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Addendum to Stability Analysis of Neutral Linear Fractional System with Distributed Delays (Filomat 30:3 (2016), 841-851)

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Abstract. In this short article we discuss the initial condition of the initial value problem for fractional differential equations with delayed argument and derivatives in Riemann-Liouville sense. We provide also a new lemma - a ”mirror” analogue of the Kilbas Lemma, concerning the right side Riemann-Liouville fractional integral, which is important for the correct setting of the initial conditions, especially in the case of equations with delay.

1. Introduction

The first goal of this short article is to correct some inaccuracies in the article [2] which can be a reason of serious misunderstandings of the initial condition of the Initial Value Problem (IVP) for fractional differential equations with delayed argument and derivatives in Riemann-Liouville sense. Without these corrections the considered IVP in [2] can be treated as not correct defined.

In response to comments from readers, our second goal is to provide a new lemma - a ”mirror” analogue of the Kilbas Lemma (Lemma 3.2 in [1]), concerning the right side Riemann-Liouville fractional integral, which is important especially in the case of equations with delay. Without this result any proof of existence of solutions in the special defined for the Riemann-Liouville case space of $(1-\alpha)$-continuous functions is and will be not complete.

2. The Initial Condition in the IVP for the Riemann-Liouville Case

In [2] is considered the following neutral linear delayed system of incommensurate type with distributed delay

$$D^{\alpha}_{0+}(X(t) - \int_{-\tau}^{0}[d_\theta V(t,\theta)]X(t+\theta)) = \int_{-\sigma}^{0}[d_\theta U(t,\theta)]X(t+\theta) + F(t), \tag{2.1}$$

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separately for both cases - for Riemann-Liouville and for Caputo fractional derivatives, where $X(t) = (x_1(t), ..., x_n(t))^T, F(t) = (f_1(t), \dots, f_n(t))^T, \alpha = (\alpha_1, \dots, \alpha_n), \alpha_k \in (0,1), \tau, \sigma \in (0, \infty), h = \max(\sigma, \tau); U : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}, U(t,\theta) = \{u^i_j(t,\theta)\}^n_{i,j=1}andV : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}, V(t,\theta) = \{v^i_j(t,\theta)\}^n_{i,j=1}$ are measurable in $(t,\theta) \in \mathbb{R} \times \mathbb{R}$ and have bounded variation in θ on $[-h, 0]$ for every $t \in (0, \infty)$.

Now we consider only the case, when in the system (2.1) the derivatives are in Riemann-Liouville sense.

Denote by $D^{\alpha}_{a\pm}X(t) = ({}_{RL}D^{\alpha_1}_{a\pm}x_1(t), \dots, {}_{RL}D^{\alpha_n}_{a\pm}x_n(t))^T, D^{-\alpha}_{a\pm}X(t) = ({}_{RL}D^{-\alpha_1}_{a\pm}x_1(t), \dots, {}_{RL}D^{-\alpha_n}_{a\pm}x_n(t))^T, a \in \mathbb{R}$ the left/right side Riemann-Liouville fractional derivative and fractional integral respectively. We recall that $(D^{\alpha}_{a\pm}D^{-\alpha}_{a\pm}f)(t) = (D^0_{a\pm}f)(t) = f(t), a \in \mathbb{R}$. With $\mathfrak{C}$ we denote the Banach space of the initial vector functions $\mathfrak{C} = \{\Phi : [-h, 0] \to \mathbb{R}^n \mid \Phi(t) = (\phi_1(t), \dots, \phi_n(t))^T, \phi_k \in C([-h, 0], \mathbb{R}), 1 \le k \le n\}$.

In the initial condition (3.3) in [2] for the IVP when the derivatives are in Riemann-Liouville sense, the left side Riemann-Liouville fractional integral should be corrected to right side Riemann-Liouville fractional integral. The initial condition (3.3) Page 844 in [2] appears correctly below

$$D^{\alpha_k - 1}_{0-}x_k(t) = \phi_k(t), t \in [-h, 0], \Phi \in \mathfrak{C},\ k = 1, 2, \dots, n \tag{2.2}$$

This technical error reflects in several places below in [2] as follow:

- The initial condition (equation (3.5), Page 844 and the explanation in the next line should be corrected as follows:

$$X(t) = D^{1-\alpha}_{0-}\Phi(t), t \in [-h, 0], \Phi \in \mathfrak{C}$$

where $\alpha = (\alpha_1, ..., \alpha_n), 1 - \alpha = (1 - \alpha_1, \dots, 1 - \alpha_n)$ and $D^{\alpha-1}_{0-}$ and $D^{1-\alpha}_{0-}$ denote the right side Riemann-Liouville fractional integral and derivative respectively".

- In Page 846, Line 3: "$G_\alpha(t) = \Phi(t)$" should be replaced by "$D^{\alpha-1}_{0-}G(t) = \Phi(t)$" and the next line should be deleted.

- In Page 846, Line 12: "$D^{\alpha_k-1}_{0+}\mathfrak{R}_k g_k(t) = \phi_k(t)$" should be replaced by "$D^{\alpha_k-1}_{0-}\mathfrak{R}_k g_k(t) = \phi_k(t)$" and in the same page Line -7: "$(\mathfrak{R}G)_\alpha(t)|_{t=0} = \Phi(0)$" should be replaced by "$D^{\alpha-1}_{0+}\mathfrak{R}G(t)|_{t=0} = \Phi(0)$".

- Everywhere in Page 849 formula (5.5), Page 850 formula (5.7) and Page 850 Line 9 "$D^{1-\alpha}_{0+}\phi_j(s)$" should be replaced by "$D^{1-\alpha}_{0-}\phi_j(s)$".

We take the opportunity to corrected also the following technical errors in [2]:

- In Page 842, in the formulas, where the Riemann-Liouville and Caputo fractional derivatives are defined, the correct value of n should be given with $n = [\alpha] + 1$.

- In Page 844, Line 13, with $C^\alpha_M(C^\alpha_\infty)$ should be denoted the space of all $(1 - \alpha)$-continuous functions (instead of α-continuous as written).

-In Page 848 formula (4.7) on the first row the expression "$\phi_k(0)$ "should be replaced by

$$\text{"}\phi_k(0) - \sum_{j=1}^{n}\int_{-\tau}^{0}\phi_j(\theta)dv^j_k(0,\theta)\text{"}$$

Remark 2.1. It is simple to be seen that without these corrections the initial condition (3.3) as written in [2] is generally speaking not clear defined for $t < 0$, which implies that the IVP (3.1),(3.3) there for the Riemann-Liouville case can be treated as not correct defined.

It must be also noted that such initial conditions appear in works of other authors too.

Remark 2.2. We emphasize that after the corrections above, all obtained results in [2] for the Riemann-Liouville case remain true.

Remark 2.3. We must also point out, that all results obtained in [2] for the Caputo case are true without any corrections.

3. The $(1-\alpha)$-Vontinuity at Zero of the Solutions

It is well known, that any solution $X(t)$ of an IVP (even without delayed argument) with Riemann-Liouville fractional derivatives D^{α}_{0+} are not continuous at zero (when the initial point or the right side of the initial interval in the case with delayed argument, coincides with the initial point of the fractional derivative), except the case when the initial condition (regardless of its type) implies that $X(0) = 0$. But in the last case this IVP can be treated as IVP with fractional derivatives in Caputo sense, i.e. in this case speaking about IVP with Riemann-Liouville fractional derivatives is pointless. That is why for IVP with Riemann-Liouville fractional derivatives D^{α}_{0+} are introduced the $(1-\alpha)$-continuous at zero solutions, where the discontinuity at zero is "natural caused" by the Riemann-Liouville derivative. A function $f(t)$ is said to be $(1-\alpha)$-continuous at zero when $|t|^{1-\alpha} f(t)$ is continuous at zero.

The following lemma proved from Kilbas at all ([1], Lemma 3.2, page 151) helps us to define correct initial conditions to obtain $(1-\alpha)$-continuous at zero solutions of the IVP with left side fractional derivatives in Riemann-Liouville sense only for differential equations without delay and gives us explicit formula to calculate them.

For convenience we will formulate this lemma in the case when α is a real number.

Lemma 3.1. ([1]) *Let $a, b \in \mathbb{R}, a < b, \alpha \in (0,1)$ and let $x(t)$ be a Lebesgue measurable function on $[a, b]$.*

(a) If there exists almost everywhere a limit $\lim_{t\to a+}[(t-a)^{1-\alpha}x(t)] = c \in \mathbb{R}$,

then there also exists almost everywhere a limit $D^{\alpha-1}_{a+}[x(s)](a+) = \lim_{t\to a+} D^{\alpha-1}_{a+}[x(s)](t) = c\Gamma(\alpha)$

(b) If there exists almost everywhere a limit $\lim_{t\to a+} D^{\alpha-1}_{a+}[x(s)](t) = b \in \mathbb{R}$ and if in addition there exists the limit $\lim_{t\to a+}[(t-a)^{1-\alpha}x(t)]$,

then $\lim_{t\to a+}[(t-a)^{1-\alpha}x(t)] = \frac{b}{\Gamma(\alpha)}$.

It is clear that Lemma 3.1 can not be used for fractional differential equations with delayed argument with derivatives in Riemann-Liouville sense (see Remark 2.1). This is the reason and motivation to study this problem for right side derivatives in Riemann-Liouville sense. Now we will formulate and prove the next Lemma 3.2 - a "mirror" analogue of Lemma 3.1, concerning the right side Riemann-Liouville integral. Moreover the statement of Corollary 3.4 of the next Lemma 3.2 will clarify that under this type of initial conditions as condition (2.2) with right side Riemann-Liouville derivatives, the IVP will have a solution, which is $(1-\alpha)$-continuous at zero.

Lemma 3.2. *Let $a \in \mathbb{R}$ be an arbitrary point, $\alpha \in (0,1)$ and there exists $h \in (0,\infty)$ such that the function $x : [a-h, a] \to \mathbb{R}$ is Lebesgue measurable on $[a-h, a]$.*

(a) If there exists almost everywhere a limit $\lim_{t\to a-}[(a-t)^{1-\alpha}x(t)] = c \in \mathbb{R}$,

then there also exists almost everywhere a limit $D^{\alpha-1}_{a-}[x(s)](a-) = \lim_{t\to a-} D^{\alpha-1}_{a-}[x(s)](t) = c\Gamma(\alpha)$

(b) If there exists almost everywhere a limit $\lim_{t\to a-} D^{\alpha-1}_{a-}[x(s)](t) = c\Gamma(\alpha)$ and if in addition there exists $h^* \in (0, h]$, such that $x \in L_1(a-h^*, a)$,

then there exists the limit $\lim_{t\to a-}[(t-a)^{1-\alpha}x(t)] = c$.

Proof. (a) Assume that $\lim_{t\to a-}(a-t)^{1-\alpha}x(t) = c \in \mathbb{R}$. Since $(D^{\alpha-1}_{a-}1)(t) = \frac{(a-t)^{\alpha-1}}{\Gamma(\alpha)} \in L_1(a-h, a)$, then according Lemma 2.6 in [1] we have that $(D^{\alpha-1}_{-a}D^{1-\alpha}_{a-}1)(t) = 1$ and hence $(D^{\alpha-1}_{-a}(a-t)^{\alpha-1})(t) = \Gamma(\alpha)$.

Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta(\varepsilon) > 0$, such that for $|a-t| < \delta$ the inequality $|(a-t)^{1-\alpha}x(t) - c| < \frac{\varepsilon}{\Gamma(\alpha)}$

holds. Then we have that

$$|(D_{-a}^{\alpha-1}x)(t) - c\Gamma(\alpha)| = |(D_{-a}^{\alpha-1}x)(t) - c(D_{-a}^{\alpha-1}(a-t)^{\alpha-1})(t)| \le$$
$$\le \frac{1}{\Gamma(1-\alpha)}\int_t^a \frac{|x(s)-c(a-s)^{\alpha-1}|}{(s-t)^{\alpha}}ds \le \frac{1}{\Gamma(1-\alpha)}\int_t^a \frac{(a-s)^{\alpha-1}|(a-s)^{1-\alpha}x(s)-c|}{(s-t)^{\alpha}}ds \le$$
$$\le \frac{\varepsilon}{\Gamma(\alpha)}\frac{1}{\Gamma(1-\alpha)}\int_t^a \frac{(a-s)^{\alpha-1}}{(s-t)^{\alpha}}ds \le \frac{\varepsilon}{\Gamma(\alpha)}(D_{-a}^{\alpha-1}(a-t)^{1-\alpha})(t) = \varepsilon.$$

Thus $\lim_{t\to a-} D_{-a}^{\alpha-1}[x(s)](t) = c\Gamma(\alpha)$.

(b) Let $\lim_{t\to a-} D_{a-}^{\alpha-1}[x(s)](t) = c\Gamma(\alpha)$. Denote $D_{a-}^{\alpha-1}x(t) = \phi(t)$ for $t \in (a-h, a)$ and set $\phi(a) = c\Gamma(\alpha)$. Moreover there exists $h^* \in (0, h)$, such that the function $\phi(t)$ is bounded on the interval $[a-h^*, a]$ and continuous from left at $t = a$. Then we have

$$x(t) = D_{a-}^{1-\alpha}[\phi(s)](t) = D_{a-}^{1-\alpha}[\phi(s)-\phi(a)+\phi(a)](t) \quad = D_{a-}^{1-\alpha}[\phi(s)-\phi(a)](t) + D_{a-}^{1-\alpha}[\phi(a)](t) =$$
$$= \frac{\phi(a)(a-t)^{\alpha-1}}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_t^a \frac{\phi(s)-\phi(a)}{(s-t)^{1-\alpha}}ds \tag{3.1}$$

From (3.1) we obtain

$$(a-t)^{1-\alpha}x(t) = (a-t)^{1-\alpha}D_{a-}^{1-\alpha}[\phi(s)](t) = \frac{\phi(a)}{\Gamma(\alpha)} - \frac{(a-t)^{1-\alpha}}{\Gamma(\alpha)}\frac{d}{dt}\int_t^a \frac{\phi(s)-\phi(a)}{(s-t)^{1-\alpha}}ds = \frac{\phi(a)}{\Gamma(\alpha)} - \frac{(a-t)^{1-\alpha}}{\Gamma(\alpha)}\frac{d}{dt}J(t), \tag{3.2}$$

where $J(t) = \int_t^a \frac{\Psi(s)}{(s-t)^{1-\alpha}}ds$ and $\Psi(s) = \phi(s) - \phi(a), s \in [a-h_*, a]$. Obviously the function $\Psi(s)$ is bounded on the interval $[a-h^*, a]$ and also continuous from left at $t = a$. Since $\Psi(a) = 0$ for $t \in [a-h^*, a]$ we have

$$|J(t)| \le \int_t^a \frac{|\Psi(s)|}{(s-t)^{1-\alpha}}ds \le \frac{(a-t)^{\alpha}}{\alpha}\sup_{s\in[t,a]}|\Psi(s)| \tag{3.3}$$

and therefore $\lim_{t\to -a} J(t) = 0$. From other hand for $t \in [a-h^*, a)$ we have

$$\frac{(a-t)^{1-\alpha}}{\Gamma(\alpha)}\frac{d}{dt}J(t) = \frac{d}{dt}[\frac{(a-t)^{1-\alpha}}{\Gamma(\alpha)}J(t)] + J(t)\frac{(1-\alpha)(a-t)^{-\alpha}}{\Gamma(\alpha)} \tag{3.4}$$

From (3.3) it follows

$$|J(t)\frac{(1-\alpha)(a-t)^{-\alpha}}{\Gamma(\alpha)}| \le \frac{(1-\alpha)(a-t)^{-\alpha}(a-t)^{\alpha}}{\Gamma(1+\alpha)}\sup_{s\in[t,a]}|\Psi(s)|$$

and hence

$$\lim_{t\to a-}|J(t)\frac{(1-\alpha)(a-t)^{-\alpha}}{\Gamma(\alpha)}| = 0. \tag{3.5}$$

For the first addend in the right side of (3.4) we have

$$\begin{aligned}\frac{d}{dt}[\frac{(a-t)^{1-\alpha}}{\Gamma(\alpha)}J(t)]|_{t=-a} &= \lim_{t\to a-}|\frac{(a-t)^{1-\alpha}J(t)}{\Gamma(\alpha)}(t-a)^{-1}| = \lim_{t\to a-}|\frac{(a-t)^{-\alpha}}{\Gamma(\alpha)}J(t)| \le \\ &\le \lim_{t\to a-}|\frac{(a-t)^{-\alpha}}{\Gamma(\alpha)}\max_{s\in[t,a]}|\Psi(s)|\frac{(a-t)^{\alpha}}{\alpha}| \le \frac{1}{\Gamma(1+\alpha)}\lim_{t\to a-}\sup_{s\in[t,a]}|\Psi(s)| = 0\end{aligned} \tag{3.6}$$

From (3.4) ,(3.5) and (3.6) it follows that

$$\lim_{t\to a-} |\frac{(a-t)^{1-\alpha}}{\Gamma(\alpha)}\frac{d}{dt}J(t)| = 0. \tag{3.7}$$

Thus from (3.2) and(3.7) it follows that

$$\lim_{t\to a-} |(a-t)^{1-\alpha}x(t) = \lim_{t\to a-} |(a-t)^{1-\alpha}D^{1-\alpha}_{a-}[\phi(s)](t) = \frac{\phi(a)}{\Gamma(\alpha)} = c.$$

□

Remark 3.3. The proof of point (a) of Lemma 3.2 is analogical of the proof of point (a) of Lemma 3.1, but the proof of point (b) of Lemma 3.2 is fully different from the proof of point (b) of Lemma 3.1. This is because the additional condition in (b) of Lemma 3.1 "and if in addition there exists the limit $\lim_{t\to a+}[(t-a)^{1-\alpha}x(t)]$" is replaced in Lemma 3.2 point (b) with the condition "and if in addition there exists $h^* \in (0,h]$, such that $x \in L_1(a-h^*,a)$" , which is generally speaking a weaker condition. It is simple to be seen that if the limit $\lim_{t\to a+}[(t-a)^{1-\alpha}x(t)]$ exists , then there exists $h^* \in (0,h]$, such that $x \in L_1(a,a+h^*)$.

Corollary 3.4. *Let $X(t)$ be an arbitrary solution of the IVP* (2.1), (2.2)*, which is continuous on* $[-h,0)\cup(0,\infty)$*. Then we have that* $D^{\alpha-1}_{0+}X(0) = D^{\alpha-1}_{0-}X(0) = \Phi(0)$*, i.e. $X(t)$ is a* $(1-\alpha)$*-continuous at zero solution of the IVP* (2.1),(2.2)*.*

Proof. Let $X(t)$ be an arbitrary solution of the IVP (2.1), (2.2), which is continuous on $[-h,0)\cup(0,\infty)$. Then from Lemma 3.3 in [2] and Lemma 3.1, point (a) it follows that $D^{\alpha-1}_{0+}X(0) = \Phi(0)$. From the initial condition (2.2) and Lemma 3.2, point (b) we obtain that $D^{\alpha-1}_{0-}X(0) = \Phi(0)$, which completes the proof. □

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