ \text{and}\ k'_{\eta_1}(\theta_2) = \frac{e^{\theta_2}\left(1+\sqrt{1-e^{\theta_2}}\right)}{2\left(1-e^{\theta_2}\right)} = m_2 > 0.$$

Setting $z = \sqrt{1-e^{\theta_2}} \in (0,1)$, we obtain the following equation

$$z^3 + (2m_2+1)z^2 - z - 1 = 0.$$

Its unique solution in the set $(0,1)$ is given by

$$z = f(m_2) = \frac{a(m_2)}{b(m_2)} - \frac{2m_2}{3} + b(m_2) - \frac{1}{3},$$

where $a(m_2) = \dfrac{(2m_2+1)^2}{9} + \dfrac{1}{3}$ and $b(m_2) = \left(\left[\left(\dfrac{m_2}{3} + \dfrac{(2m_2+1)^3}{27} - \dfrac{1}{3}\right)^2 - (a(m_2))^3\right]^{\frac{1}{2}} - \dfrac{m_2}{3} - \dfrac{(2m_2+1)^3}{27} + \dfrac{1}{3}\right)^{\frac{1}{3}}$.

Its follows that $\theta_2 = \psi_{\eta_1}(m_2) = \ln(1-(f(m_2))^2)$. Inserting this in (17), we get the result.

2. For all $\theta_2 < 0$ $k'_{Q_1}(\theta_2) = \dfrac{1}{2\sqrt{-\theta_2}}$ and for all $m_2 > 0$, $V_{F(Q_1)}(m_2) = 2m_2^3$ (see [10]). This with (14) imply that the variance function of the NEF generated the inverse Gaussian process time-changed by the first-exit time of the inverse Gaussian subordinator is equal to

$$V_{F(\eta_1)}(m_2) = -\frac{1}{4\psi_{\eta_1}(m_2)} V_{F(\rho_1)}\left(2m_2\sqrt{-\psi_{\eta_1}(m_2)}\right) - \frac{m_2}{2\psi_{\eta_1}(m_2)}, \text{ for all } m_2 > 0. \tag{18}$$

According to (4), we obtain, for all $\theta_2 < 0$,

$$k_{\eta_1}(\theta_2) = -\frac{1}{4}\ln(-\theta_2) - (-\theta_2)^{\frac{1}{4}} \text{ and } k'_{\eta_1}(\theta_2) = -\frac{1}{4\theta_2}\left(1 + (-\theta_2)^{\frac{1}{4}}\right) = m_2 > 0.$$

Setting $z = (-\theta_2)^{\frac{1}{4}}$, we get the following equation

$$4m_2 z^4 - z - 1 = 0.$$

Its unique positive solution is given by

$$z = g(m_2) = \frac{\left[\sqrt{6}\left(\frac{27}{16m_2^2} + 3\sqrt{3}u(m_2)\right)^{\frac{3}{2}} - 36\sqrt{3}m_2(l(m_2))^{\frac{2}{3}}w(m_2) + 12\sqrt{3}w(m_2)\right]^{\frac{1}{2}}}{12\sqrt{m_2}(l(m_2))^{\frac{1}{6}}(9(l(m_2))^{\frac{2}{3}} - \frac{3}{m_2})^{\frac{1}{4}}} + \frac{\sqrt{3}w(m_2)}{6(l(m_2))^{\frac{1}{6}}},$$

where $u(m_2) = \sqrt{\dfrac{4}{m_2^3} + \dfrac{27}{256m_2^4}}$, $l(m_2) = \dfrac{1}{32m_2^2} + \dfrac{\sqrt{3}u(m_2)}{18}$ and $w(m_2) = \sqrt{3(l(m_2))^{\frac{2}{3}} - \dfrac{1}{m_2}}$.

Its follows that $\theta_2 = \psi_{\eta_1}(m_2) = -(g(m_2))^4$. Inserting this in (18), we achieve the result. □

Conclusion

The time-changed Lévy processes are an important and useful tools in the theory of diffusion equations. Furthermore, this class of mixture added more flexibility to the data analysis. The results of this paper give some characterizations of the NEFs governed by these processes. These characterizations are obtained by means of variance functions which have an important role in the estimation of the parameters of exponential dispersion models. Since these models has been extensively developed in the field of statistics and classification literature, then we propose, in a future work, to use these variance functions in the estimation of the parameters of some exponential dispersion models and to apply the results of estimation in the fields of image segmentation and signal path loss.

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