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Topological Aspect of Monodromy Groupoid for a Group-Groupoid

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Abstract. In this paper we develop star topological and topological group-groupoid structures of monodromy groupoid and prove that the monodromy groupoid of a topological group-groupoid is also a topological group-groupoid.

1. Introduction

As enunciated by Chevalley in [7, Theorem 2, Chapter 2], the general idea of the monodromy principle is that of extending a local morphism f on a topological structure G, or extending a restriction of f, not to G itself but to some simply connected cover of G. A form of this for topological groups was given in [7, Theorem 3], and developed by Douady and Lazard in [8] for Lie groups, generalized to topological groupoid case in [14] and [4].

The notion of monodromy groupoid was indicated by J. Pradines in [17] as part of his grand scheme announced in [17–20] to generalize the standard construction of a simply connected Lie group from a Lie algebra to a corresponding construction of a Lie groupoid from a Lie algebroid (see also [11, 12, 16]).

One construction of the monodromy groupoid for a topological groupoid G and an open subset W including the identities is given via free groupoid concept and denoted by $\mathrm{M(G,W)}$ as a generalization of the construction in [8].

Another construction of the monodromy groupoid for a topological groupoid G in which each star G_x has a universal cover is directly given in Mackenzie [11, p. 67-70] as a disjoint union of the universal covers of the stars G_x's and denoted by $\mathrm{Mon}(G)$.

These two monodromy groupoids $\mathrm{M(G,W)}$ and $\mathrm{Mon}(G)$ are identified as star Lie groupoids in [4, Theorem 4.2] using Theorem 4.2 which is originally [1, Theorem 2.1] to get an appropriate topology.

In particularly if G is a connected topological group which has a universal cover, then the monodromy groupoid $\mathrm{Mon}(G)$ is the universal covering group, while if G is the topological groupoid $X \times X$, for a semi-locally simply connected topological space X, then the monodromy groupoid $\mathrm{Mon}(G)$ is the fundamental groupoid πX. Hence the monodromy groupoid generalizes both the concepts of universal covering group and the fundamental groupoid. For further discussion on monodromy and holonomy groupoids see [2].

The notion of monodromy groupoid for topological group-groupoid was recently introduced and investigated in [15]; and then it has been generalized to the internal groupoid case in [13]. Motivated by the referee's comments of latter paper, in this paper, we aim to develop the topological aspect of the monodromy groupoid $\mathrm{Mon}(G)$ as a group-groupoid for a topological group-groupoid G.

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The organization of the paper is as follows. In Section 1 we give a preliminary of groupoid, group-groupoid, topological groupoid and the constructions of monodromy groupoid. In Section 2 star topological group-groupoid structure of monodromy groupoid with some results are given. Section 3 is devoted to develop topological aspect of the monodromy groupoid Mon(G) as group-groupoid together with a strong monodromy principle for topological group-groupoids.

2. Preliminaries on Monodromy Groupoid

A *groupoid* is a small category in which each morphism is an isomorphism (see for example [5] and [11]). So a groupoid G has a set G of morphisms and a set Ob(G) of *objects* together with *source* and *target* point maps $\alpha, \beta\colon G \to \mathrm{Ob}(G)$ and *object inclusion* map $\epsilon\colon \mathrm{Ob}(G) \to G$ such that $\alpha\epsilon = \beta\epsilon = 1_{\mathrm{Ob}(G)}$. There exists a partial composition defined by $G_\beta \times_\alpha G \to G, (g, h) \mapsto g \circ h$, where $G_\beta \times_\alpha G$ is the pullback of β and α. Here if $g, h \in G$ and $\beta(g) = \alpha(h)$, then the *composite* $g \circ h$ exists such that $\alpha(g \circ h) = \alpha(g)$ and $\beta(g \circ h) = \beta(h)$. Further, this partial composition is associative, for $x \in \mathrm{Ob}(G)$ the morphism $\epsilon(x)$ acts as the identity and it is denoted by 1_x, and each element g has an inverse g^{-1} such that $\alpha(g^{-1}) = \beta(g)$, $\beta(g^{-1}) = \alpha(g)$, $g \circ g^{-1} = \epsilon(\alpha(g))$, $g^{-1} \circ g = \epsilon(\beta((g))$. The map $G \to G$, $g \mapsto g^{-1}$ is called the *inversion*. In a groupoid G, the source and target points, the object inclusion, the inversion maps and the partial composition are called *structural maps*. An example of a groupoid is fundamental groupoid of a topological space X, where the objects are points of X and morphisms are homotopy classes of the paths relative to the end points. A group is a groupoid with one object.

In a groupoid G each object x is identified with unique identity $\epsilon(x) = 1_x$ and hence we sometimes write Ob(G) for the set of identities. For $x, y \in \mathrm{Ob}(G)$ we write $G(x, y)$ for $\alpha^{-1}(x) \cap \beta^{-1}(y)$. The difference map $\delta\colon G \times_\alpha G \to G$ is given by $\delta(g, h) = g^{-1} \circ h$, and is defined on the double pullback of G by α. If $x \in \mathrm{Ob}(G)$, and $W \subseteq G$, we write W_x for $W \cap \alpha^{-1}(x)$, and call W_x the *star* of W at x. Especially we write G_x for $\alpha^{-1}(x)$ and call *star* of G at x. We denote the set of inverses of the morphisms in W by W^{-1}.

A *star topological groupoid* is a groupoid in which the stars G_x's have topologies such that for each $g \in G(x, y)$ the left (and hence right) translation

$$L_g\colon G_y \to G_x, h \mapsto g \circ h$$

is a homeomorphism and G is the topological sum of the G_x's. A *topological groupoid* is a groupoid in which G and Ob(G) have both topologies such that the structural maps of groupoid are continuous.

A *group-groupoid* is a groupoid G in which the sets of objects and morphisms have both group structures and the product map $G \times G \to G, (g, h) \mapsto gh$, inverse $G \to G, g \mapsto g^{-1}$ and, the unit maps : $\{\star\} \to G$, where $\{\star\}$ is singleton, are morphisms of groupoids.

In a group-groupoid G, we write $g \circ h$ for the composition of morphisms g and h in groupoid while gh for the product in group and write $\bar{g}$ for the inverse of g in groupoid and g^{-1} for the one in group. Here note that the product map is a morphism of groupoids if and only if the *interchange rule*

$$(gh) \circ (kl) = (g \circ k)(h \circ l)$$

is satisfied for $g, h, k, l \in G$ whenever one side composite is defined.

A *topological group-groupoid* is defined in [10, Definition 1] as a group-groupoid which is also a topological groupoid and the structural maps of group multiplication are continuous. We define a *star topological group-groupoid* as a group-groupoid which is also a star topological groupoid.

Let X be a topological space admitting a simply connected cover. A subset U of X is called *liftable* if U is open, path-connected and the inclusion $U \to X$ maps each fundamental group of U trivially. If U is liftable, and $q\colon Y \to X$ is a covering map, then for any $y \in Y$ and $x \in U$ such that $qy = x$, there is a unique map $\hat{\imath}\colon U \to Y$ such that $\hat{\imath}x = y$ and $q\hat{\imath}$ is the inclusion $U \to X$. A space X is called *semi-locally simply connected* if each point has a liftable neighborhood and *locally simply connected* if it has a base of simply connected sets.

Let X be a topological space such that each path component of X admits a simply connected covering space. It is standard that if the fundamental groupoid πX is provided with a topology as in [6], then for an

$x \in X$ the target point map $t\colon (\pi X)_x \to X$ is the universal covering map of X based at x (see also Brown [5, 10.5.8]).

Let G be a topological groupoid and W an open subset of G including all the identities. As a generalization of the construction in [8], the monodromy groupoid $\mathrm{M(G, W)}$ is defined as the quotient groupoid $\mathrm{F(W)}/N$, where $\mathrm{F(W)}$ is the free groupoid on W and N is the normal subgroupoid of $\mathrm{F(W)}$ generated by the elements of the form $[uv]^{-1}[u][v]$ whenever $uv \in W$ for $u, v \in W$. Then $\mathrm{M(G, W)}$ has a universal property that any local morphism $f\colon W \to H$ globalizes to a unique morphism $\tilde{f}\colon \mathrm{M(G, W)} \to H$ of groupoids.

Let G be a star topological groupoid such that each star G_x has a universal cover. The groupoid $\mathrm{Mon}(G)$ is defined in [11] as the disjoint union of the universal covers of stars G_x's at the base points identities. Hence $\mathrm{Mon}(G)$ is disjoint union of the stars $(\pi(G_x))_{\epsilon(x)}$. The object set X of $\mathrm{Mon}(G)$ is the same as that of G. The source point map $\alpha\colon \mathrm{Mon}(G) \to X$ maps all stars $(\pi(G_x))_{\epsilon(x)}$ to x, while the target point map $\beta\colon \mathrm{Mon}(G) \to X$ is defined on each star $(\pi(G_x))_{\epsilon(x)}$ as the composition of the two target point maps

$$(\pi(G_x))_{\epsilon(x)} \xrightarrow{\beta} G_x \xrightarrow{\beta} X.$$

As explained in Mackenzie [11, p.67] there is a partial composition on $\mathrm{Mon}(G)$ defined by

$$[a] \bullet [b] = [a \star (a(1) \circ b)]$$

where $\star$, inside the bracket, denotes the usual composition of paths and $\circ$ denotes the composition in the groupoid. Here $a(1) \circ b$ is the path defined by $(a(1) \circ b)(t) = a(1) \circ b(t)$ $(0 \leq t \leq 1)$. Here we point that since G is a star topological groupoid, the left translation is a homeomorphism. Hence the path $a(1) \circ b$, which is a left translation of b by $a(1)$, is defined when b is a path. So the path $a \star (a(1) \circ b)$ is defined by

$$(a \star (a(1) \circ b))(t) = \begin{cases} a(2t), & 0 \leq t \leq \frac{1}{2} \\ a(1) \circ b(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Here if a is a path in G_x from $\epsilon(x)$ to $a(1)$, where $\beta(a(1)) = y$, say, and b is a path in G_y from $\epsilon(y)$ to $b(1)$, then for each $t \in [0, 1]$ the composite $a(1) \circ b(t)$ is defined in G_y, yielding a path $a(1) \circ b$ from $a(1)$ to $a(1) \circ b(1)$. It is straightforward to prove that in this way a groupoid is defined on $\mathrm{Mon}(G)$ and that the target point map of paths induces a morphism of groupoids $p\colon \mathrm{Mon}(G) \to G$.

The following theorem whose Lie version is given in [4, Theorem 4.2], identifies two monodromy groupoids $\mathrm{M(G, W)}$ and $\mathrm{Mon}(G)$ as star topological groupoids.

Theorem 2.1. *Let G be a star connected topological groupoid such that each star G_x has a simply connected cover. Suppose that W is a star path connected neighborhood of* $\mathrm{Ob}(G)$ *in G and W^2 is contained in a star path connected neighbourhood V of* $\mathrm{Ob}(G)$ *such that for all $x \in \mathrm{Ob}(G)$, V_x is liftable. Then there is an isomorphism of star topological groupoids* $\mathrm{M(G, W)} \to \mathrm{Mon}(G)$ *and hence the morphism* $\mathrm{M(G, W)} \to G$ *is a star universal covering map.*

3. Monodromy Groupoids as Star Topological Group-Groupoids

In [15, Theorem 3.10] it was proved that if G is a topological group-groupoid in which each star G_x has a universal cover, then $\mathrm{Mon}(G)$ is a group-groupoid. We can now state the following theorem in terms of star topological group-groupoids.

Theorem 3.1. *Let G be a topological group-groupoid such that each star G_x has a universal cover. Then the monodromy groupoid* $\mathrm{Mon}(G)$ *is a star topological group-groupoid.*

Proof. If each star G_x admits a universal cover at $\epsilon(x)$, then each star $\mathrm{Mon}(G)_x$ may be given a topology so that it is the universal cover of G_x based at $\epsilon(x)$, and then $\mathrm{Mon}(G)$ becomes a star topological groupoid. Further by the detailed proof of the Theorem [15, Theorem 3.10], we define a group structure on $\mathrm{Mon}(G)$ by

$$\mathrm{Mon}(G) \times \mathrm{Mon}(G) \to \mathrm{Mon}(G), ([a], [b]) \mapsto [ab]$$

such that $\mathrm{Mon}(G)$ is a group-groupoid. The other details of the proof follow from the cited theorem. □

The following corollary is a result of Theorem 2.1 and Theorem 3.1.

Corollary 3.2. *Let G be a topological group-groupoid and W an open subset of G satisfying the conditions in Theorem 2.1, then the monodromy groupoid* M(G, W) *is a star topological group-groupoid.*

Let TopGrpGpd be the category whose objects are topological group-groupoids and morphisms are the continuous groupoid morphisms preserving group operation; and let STopGrpGpd be the full subcategory of TopGrpGpd on those objects which are topological group-groupoids whose stars have universal covers. Let StarTopGrpGpd be the category whose objects are star topological group-groupoids and the morphisms are those of group-groupoids which are continuous on stars. Then we have the following.

Proposition 3.3.

$$\mathrm{Mon}\colon \mathsf{STopGrpGpd} \to \mathsf{StarTopGrpGpd}$$

which assigns the monodromy groupoid $\mathrm{Mon}(G)$ *to such a topological group-groupoid G is a functor.*

Proof. We know from Theorem 3.1 that if G is a topological group-groupoid in which the stars have universal covers, then $\mathrm{Mon}(G)$ is also a star topological group-groupoid. Let $f\colon G \to H$ be a morphism of STopGrpGpd. Then the restriction $f\colon G_x \to H_{f(x)}$ is continuous and hence by [6, Proposition 3], the induced morphism $\pi(f)\colon \pi(G_x) \to \pi(H_{f(x)})$, which is a morphism of topological groupoids, is continuous. Latter morphism is restricted to the continuous map $\pi(f)\colon \pi(G_x)_{1_x} \to \pi(H_{f(x)})_{1_{f(x)}}$ which is $\mathrm{Mon}(f)\colon (\mathrm{Mon}(G))_x \to (\mathrm{Mon}(H))_{f(x)}$. That means $\mathrm{Mon}(f)$ is a morphism of star topological group-groupoids. The other details of the proof is straightforward. □

We need the following results in the proof of Theorem 3.6.

Proposition 3.4. ([6, Theorem 1]) *If X is a locally path connected and semi-locally simply connected space, then the fundamental groupoid* πX *may be given a topology making it a topological groupoid.*

Theorem 3.5. ([15, Theorem 3.8]) *If X and Y are locally path connected and locally simply connected topological spaces, then* $\pi(X \times Y)$ *and* $\pi(X) \times \pi(Y)$ *are isomorphic as topological groupoids.*

Theorem 3.6. *For the topological group-groupoids G and H whose stars have universal covers, the monodromy groupoids* $\mathrm{Mon}(G \times H)$ *and* $\mathrm{Mon}(G) \times \mathrm{Mon}(H)$ *as star topological group-groupoids are isomorphic.*

Proof. Let G and H be the topological group-groupoids such that the stars have the universal covers. Then by Theorem 3.1, $\mathrm{Mon}(G \times H)$ and $\mathrm{Mon}(G) \times \mathrm{Mon}(H)$ are star topological group-groupoids and by [15, Theorem 2.1] we know that these groupoids $\mathrm{Mon}(G \times H)$ and $\mathrm{Mon}(G) \times \mathrm{Mon}(H)$ are isomorphic. By the fact that the star $(\mathrm{Mon}(G))_x$ is the star $(\pi(G_x))_{\epsilon(x)}$ of the fundamental groupoid $\pi(G_x)$ we have the following evaluation

$$\begin{aligned}
(\mathrm{Mon}(G \times H))_{(x,y)} &= \pi((G \times H)_{(x,y)})_{\epsilon(x,y)} \\
&= (\pi(G_x \times H_y))_{\epsilon(x,y)} \\
&\simeq (\pi(G_x))_{\epsilon(x)} \times (\pi(H_y))_{\epsilon(y)} && \text{(by Theorem 3.5)} \\
&= (\mathrm{Mon}(G))_x \times (\mathrm{Mon}(H))_y
\end{aligned}$$

Hence we have a homeomorphism $(\mathrm{Mon}(G \times H))_{(x,y)} \to (\mathrm{Mon}(G))_x \times (\mathrm{Mon}(H))_y$ on the stars and then by gluing these homeomorphisms on the stars we have an isomorphism $f\colon \mathrm{Mon}(G \times H) \to \mathrm{Mon}(G) \times \mathrm{Mon}(H)$ defined by $f([a]) = ([p_1 a], [p_2 a])$ for $[a] \in \mathrm{Mon}(G \times H)$, which is identity on objects. Here f is reduced to the homeomorphisms on the stars and it is also a morphism of group-groupoids. Hence f is an isomorphism of star topological group-groupoids and therefore the star topological group-groupoids $\mathrm{Mon}(G \times H)$ and $\mathrm{Mon}(G) \times \mathrm{Mon}(H)$ are isomorphic. □

Before giving the group-groupoid version of the monodromy principle we give definition of local morphism for group-groupoids adapted from definition of local morphism of groupoids in [14].

Definition 3.7. Let G and H be group-groupoids. A *local morphism* from G to H is a map $f\colon W \to H$ from a subset W of G, including the identities, satisfying the conditions $\alpha_H(f(u)) = f(\alpha_G(u))$, $\beta_H(f(u)) = f(\beta_G(u))$, $f(u \circ v) = f(u) \circ f(v)$ and $f(uv) = f(u)f(v)$ whenever $u, v \in W$, $u \circ v \in W$ and $uv \in W$.

A *local morphism of star topological group-groupoids* is a local morphism of group-groupoids which is continuous on the stars.

Let G and H be topological group-groupoids and W an open neighborhood of $\mathrm{Ob}(G)$. A *local morphism* from G to H is a continuous local morphism $f\colon W \to H$ of group-groupoids.

We can now prove a weak monodromy principle for star topological group-groupoids.

Theorem 3.8. (Weak Monodromy Principle) *Let G be a star connected topological group-groupoid and W an open and star connected subgroup of G containing $\mathrm{Ob}(G)$ and satisfying the condition in Theorem 2.1. Let H be a star topological group-groupoid and $f\colon W \to H$ a local morphism of star topological group-groupoids which is identity on $\mathrm{Ob}(G)$. Then f globalizes uniquely to a morphism $\tilde{f}\colon \mathrm{M(G, W)} \to H$ of star topological group-groupoids.*

Proof. Here remark that by Corollary 3.2, $\mathrm{M(G, W)}$ is a star topological group-groupoid and by the construction of $\mathrm{M(G, W)}$ we have an inclusion map $\imath\colon W \to \mathrm{M(G, W)}$ and W is homeomorphic to $\imath(W) = W'$ which generates $\mathrm{M(G, W)}$. The existence of $\tilde{f}\colon \mathrm{M(G, W)} \to H$ as a groupoid morphism follows from the universal property of free groupoid $\mathrm{F(W)}$ and the fact that $\mathrm{M(G, W)}$ is generated by W'. Hence one needs to show that $\tilde{f}$ is a group-groupoid morphism, i.e., it preserves the group operation. Let a and b be the morphisms of $\mathrm{M(G, W)}$. Since W' generates $\mathrm{M(G, W)}$, a and b are written as $a = u_1 \circ u_2 \ldots \circ u_n$ and $b = v_1 \circ v_2 \ldots \circ v_m$ for $u_i, v_j \in W'$. Since W is group and so also is W' we have $u_i v_i \in W'$ for all i. Then by the interchange rule we have the following evaluation for $m \geq n$

$$\begin{aligned}
\tilde{f}(ab) &= \tilde{f}((u_1 \circ u_2 \ldots \circ u_n)(v_1 \circ v_2 \ldots \circ v_m)) \\
&= \tilde{f}(u_1 v_1 \circ \ldots \circ u_n v_n \circ 1_{\beta(u_n)} v_{n+1} \circ \ldots \circ 1_{\beta(u_n)} v_m) \\
&= f(u_1 v_1 \circ \ldots \circ u_n v_n \circ 1_{\beta(u_n)} v_{n+1} \circ \ldots \circ 1_{\beta(u_n)} v_m) \\
&= f(u_1) f(v_1) \circ \ldots \circ f(1_{\beta(u_n)}) f(v_m) \\
&= (f(u_1) \circ \ldots \circ f(u_n))(f(v_1) \circ \ldots \circ f(v_m)) \\
&= f(u_1 \circ \ldots \circ u_n) f(v_1 \circ \ldots \circ v_m) \\
&= \tilde{f}(a)\tilde{f}(b).
\end{aligned}$$

Since $\mathrm{M(G, W)}$ is generated by W', the continuity of $\tilde{f}\colon \mathrm{M(G, W)} \to H$ on stars follows by the continuity of $f\colon W \to H$. □

As a result of Theorem 3.8 we have the following corollary.

Corollary 3.9. *Let G be a star topological group-groupoid which is star connected and star simply connected and let W be an open and star connected subgroup of G containing $\mathrm{Ob}(G)$ and satisfying the condition in Theorem 2.1. Let H be a star topological group-groupoid and let $f\colon W \to H$ be a local morphism of star topological group-groupoids which is identity on $\mathrm{Ob}(G)$. Then f globalizes uniquely to a morphism $\tilde{f}\colon G \to H$ of star topological group-groupoids.*

Proof. Since G is star connected and star simply connected, $\mathrm{Mon}(G)$, as a star topological groupoid, becomes isomorphic to G; and by Theorem 2.1 and Corollary 3.2 $\mathrm{M(G, W)}$ and $\mathrm{Mon}(G)$ are isomorphic as star topological groupoids. Hence the rest of the proof follows from Theorem 3.8. □

4. Topological Structure on Monodromy Groupoid as Group-Groupoid

In this section we prove that if G is a topological group-groupoid in which each star has a universal cover and W is a useful open subset of G, including the identities, then the monodromy groupoid M(G, W) becomes a topological group-groupoid with the topology obtained by Theorem 4.2.

Let G be a groupoid and $X = \mathrm{Ob}(G)$ a topological space. An *admissible local section* of G, which is due to Ehresmann [9], is a function $\sigma\colon U \to G$ from an open subset of X such that the following holds.

1. $\alpha\sigma(x) = x$ for all $x \in U$;
2. $\beta\sigma(U)$ is open in X;
3. $\beta\sigma$ maps U topologically to $\beta\sigma(U)$.

Here the set U is called *domain* of s and written as D_s. Let $\Gamma(G)$ be the set of all admissible local sections of G. A product defined on $\Gamma(G)$ as follows: for any two admissible local sections

$$(st)x = (sx)(t\beta sx)$$

s and t are composable if $D_{st} = D_s$. If s is admissible local section then s^{-1} is also an admissible local section $\beta sD_s \to G$, $\beta sx \mapsto (sx)^{-1}$.

Let W be a subset of G and let W have a topology such that X is a subspace. (α, β, W) is called *enough continuous admissible local sections* or *locally sectionable* if

1. $s\alpha(w) = w$;
2. $s(U) \subseteq W$;
3. s is continuous from D_s to W. Such s is called continuous admissible local section.

Holonomy groupoid is constructed for a locally topological groupoid whose definition is as follows (see [3] for a locally topological groupoid structure on a foliated manifold):

Definition 4.1. ([1, Definition 1.2]) A locally topological groupoid is a pair (G, W) where G is a groupoid and W is a topological space such that

1. $\mathrm{Ob}(G) \subseteq W \subseteq G$;
2. $W = W^{-1}$;
3. W generates G as a groupoid;
4. The set $W_\delta = (W \times_\alpha W) \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$ and the restriction to W_δ of the difference map $\delta\colon G \times_\alpha G \to G$ is continuous;
5. the restrictions to W of the source and target point maps α, β are continuous and (α, β, W) has enough continuous admissible local sections.

Note that a topological groupoid is a locally topological groupoid but converse is not true.

The following globalization theorem assigns a topological groupoid called *holonomy groupoid* and denoted by $\mathrm{Hol}(G, W)$ or only H to a locally topological groupoid (G, W) and hence it is more useful to obtain an appropriate topology on the monodromy groupoid. We give an outline of the proof since some details of the construction in the proof are needed for Proposition 4.3 and Theorem 4.4.

Theorem 4.2. ([1, Theorem 2.1]) *Let (G, W) be a locally topological groupoid. Then there is a topological groupoid H, a morphism $\phi : H \to G$ of groupoids and an embedding $i : W \to H$ of W to an open neighborhood of* $\mathrm{Ob}(H)$ *such that the following conditions are satisfied:*

i) ϕ is the identity on objects, $\phi i = id_W, \phi^{-1}(W)$ is open in H,and the restriction $\phi_W : \phi^{-1}(W) \to W$ of ϕ is continuous;

ii) if A is a topological groupoid and $\xi : A \to G$ is a morphism of groupoids such that:

a) ξ is the identity on objects;

b) the restriction $\xi_W : \xi^{-1}(W) \to W$ of ξ is continuous and $\xi^{-1}(W)$ is open in A and generates A;

c) the triple (α_A, β_A, A) is locally sectionable,

then there is a unique morphism $\xi' : A \to H$ of topological groupoids such that $\phi\xi' = \xi$ and $\xi' a = i\xi a$ for $a \in \xi^{-1}(W)$.

Proof. Let $\Gamma(G)$ be the set of all admissible local sections of G. Define a product on $\Gamma(G)$ by

$$(st)x = (sx)(t\beta sx)$$

for two admissible local sections s and t. If s is an admissible local section then write s^{-1} for the admissible local section $\beta sD_s \to G, \beta sx \mapsto (sx)^{-1}$. With this product $\Gamma(G)$ becomes an inverse semigroup. Let $\Gamma^c(W)$ be the subset of $\Gamma(G)$ consisting of admissible local sections which have values in W and are continuous. Let $\Gamma^c(G,W)$ be the subsemigroup of $\Gamma(G)$ generated by $\Gamma^c(W)$. Then $\Gamma^c(G,W)$ is again an inverse semigroup. Intuitively, it contains information on the iteration of local procedures.

Let $J(G)$ be the sheaf of germs of admissible local sections of G. Thus the elements of $J(G)$ are the equivalence classes of pairs (x,s) such that $s \in \Gamma(G), x \in D_s$, and (x,s) is equivalent to (y,t) if and only if $x = y$ and s and t agree on a neighbourhood of x. The equivalence class of (x,s) is written $[s]_x$. The product structure on $\Gamma(G)$ induces a groupoid structure on $J(G)$ with X as the set of objects, and source and target point maps are $[s]_x \mapsto x, [s]_x \mapsto \beta sx$ respectively. Let $J^c(G,W)$ be the subsheaf of $J(G)$ of germs of elements of $\Gamma^c(G,W)$. Then $J^c(G,W)$ is generated as a subgroupoid of $J(G)$ by the sheaf $J^c(W)$ of germs of elements of $\Gamma^c(W)$. Thus an element of $J^c(G,W)$ is of the form

$$[s]_x = [s_1]_{x_1} \dots [s_n]_{x_n}$$

where $s = s_1 \dots s_n$ with $[s_i]_{x_i} \in J^c(W), x_{i+1} = \beta s_i x_i, i = 1, \dots, n$ and $x_1 = x \in D_s$.

Let $\psi : J(G) \to G$ be the map defined by $\psi([s]_x) = s(x)$, where s is an admissible local section. Then $\psi(J^c(G,W)) = G$. Let $J_0 = J^c(W) \cap \ker\psi$. Then J_0 is a normal subgroupoid of $J^c(G,W)$; the proof is the same as in [1, Lemma 2.2] The holonomy groupoid $H = \mathrm{Hol}(G,W)$ is defined to be the quotient groupoid $J^c(G,W)/J_0$. Let $p\colon J^c(G,W) \to H$ be the quotient morphism and let $p([s]_x)$ be denoted by $< s >_x$. Since $J_0 \subseteq \ker\psi$ there is a surjective morphism $\phi : H \to G$ such that $\phi p = \psi$.

The topology on the holonomy groupoid H such that H with this topology is a topological groupoid is constructed as follows. Let $s \in \Gamma^c(G,W)$. A partial function $\sigma_s : W \to H$ is defined as follows. The domain of σ_s is the set of $w \in W$ such that $\beta w \in D_s$. A continuous admissible local section f through w is chosen and the value $\sigma_s w$ is defined to be

$$\sigma_s w = < f >_{\alpha w} < s >_{\beta w} = < fs >_{\alpha w} .$$

It is proven that $\sigma_s w$ is independent of the choice of the local section f and that these σ_s form a set of charts. Then the initial topology with respect to the charts σ_s is imposed on H. With this topology H becomes a topological groupoid. Again the proof is essentially the same as in Aof-Brown [1]. □

From the construction of the holonomy groupoid the following extendibility condition is obtained.

Proposition 4.3. *The locally topological groupoid (G,W) is extendible to a topological groupoid structure on G if and only if the following condition holds:*
(1)*: if $x \in \mathrm{Ob}(G)$, and s is a product $s_1 \dots s_n$ of local sections about x such that each s_i lies in $\Gamma^c(W)$ and $s(x) = 1_x$, then there is a restriction s' of s to a neighbourhood of x such that s' has image in W and is continuous, i.e. $s' \in \Gamma^c(W)$.*

To prove that M(G, W) is a topological group-groupoid, we first prove a more general result.

Theorem 4.4. *Let G be a topological group-groupoid and W an open subset of G such that*

1. $\mathrm{Ob}(G) \subseteq W$
2. $W = W^{-1}$
3. *W generates G and*
4. *(α_W, β_W, W) has enough continuous admissible local sections.*

Let $p\colon M \to G$ be a morphism of group-groupoids such that $\mathrm{Ob}(p)\colon \mathrm{Ob}(M) \to \mathrm{Ob}(G)$ is identity and assume that $\imath\colon W \to M$ is an inclusion such that $p\imath = i\colon W \to G$ and $W' = \imath(W)$ generates M.

Then M admits the structure of a topological group-groupoid such that $p\colon M \to G$ is a morphism of topological group-groupoids and maps W' to W homeomorphically.

Proof. As it was proved in [4, Corollary 5.6], (M, W') is a locally topological groupoid and by Proposition 4.3 it is extendible, i.e., the holonomy groupoid $H = \mathrm{Hol}(M, W')$ is isomorphic to M. Hence by Theorem 4.2, M becomes a topological groupoid such that M has the chart topology from W'. Hence the chart open subsets of M form a base for this topology. We now prove that the difference map of product $m\colon M \times M \to M, (a, b) \to ab^{-1}$ is continuous. We now consider the following diagram.

$$\begin{array}{ccc} m^{-1}(W') \cap (W' \times W') & \xrightarrow{m_{W'}} & W' \\ \downarrow & & \downarrow \\ M \times M & \xrightarrow{m} & M \end{array}$$

Let U be a base open subset, i.e. a chart open subset of M and U' be the open subset of W', which is homeomorphic to U. Since the restriction $m_{W'}: m^{-1}(W') \cap (W' \times W') \to W'$ is continuous, the inverse image $(m_{W'})^{-1}(U')$ is open in $m^{-1}(W') \cap (W' \times W')$ and it is homeomorphic to an open neighbourhood of $M \times M$. That means the inverse image $m^{-1}(U)$ is open in $M \times M$ and hence m is continuous.

Since the locally topological groupoid (M, W') is extendible the holonomy groupoid $H = \mathrm{Hol}(M, W')$ is isomorphic to M and hence by Theorem 4.2, $p\colon M \to G$ becomes a morphism of topological groupoids. Further by assumption it is a morphism of group-groupoids. Hence $p\colon M \to G$ becomes a morphism of topological group-groupoids. □

As a result of Theorems 2.1 and 4.4 we can state the following corollary.

Corollary 4.5. *Let G be a topological group-groupoid such that each star G_x has a universal cover. Suppose that W is a star path connected neighborhood of* $\mathrm{Ob}(G)$ *in G satisfying the conditions in Theorem 2.1 and Theorem 4.4. Then the monodromy groupoid* $\mathrm{Mon}(G)$ *is a topological group-groupoid such that the projection $p\colon$* $\mathrm{Mon}(G) \to G$ *is a morphism of topological group-groupoids.*

Proof. By Theorem 2.1, M(G, W) and $\mathrm{Mon}(G)$ are isomorphic as star topological groupoids. By Theorem 4.4, M(G, W) is a topological group-groupoid and so also is $\mathrm{Mon}(G)$ as required. □

As a result of Corollaries 3.9 and 4.5 we can give the following theorem which we call as *strong monodromy principle* for topological group-groupoids.

Theorem 4.6. (Strong Monodromy Principle) *Let G be a star connected and star simply connected topological group-groupoid and let W be an open and star connected subgroup of G satisfying the conditions of Theorem 2.1 and Theorem 4.4. Let H be a topological group-groupoid and let $f\colon W \to H$ be a local morphism of topological group-groupoids which is the identity on* $\mathrm{Ob}(G)$*. Then f extends uniquely to a morphism $\tilde{f}\colon G \to H$ of topological group-groupoids.*

Proof. By Corollary 3.9, the local morphism $f\colon W \to H$ extends to $\tilde{f}\colon G \to H$; and by Corollary4.5 $\mathrm{Mon}(G)$ and M(G, W) are isomorphic as topological group-groupoids. The continuity of $\tilde{f}$ follows from the fact that $\tilde{f}$ is continuous on an open subset W' which generates M(G, W). □

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