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Anti-Invariant Riemanian Submersions from Nearly-K-Cosymplectic Manifolds

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Abstract. In this paper, we introduce anti-invariant Riemannian submersions from nearly-K-cosymplectic manifolds onto Riemannian manifolds. We study the integrability of horizontal distributions. And we investigate the necessary and sufficient condition for an anti-invariant Riemannian submersion to be totally geodesic and harmonic. Moreover, we give examples of anti-invariant Riemannian submersions such that characteristic vector field ξ is vertical or horizontal.

1. Introduction

Let π be a C^∞-submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N). Then according to the different conditions on the map $\pi : (M, g_M) \longrightarrow (N, g_N)$, we have the following submersions: Lorentzian submersion and semi-Riemannian submersion [7], slant submersion ([4, 19]), contact-complex submersion [8], almost h-slant submersion and h-slant submersion [16] quaternionic submersion [9], semi-invariant submersion [18], h-semi-invariant submersion [15], etc. In [17], Sahin introduced anti-invariant Riemannian submersions from almost hermitian manifolds onto Riemannian manifolds. Recently, C. Murathan and I. K*ü*peli Erken have investigated anti-invariant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds and from cosymplectic manifolds onto Riemannian manifolds ([11, 12]). Furthermore, anti-invariant Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds have also been studied in [2].

In this paper, we study anti-invariant Riemannian submersions from nearly-K-cosymplectic manifolds onto Riemannian manifolds. The paper is organized as follows: In section 2, we present some basic facts about Riemannian submersions. Nearly-K-cosymplectic manifolds are introduced in section 3. In section 4, we give the definition of anti-invariant Riemannian submersions and introduce anti-invariant Riemannian submersions from nearly-K-cosymplectic manifolds onto Riemannian manifolds. Moreover, we investigate the geometry of leaves of the distributions. In addition, we give two examples of anti-invariant Riemannian submersions such that characteristic vector field ξ is vertical and horizontal respectively.

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2. Preliminaries

Let (M, g_M) be an m-dimensional Riemannian manifold, Let (N, g_N) be an n-dimensional Riemannian manifold. A smooth surjective mapping $F : (M, g_M) \longrightarrow (N, g_N)$ is called a Riemannian submersion if the following conditions are satisfied:

- F has maximal rank ,
- The differential F_* preserves the lengths of horizontal vectors.

In ([13, 14]), O'Neil have defined the fundamental tensors of a submersion, which are (1, 2)-tensors on M and are given by the following formulas:

$$\mathcal{T}(E, F) = \mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{1}$$

$$\mathcal{A}(E, F) = \mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2}$$

for any vector field E and F on M. Here ∇ denotes the Levi-Civita connection of (M, g_M). Note that we denote the projection morphism on the distributions $kerF_*$ and $(kerF_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. And we have the following lemma ([13, 14]).

Lemma 2.1. *For any U, W vertical and X, Y horizontal vector fields, the tensor fields $\mathcal{T}$, $\mathcal{A}$ satisfy :*

$$\mathcal{T}(U, W) = \mathcal{T}(W, U), \tag{3}$$

$$\mathcal{A}(X, Y) = -\mathcal{A}(Y, X) = \frac{1}{2}\mathcal{V}[X, Y], \tag{4}$$

Obviously, $\mathcal{T}$ is vertical, i.e. $(\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E})$ And $\mathcal{A}$ is horizontal, i,e. $(\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E})$.

For each $q \in N$, $F^{-1}(q)$ is a submanifold of M of dimension dim$M$-dim$N$. The submanifolds $F^{-1}(q), q \in N$ are called fibers, and a vector field on M is vertical if it is always tangent to fibers, horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and F-related to a vector field X on N, i,e. $(\forall P \in M, F_* X_P = X_{*F(P)})$

From (2.1) and (2.2) we have the following basic equations:

$$\nabla_V W = \mathcal{T}_V W + \mathcal{V}\nabla_V W, \tag{5}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X, \tag{6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y. \tag{8}$$

where X, Y are horizontal vector fields and V, W are vertical vector fields.

From (2.1) and (2.2), we can also deduce the following formulas:

$$g(\mathcal{T}_E F, G) + g(\mathcal{T}_E G, F) = 0, \tag{9}$$

$$g(\mathcal{A}_E F, G) + g(\mathcal{A}_E G, F) = 0, \tag{10}$$

for any $E, F, G \in \Gamma(TM)$. Moreover, $\mathcal{T}_E, \mathcal{A}_E$ reverse the horizontal and the vertical distributions.

It is well-known that a Riemannia submersion has totally geodesic fiber if and only if $\mathcal{T} = 0$; Horizontal distribution $\mathcal{H}$ is totally geodesic if and only if $\mathcal{A} = 0$ (see [10]). Suppose $e_1, ..., e_{m-n}$ be an orthogonal frame of $\Gamma(kerF_*)$, then the horizontal vector field $H = \frac{1}{m-n}\sum_{i=1}^{m-n}\Gamma_{e_i} e_i$ is called the mean curvature vector field of the fiber. If $H = 0$ the Riemannian submersion is called minimal.

Mow, we recall the notion of harmonic maps between Riemannian manifolds. If $F : M \longrightarrow N$ is a smooth map between Riemannian manifolds. Then the differential F_* of F can be viewed a section of the bundle $Hom(TM, F^{-1}TN) \longrightarrow M$, where $F^{-1}TN$ is the pullback bundle which has fibres $(F^{-1}TN)_p = T_{F(p)}N, p \in M$.

$Hom(TM, F^{-1}TN)$ has a connection ∇ induced from the pullback connection and the Levi-Civita connection ∇^M. Then the second fundamental form of F is given by

$$(\nabla F_*)(X, Y) = \nabla^F_X F_*(Y) - F_*(\nabla^M_X Y), \tag{11}$$

for any $X, Y \in \Gamma(TM)$, where ∇^F is the pullback connection. It is known that the second fundamental form is symmetric. For a Riemannian submersion F, one can easily obtain:

$$(\nabla F_*)(X, Y) = 0, \tag{12}$$

for any $X, Y \in \Gamma((kerF_*)^\perp)$. A smooth map $F : M \longrightarrow N$ is said to be harmonic if $trace(\nabla F_*) = 0$. On the other hand, the tension field of F is the section $\tau(F)$ of $\Gamma(F^{-1}TN)$ defined by

$$\tau(F) = divF_* = \sum_{i=1}^{m} (\nabla F_*)(e_i, e_i), \tag{13}$$

where $\{e_1, \ldots, e_m\}$ is the orthonormal frame on M. Then it follows that F is harmonic if and only if $\tau(F) = 0$, (for details, see [1]).

3. Nearly-K-cosymplectic manifolds

A (2n+1)-dimensional C^∞ differential manifold M is said to have an almost contact structure or (ϕ, ξ, η)-structure if there exist on M a tensor field ϕ of type (1, 1), a vector field ξ and 1-form η satifying:

$$\eta(\xi) = 1, \phi^2 = -I + \eta \otimes \xi, \tag{14}$$

here I denote the identity tensor, ξ is called characteristic vector field. And we have the following proposition [3].

Proposition 3.1. *Suppose M^{2n+1} has a (ϕ, ξ, η)-structure. Then $\phi \cdot \xi = 0$ and $\eta \cdot \phi = 0$. Furthermore, the endomorphism ϕ has rank $2n$.*

M is said to have a (ϕ, ξ, η, g)-structure or an almost contact metric structure if the manifold M with a (ϕ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{15}$$

here X, Y are vector fields on M. Obviously, set $Y = \xi$, We get $\eta(X) = g(X, \xi)$.

We define an almost complex structure J on $M \times R$:

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}), \tag{16}$$

here $M \times R$ is considered as the product manifold, And M have an almost contact structure (ϕ, ξ, η), f denotes the C^∞-function on $M \times R$, X is tangent to M. Now we define a Riemannian metric on $M \times R$ by

$$h((X, f\frac{d}{dt}), (Y, g\frac{d}{dt})) = g(X, Y) + fg.$$

From [7], We have the following proposition:

Proposition 3.2. *M have an almost contact metric structure if and only if h is a Hermitian metric on $(M \times R, J)$; An (ϕ, ξ, η, g)-structure is called cosymplectic structure if and only if the structure (J, h) in $M \times R$ is Kählerian; An (ϕ, ξ, η, g)-structure is called a nearly-K-cosymplectic structure if (J, h) is nearly Kählerian.*

A manifold M endowed with a nearly-K-cosymplectic structure is called nearly-K-cosymplectic manifold. And from [7], M is nearly-K-cosymplectic manifold if and only if it satisfies the following formula:

$$(\nabla_X\phi)X = 0, \tag{17}$$

$$\nabla_X\xi = 0, \tag{18}$$

here X is tangent to M. Obviously, the first equation is equivalent to

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0. \tag{19}$$

It is obvious that a cosymplectic manifold is nearly-K-cosymplectic manifold. The canonical example of nearly-K-cosymplectic manifolds is given by the product $S^6 \times R$ nearly Kähler manifold $S^6(J, g)$ with real R line [5]. Now we introduce a nearly-K-cosymplectic manifold example.

Example 3.3. *Let L be a $(2n+1)$ dimensional Lie algebra, and choose a basis $\{e_0, e_1, \ldots, e_{2n}\}$ of L. The non-vanishing Lie bracket relations are following:*

$$[e_0, e_i] = -a_i e_{n+i},$$
$$[e_0, e_{n+i}] = a_i e_i.$$

for $i = 1, \ldots, n$, $a_1^2 + \ldots + a_n^2 > 0$.

Consider a connected Lie subgroup G of general linear group $GL(k, R)$, for certain k, such that the Lie algebra g of G is isomorphic with L. Let $\sigma : L \to g$ be the isomorphism. Let $\{E_0, E_1, \ldots, E_{2n}\}$ be the basis of G formed by left invariant vector fields on G such that $E_j = \sigma(e_j)$ for $j = 0, 1, \ldots, 2n$. Then, the non-vanishing Lie bracket relations on Lie algebra g are following:

$$[E_0, E_i] = -a_i E_{n+i},$$
$$[E_0, E_{n+i}] = a_i E_i.$$

Define a left invariant Riemannian metric g on G by $g(E_j, E_k) = \delta_{jk}$, $j, k = 0, 1, \ldots, 2n$. Then the Levi-Civita connection on G with respect to g is:

$$\nabla_{E_0} E_i = -a_i E_{n+i},$$
$$\nabla_{E_0} E_{n+i} = a_i E_i.$$

Define a 1-form η and $(1,1)$-tensor field ϕ on G by $\eta(E_j) = \delta_{0j}$, for $j = 0, 1, \ldots, 2n$, and $\phi E_0 = 0$, $\phi E_i = E_i$, $\phi E_{n+i} = -E_{n+i}$, for $i = 1, \ldots, n$. Set $\xi = E_0$. Then (ϕ, ξ, η, g) is an almost contact metric structure on G. Notice $\nabla\xi = 0$ and $(\nabla_{E_i}\phi)E_i = 0$, for $i = 0, 1, \ldots, 2n$, So (ϕ, ξ, η, g) is an nearly-K-cosymplectic structure. And

$$\begin{aligned}(\nabla_{E_0}\phi)E_i &= \nabla_{E_0}(\phi E_i) - \phi(\nabla_{E_0}E_i) \\ &= \nabla_{E_0}E_i + \phi(a_i E_{n+i}) \\ &= -2a_i E_{n+i} \neq 0.\end{aligned}$$

Thus G is not a non-trivial nearly-K-cosymplectic manifold. Moreover, there is a global system of coordinates (x_i, y_i, z), $1 \leq i \leq n$ on nearly-K-cosymplectic manifold G such that

$$E_i = \frac{\partial}{\partial x_i}, \quad E_{n+i} = \frac{\partial}{\partial y_i},$$
$$E_0 = \frac{\partial}{\partial z} + \sum_{j=1}^{n} a_j x_j \frac{\partial}{\partial y_j} - \sum_{j=1}^{n} a_j y_j \frac{\partial}{\partial x_j}.$$

4. Anti-invariant Riemannian Submersions

Definition 4.1. *Let F is a Riemannian submersion from nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to Riemannian manifold (N, g_N). We say F is an anti-invariant Riemannnian submersion if the following condition is satisfied:*

$$\phi(kerF_*) \subseteq (kerF_*)^{\perp}$$

We denote the complementary orthogonal distribution to $\phi(kerF_*)$ in $(kerF_*)^{\perp}$ by μ. Then it is easy to prove that μ is an invariant distribution of $(kerF_*)^{\perp}$, under the action of endomorphism ϕ.
Now we will give two examples.

Example 4.2. *Let G be a nearly-K-cosymplectic manifold with dimension seven as in Example 3.3. And set $a_1 = 1, a_2 = 0$, then $\xi = E_0 = \frac{\partial}{\partial z} + x_1E_3 - y_1E_1$. Let $N = \{(u, v, w)|u, v, w \in \mathbb{R}, u > 0\}$. The Riemannian metric tensor field g_N is defined by $g_N = \frac{1}{u}du^2 + dv^2 + dw^2$ on N.*

Let $F : G \to N$ be a map defined by $F(x_1, x_2, x_3, y_1, y_2, y_3, z) = (\frac{x_1^2+y_1^2}{2}, \frac{x_2+y_2}{\sqrt{2}}, \frac{x_3+y_3}{\sqrt{2}})$, $(x_1y_1 = 0)$. Then by direct calculation, we have

$$KerF_* = span\{V_1 = \frac{1}{\sqrt{2}}(E_2 - E_5), V_2 = \frac{1}{\sqrt{2}}(E_3 - E_6), V_3 = E_0 = \xi\}$$

and

$$(KerF_*)^{\perp} = span\{H_i = \frac{1}{\sqrt{2}}(E_{i+1} + E_{i+4}), i = 1, 2, H_3 = \frac{1}{\sqrt{2}}(E_1 + E_4), H_4 = \frac{1}{\sqrt{2}}(E_1 - E_4)\}$$

Obviously, F is a Riemannian submersion. Furthermore, $\phi V_1 = H_1$, $\phi V_2 = H_2$ imply that $(KerF_*)^{\perp} = \phi(KerF_*) \oplus span\{H_3, H_4\}$. Thus F is an anti-invariant Riemannian submersion such that ξ is vertical.

Example 4.3. *Let G be a nearly-K-cosymplectic manifold with dimension seven as in Example 3.3. And set $a_1 = a_2 = 0, a_3 = 1$, then $\xi = E_0 = \frac{\partial}{\partial z} + x_3E_6 - y_3E_3$. Let $N = \{(u_1, u_2, u_3, u_4, u_5)|u_3^2 + u_4^2 < 1, u_i \in \mathbb{R}, i = 1, 2, 3, 4, 5\}$. The Riemannian metric tensor field g_N is defined by $g_N = \sum_{i=1}^{4} du_i^2 + (1 - u_3^2 - u_4^2)du_5^2$ on N.*

Let $F : G \to N$ be a map defined by $F(x_1, x_2, x_3, y_1, y_2, y_3, z) = (\frac{x_1+y_1}{\sqrt{2}}, \frac{x_2+y_2}{\sqrt{2}}, \frac{x_3+y_3}{\sqrt{2}}, \frac{x_3-y_3}{\sqrt{2}}, z)$. Then by direct calculation, we have

$$KerF_* = span\{V_1 = \frac{1}{\sqrt{2}}(E_1 - E_4), V_2 = \frac{1}{\sqrt{2}}(E_2 - E_5)\}$$

and

$$(KerF_*)^{\perp} = span\{H_i = \frac{1}{\sqrt{2}}(E_i + E_{3+i}), i = 1, 2, 3, H_4 = \frac{1}{\sqrt{2}}(E_3 - E_6), H_5 = \xi\}$$

Obviously, F is a Riemannian submersion. Furthermore, $\phi V_1 = H_1, \phi V_2 = H_2$ imply that $\phi(KerF_*) \subseteq (KerF_*)^{\perp}$. And F is an anti-invariant Riemannian submersion such that ξ is horizontal.

4.1. *Anti-invariant submersions admitting vertical characteristic vector field*

In this subsection, we will discuss anti-invariant submersions from a nearly-K-cosymplectic manifold onto a Riemannian manifold such that the characteristic vector field ξ is vertical.

On the one hand, because of the invariance of μ under the action of ϕ, we can get

$$\phi X = BX + CX, \tag{20}$$

here $X \in \Gamma((kerF_*)^{\perp}), BX \in \Gamma(kerF_*), CX \in \Gamma(\mu)$. On the other hand, since F is a Riemannian submersion and $F_*((kerF_*)^{\perp}) = TN$, We get $g_N(F_*\phi V, F_*CX) = 0$, for $X \in \Gamma((kerF_*)^{\perp})$, $V \in \Gamma(kerF_*)$. And, we have

$$TN = F_*(\phi(kerF_*)) \oplus F_*(\mu). \tag{21}$$

By (3.14) and (4.20), it is easy to obtain the following proposition.

Proposition 4.4. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then we have*

$$BCX = 0, \quad \eta(BX) = 0, \quad C^2X = -X - \phi(BX),$$
$$C\phi V = 0, \quad C^3X + CX = 0, \quad B\phi V = -V + \eta(V)\xi,$$

where $X \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$.

Lemma 4.5. *Let ∇ be the connection of a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$. Then we have*

$$\nabla_X Y = -\phi\nabla_X\phi Y + \phi((\nabla_X\phi)Y), \tag{22}$$

$$\nabla_X Y + \nabla_Y X = -\phi\nabla_X\phi Y - \phi\nabla_Y\phi X, \tag{23}$$

here $X, Y \in \Gamma((kerF_*)^\perp)$.

Proof. Denote $g_M(,)$ by $\langle,\rangle$. Since ξ is vertical and $\nabla_X\xi = 0$, by (2.7), (2.8) and (2.10), we have:

$$\begin{aligned}\eta(\nabla_X Y) &= \langle \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \xi\rangle \\ &= \langle \mathcal{A}_X Y, \xi\rangle \\ &= -\langle Y, \mathcal{A}_X\xi\rangle \\ &= -\langle Y, \nabla_X\xi - \mathcal{V}\nabla_X\xi\rangle \\ &= 0.\end{aligned}$$

And

$$\nabla_X(\phi Y) = (\nabla_X\phi)Y + \phi(\nabla_X Y),$$

So

$$\phi(\nabla_X\phi Y) = \phi((\nabla_X\phi)Y) + \phi^2(\nabla_X Y) = \phi((\nabla_X\phi)Y) - \nabla_X Y + \eta(\nabla_X Y)\xi,$$

Thus we obtain (4.22). To see (4.23), By (3.19) and (4.22), we have

$$\begin{aligned}\nabla_X Y &= -\phi\nabla_X\phi Y - \phi((\nabla_Y\phi)X) \\ &= -\phi\nabla_X\phi Y - \phi(\nabla_Y\phi X) + \phi^2(\nabla_Y X) \\ &= -\phi\nabla_X\phi Y - \phi(\nabla_Y\phi X) - \nabla_Y X + \eta(\nabla_Y X)\xi.\end{aligned}$$

Hence, we get

$$\nabla_X Y + \nabla_Y X = -\phi\nabla_X\phi Y - \phi\nabla_Y\phi X.$$

□

Lemma 4.6. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{A}_X\xi = 0, \tag{24}$$

$$\mathcal{T}_U\xi = 0, \tag{25}$$

$$g_M(CX, \phi U) = 0, \tag{26}$$

$$g_M(\nabla_X CY, \phi U) = g_M(CY, \nabla_U\phi X) - 2g_M(CY, \phi(\nabla_U X)), \tag{27}$$

here $X, Y \in \Gamma((kerF_*)^\perp)$, $U \in \Gamma(kerF_*)$.

Proof. By (3.18) and (2.7), (2.5), and notice $\mathcal{A}_X, \mathcal{T}_U$ reverse the distributions, we get (4.24) and (4.25).

By (3.15) and (4.20), we have

$$\begin{aligned} g_M(CX, \phi U) &= g_M(\phi X - BX, \phi U) \\ &= g_M(X, U) - \eta(X)\eta(U) + g_M(\phi BX, \phi(\phi U)). \end{aligned}$$

Since $\phi BX \in \Gamma((kerF_*)^\perp), U, \xi \in \Gamma(kerF_*)$, we get (4.26).

Since $[X, U] \in \Gamma(kerF_*)$, We have $g_M(CY, \phi([X, U])) = 0$ and $g_M(CY, \phi\nabla_X U) = g_M(CY, \phi\nabla_U X)$. By (4.26) and (3.19), we obtain

$$\begin{aligned} g_M(\nabla_X CY, \phi U) &= -g_M(CY, \nabla_X(\phi U)) \\ &= g_M(CY, (\nabla_U\phi)X) - g_M(CY, \phi(\nabla_X U)) \\ &= g_M(CY, \nabla_U\phi X) - 2g_M(CY, \phi(\nabla_U X)). \end{aligned}$$

□

Next, we study the integrability of the horizontal distribution and then we investigate the geometry of leaves of $KerF_*$ and $(KerF_*)^\perp$.

Theorem 4.7. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold* $M(\phi, \xi, \eta, g_M)$ *to a Riemannian manifold* (N, g_N). *Then the following criteria are equivalent:*

1. $(kerF_*)^\perp$ *is integrable,*
2. $$\begin{aligned} g_N((\nabla F_*)(Y, BX), F_*\phi V) &= g_N((\nabla F_*)(X, BY), F_*\phi V) - g_M(CY, \nabla_V\phi X) \\ &\quad + g_M(CX, \nabla_V\phi Y) - 2g_M((\nabla_Y\phi)X, \phi V) \\ &\quad + 2g_M(CY, \phi(\nabla_V X)) - 2g_M(CX, \phi(\nabla_V Y)), \end{aligned}$$
3. $$\begin{aligned} g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, \phi V) &= -g_M(CY, \nabla_V\phi X) \\ &\quad + g_M(CX, \nabla_V\phi Y) - 2g_M((\nabla_Y\phi)X, \phi V) \\ &\quad + 2g_M(CY, \phi(\nabla_V X)) - 2g_M(CX, \phi(\nabla_V Y)), \end{aligned}$$

here $X, Y \in \Gamma((kerF_*)^\perp), V \in \Gamma(kerF_*)$.

Proof. For $X, Y \in \Gamma((kerF_*)^\perp), V \in \Gamma(kerF_*)$, we have

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X Y, V) - g_M(\nabla_Y X, V) \\ &= g_M(\phi\nabla_X Y, \phi V) - g_M(\phi\nabla_Y X, \phi V). \end{aligned}$$

Then from (4.20), we have

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X\phi Y, \phi V) - g_M(\nabla_Y\phi X, \phi V) + g_M((\nabla_Y\phi)X - (\nabla_X\phi)Y, \phi V) \\ &= g_M(\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - g_M(\nabla_Y BX, \phi V) \\ &\quad - g_M(\nabla_Y CX, \phi V) + 2g_M((\nabla_Y\phi)X, \phi V). \end{aligned}$$

Since F is a Riemannian submersion and $\phi V \in \Gamma((kerF_*)^\perp$, we get

$$g_M(\nabla_X BY, \phi V) = g_N(F_*\nabla_X BY, F_*\phi V), \ g_M(\nabla_Y BX, \phi V) = g_N(F_*\nabla_Y BX, F_*\phi V).$$

From (2.11) and (4.27), we get

$$\begin{aligned} g_M([X, Y], V) &= -g_N((\nabla F_*)(X, BY), F_*\phi V) + g_M(CY, \nabla_V\phi X) \\ &\quad - g_M(CX, \nabla_V\phi Y) + 2g_M((\nabla_Y\phi)X, \phi V) \\ &\quad - 2g_M(CY, \phi(\nabla_V X)) + 2g_M(CX, \phi(\nabla_V Y)) \\ &\quad + g_N((\nabla F_*)(Y, BX), F_*\phi V), \end{aligned}$$

which proves (1) ⇔ (2). On the other hand, by (2.11), we have

$$(\nabla F_*)(Y, BX) - (\nabla F_*)(X, BY) = -F_*(\nabla_Y BX - \nabla_X BY)$$

Then, according to (2.7), we get

$$(\nabla F_*)(Y, BX) - (\nabla F_*)(X, BY) = -F_*(\mathcal{A}_Y BX - \mathcal{A}_X BY).$$

Notice $\mathcal{A}_Y BX - \mathcal{A}_X BY \in \Gamma((kerF_*)^\perp$, this implies that (2) ⇔ (3). □

If $\phi(kerF_*) = (kerF_*)^\perp$, then we can get $C = 0$ and $TN = F_*(\phi(kerF_*))$. We have the following corollary.

Corollary 4.8. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N), and $\phi(kerF_*) = (kerF_*)^\perp$. Then the following assertions are equivalent to each other:*

1. $(kerF_*)^\perp$ *is integrable,*
2. $(\nabla F_*)(X, \phi Y) - (\nabla F_*)(Y, \phi X) = 2F_*((\nabla_Y \phi)X)$,
3. $\mathcal{A}_X \phi Y - \mathcal{A}_Y \phi X = -2\mathcal{H}((\nabla_Y \phi)X)$ *for* $X, Y \in \Gamma((kerF_*)^\perp)$.

Theorem 4.9. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) .Then the following criteria are equivalent:*

1. $(kerF_*)^\perp$ *defines a totally geodesic foliation on M,*
2. $g_M(\mathcal{A}_X BY, \phi V) = g_M((\nabla_X \phi)Y, \phi V) - g_M(CY, \nabla_V \phi X)) + 2g_M(CY, \phi(\nabla_V X))$,
3. $g_N(\nabla F_*(X, \phi Y), F_* \phi V) = -g_M((\nabla_X \phi)Y, \phi V) + g_M(CY, \nabla_V \phi X)) - 2g_M(CY, \phi(\nabla_V X))$,

for $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(kerF_*)$.

Proof. For $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(kerF_*)$, by (3.15), we get

$$g_M(\nabla_X Y, V) = g_M(\nabla_X \phi Y, \phi V) - g_M((\nabla_X \phi)Y, \phi V).$$

And using (2.7), (4.20) and (4.27), we have

$$\begin{aligned} g_M(\nabla_X Y, V) &= g_M(\mathcal{A}_X BY + \mathcal{V}\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - g_M((\nabla_X \phi)Y, \phi V) \\ &= g_M(\mathcal{A}_X BY, \phi V) + g_M(CY, \nabla_V \phi X) - g_M((\nabla_X \phi)Y, \phi V) - 2g_M(CY, \phi(\nabla_V X)). \end{aligned}$$

The above equation shows (1) ⇔ (2).

Since F is a Riemannian submersion and $\phi V \in \Gamma((kerF_*)^\perp$, we have

$$\begin{aligned} g_M(\mathcal{A}_X BY, \phi V) &= g_M(\nabla_X BY, \phi V) \\ &= g_N(F_* \nabla_X BY, F_* \phi V). \end{aligned}$$

Using (2.11) and (2.12), we get

$$\begin{aligned} g_M(\mathcal{A}_X BY, \phi V) &= -g_N((\nabla F_*)(X, BY), F_* \phi V) \\ &= -g_N((\nabla F_*)(X, \phi Y), F_* \phi V), \end{aligned}$$

which shows that (2) ⇔ (3). □

Corollary 4.10. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) such that $\phi(kerF_*) = (kerF_*)^\perp$. Then the following assertions are equivalent to each other:*

1. $(kerF_*)^\perp$ *defines a totally geodesic foliation on M,*
2. $\mathcal{A}_X \phi Y = \mathcal{H}((\nabla_X \phi)Y)$,
3. $(\nabla F_*)(X, \phi Y) = -F_*((\nabla_X \phi)Y)$ *for* $X, Y \in \Gamma((kerF_*)^\perp)$.

Theorem 4.11. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold* $M(\phi, \xi, \eta, g_M)$ *to a Riemannian manifold* (N, g_N)*. Then the following criteria are equivalent:*

1. $(kerF_*)$ *defines a totally geodesic foliation on M,*
2. $g_N((\nabla F_*)(V, \phi X), F_*\phi W) + g_M(\phi W, (\nabla_V \phi)X) = 0,$
3. $\mathcal{H}((\nabla_V \phi)X) - \mathcal{T}_V BX - \mathcal{A}_{CX} V \in \Gamma(\mu),$

here $X \in \Gamma((kerF_*)^\perp), V, W \in \Gamma(kerF_*)$.

Proof. For $X \in \Gamma((kerF_*)^\perp), V, W \in \Gamma(kerF_*)$, since $\xi \in \Gamma(kerF_*)$, by (2.6) and (4.25), it is easy to obtain $g_M(\nabla_V X, \xi) = 0$. Then by (3.15) and (2.6), we have

$$\begin{aligned} g_M(\nabla_V W, X) &= -g_M(W, \nabla_V X) \\ &= -g_M(\phi W, \phi \nabla_V X) \\ &= -g_M(\phi W, \mathcal{H}\nabla_V \phi X) + g_M(\phi W, (\nabla_V \phi)X). \end{aligned}$$

Since $[V, \phi X] \in \Gamma(kerF_*), \phi W \in \Gamma((kerF_*)^\perp)$, then $g_M([V, \phi X], \phi W) = 0$. By (2.11), we have

$$\begin{aligned} g_M(\nabla_V W, X) &= -g_N(F_*\phi W, F_*\mathcal{H}\nabla_V \phi X) + g_M(\phi W, (\nabla_V \phi)X) \\ &= g_N((\nabla F_*)(V, \phi X), F_*\phi W) + g_M(\phi W, (\nabla_V \phi)X), \end{aligned}$$

which shows (1) ⇔ (2). Next, by some calculation, we get

$$g_N((\nabla F_*)(V, \phi X), F_*\phi W) = -g_M(\phi W, \nabla_V \phi X).$$

Using (4.20), we have

$$g_N((\nabla F_*)(V, \phi X), F_*\phi W) = -g_M(\phi W, \nabla_V BX + \nabla_V CX).$$

Hence, we have

$$g_N((\nabla F_*)(V, \phi X), F_*\phi W) = -g_M(\phi W, \nabla_V BX + [V, CX] + \nabla_{CX} V).$$

Since $[V, CX] \in \Gamma((kerF_*)$, using (2.5) and (2.7), we get

$$g_N((\nabla F_*)(V, \phi X), F_*\phi W) = -g_M(\phi W, \mathcal{T}_V BX + \mathcal{A}_{CX} V).$$

This shows (2) ⇔ (3). □

Corollary 4.12. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold* $M(\phi, \xi, \eta, g_M)$ *to a Riemannian manifold* (N, g_N) *such that* $\phi(kerF_*) = (kerF_*)^\perp$*. Then the following assertions are equivalent to each other:*

1. $(kerF_*)$ *defines a totally geodesic foliation on M,*
2. $(\nabla F_*)(V, \phi X) + F_*((\nabla_V \phi)X) = 0,$
3. $\mathcal{H}((\nabla_V \phi)X) = \mathcal{T}_V \phi X$*, for* $X \in \Gamma((kerF_*)^\perp), V \in \Gamma(kerF_*)$.

We recall that a C^∞ map F between two Riemannian manifolds is called totally geodesic if $\nabla F_* = 0$. For an anti-invariant Riemannian submersion such that $\phi(kerF_*) = (kerF_*)^\perp$, we have the following theorem.

Theorem 4.13. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold* $M(\phi, \xi, \eta, g_M)$ *to a Riemannian manifold* (N, g_N) *such that* $\phi(kerF_*) = (kerF_*)^\perp$*. Then F is a totally geodesic map if and only if*

$$\phi\mathcal{T}_W \phi V + \mathcal{H}((\nabla_W \phi)\phi V) = 0, \tag{28}$$

and

$$\phi\mathcal{A}_X \phi W + \mathcal{H}((\nabla_X \phi)\phi W) = 0, \tag{29}$$

for $V, W \in \Gamma(kerF_*), X \in \Gamma((kerF_*)^\perp)$.

Proof. For $V, W \in \Gamma(kerF_*)$, $X \in \Gamma((kerF_*)^\perp)$, since $\phi(kerF_*) = (kerF_*)^\perp$ and ξ is vertical, by (2.6) and (3.18), it is easy to obtain

$$(\nabla F_*)(W, V) = F_*(\phi \mathcal{T}_W \phi V) + F_*((\nabla_W \phi)\phi V). \tag{30}$$

One the other hand, by (3.14) and (2.11), we have

$$F_*(\phi \nabla_X \phi W) = (\nabla F_*)(X, W) - F_*((\nabla_X \phi)\phi W).$$

Then, by (2.8), we get

$$(\nabla F_*)(X, W) = F_*((\nabla_X \phi)\phi W) + F_*(\phi \mathcal{A}_X \phi W). \tag{31}$$

Hence, proof comes from (2.12) (4.30) and (4.31). □

Finally, we study the necessary and sufficient condition for an anti-invariant Riemannian submersion such that $\phi(kerF_*) = (kerF_*)^\perp$ to be harmonic.

Theorem 4.14. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) such that $\phi(kerF_*) = (kerF_*)^\perp$. Then F is harmonic if and only if trace$(\phi \mathcal{T}_V) = 0$, for $V \in \Gamma(kerF_*)$.*

Proof. From [6] we know that F is harmonic if and only if F has minimal fibres. Thus F is harmonic if and only if $\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = 0$, where k denotes the dimension of $kerF_*$. On the other hand, by (3,15), we get

$$\mathcal{H}(\phi \nabla_V W) = \phi(\mathcal{V} \nabla_V W), \tag{32}$$

for $V, W \in \Gamma(kerF_*)$. By (4.32) and some calculations, we obtain

$$\mathcal{T}_V \phi W - \phi \mathcal{T}_V W = \mathcal{V}((\nabla_V \phi)W).$$

Then, by (3.17), we have

$$\begin{aligned} \sum_{i=1}^{k} g_M(\mathcal{T}_{e_i} \phi e_i, V) &= \sum_{i=1}^{k} g_M(\phi \mathcal{T}_{e_i} e_i, V) + \sum_{i=1}^{k} g_M((\nabla_{e_i} \phi) e_i, V) \\ &= -\sum_{i=1}^{k} g_M(\mathcal{T}_{e_i} e_i, \phi V) \end{aligned}$$

for any $V \in \Gamma(kerF_*)$. And by (2.9), we get

$$\sum_{i=1}^{k} g_M(\phi e_i, \mathcal{T}_{e_i} V) = \sum_{i=1}^{k} g_M(\mathcal{T}_{e_i} e_i, \phi V).$$

By (2.3) and (3.14), we have

$$\sum_{i=1}^{k} g_M(e_i, \phi \mathcal{T}_V e_i) = -\sum_{i=1}^{k} g_M(\mathcal{T}_{e_i} e_i, \phi V).$$

This completes the proof. □

4.2. *Anti-invariant submersions admitting horizontal characteristic vector field.*

In this subsection,we will discuss anti-invariant submersions from a nearly-K-cosymplectic manifold onto a Riemannian manifold such that the characteristic vector field ξ is horizontal. Since $\phi\mu \subseteq \mu$, by (3.14), it is easy to obtain: $\mu = \phi\mu \oplus \{\xi\}$. For any horizontal vector field X, we write

$$\phi X = BX + CX, \tag{33}$$

where $BX \in \Gamma(kerF_*), CX \in \Gamma(\mu)$.

Now we suppose that X is horizontal and V is vertical vector field. From $g_M(\phi V, CX) = 0$, we can obtain $g_N(F_*\phi V, F_*CX) = 0$, which implies that

$$TN = F_*(\phi(kerF_*)) \oplus F_*(\mu). \tag{34}$$

By (3.14) and (4.33), we have the following proposition.

Proposition 4.15. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then we have*

$$BCX = 0, \quad C^2X = \phi^2X - \phi(BX), \quad C\phi V = 0,$$
$$C^3X + CX = 0, \quad B\phi V = -V,$$

where $X \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$.

By (3.14), it is easy to get

$$\nabla_X Y = -\phi(\nabla_X \phi Y) + \phi((\nabla_X \phi)Y) + \eta(\nabla_X Y)\xi, \ \forall X, Y \in \Gamma((kerF_*)^\perp) \tag{35}$$

Lemma 4.16. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{A}_X\xi = 0, \tag{36}$$

$$\mathcal{T}_U\xi = 0, \tag{37}$$

$$g_M(CX, \phi U) = 0, \tag{38}$$

$$g_M(\nabla_X CY, \phi U) = g_M(CY, (\nabla_U\phi)X) - g_M(CY, \phi\mathcal{A}_X U), \tag{39}$$

where $X, Y \in \Gamma((kerF_*)^\perp), U \in \Gamma(kerF_*)$.

Proof. Assume that $X, Y \in \Gamma((kerF_*)^\perp), U \in \Gamma(kerF_*)$. By (2.8), (2.6) and (3.18), we obtain (4.36) and (4.37).

Using (3.15) and (4.33), $\eta \cdot \phi = 0$, since $\phi BX, \xi \in \Gamma((kerF_*)^\perp), U \in \Gamma(kerF_*)$, we have

$$\begin{aligned} g_M(CX, \phi U) &= g_M(\phi X - BX, \phi U) \\ &= g_M(X, U) - \eta(X)\eta(U) + g_M(\phi BX, \phi(\phi U)) \\ &= g_M(\phi BX, -U + \eta(U)\xi) \\ &= 0 \end{aligned}$$

For $X, Y \in \Gamma((kerF_*)^\perp), U \in \Gamma(kerF_*)$, by (3.19) we have

$$\begin{aligned} g_M(\nabla_X CY, \phi U) &= -g_M(CY, \nabla_X(\phi U)) \\ &= -g_M(CY, (\nabla_X\phi)U + \phi(\nabla_X U)) \\ &= g_M(CY, (\nabla_U\phi)X) - g_M(CY, \phi(\nabla_X U)). \end{aligned}$$

Since $\phi(\mathcal{V}\nabla_X U) \in \phi(kerF_*)$, by (2.7), we have

$$\begin{aligned} g_M(\nabla_X CY, \phi U) &= g_M(CY, (\nabla_U\phi)X) - g_M(CY, \phi\mathcal{A}_X U) - g_M(CY, \phi(\mathcal{V}\nabla_X U)) \\ &= g_M(CY, (\nabla_U\phi)X) - g_M(CY, \phi\mathcal{A}_X U). \end{aligned}$$

□

Next, we study the integrability of the horizontal distribution and then we investigate the geometry of leaves of $KerF_*$ and $(KerF_*)^\perp$.

Theorem 4.17. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other:*

1. $(kerF_*)^\perp$ *is integrable,*
2. $$\begin{aligned} g_N((\nabla F_*)(Y, BX), F_*\phi V) &= g_N((\nabla F_*)(X, BY), F_*\phi V) - g_M(CY, (\nabla_V\phi)X) \\ &+ g_M(CX, (\nabla_V\phi)Y) - 2g_M((\nabla_Y\phi)X, \phi V) \\ &+ g_M(CY, \phi\mathcal{A}_X V) - g_M(CX, \phi\mathcal{A}_Y V), \end{aligned}$$
3. $$\begin{aligned} g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, \phi V) &= -g_M(CY, (\nabla_V\phi)X) \\ &+ g_M(CX, (\nabla_V\phi)Y) - 2g_M((\nabla_Y\phi)X, \phi V) \\ &+ g_M(CY, \phi\mathcal{A}_X V) - g_M(CX, \phi\mathcal{A}_Y V), \end{aligned}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(kerF_*)$.

Proof. For $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(kerF_*)$, we have

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X Y, V) - g_M(\nabla_Y X, V) \\ &= g_M(\phi\nabla_X Y, \phi V) - g_M(\phi\nabla_Y X, \phi V). \end{aligned}$$

Then from (4.20), we have

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X\phi Y, \phi V) - g_M(\nabla_Y\phi X, \phi V) + g_M((\nabla_Y\phi)X - (\nabla_X\phi)Y, \phi V) \\ &= g_M(\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - g_M(\nabla_Y BX, \phi V) \\ &- g_M(\nabla_Y CX, \phi V) + 2g_M((\nabla_Y\phi)X, \phi V). \end{aligned}$$

Since F is a Riemannian submersion and $\phi V \in \Gamma((kerF_*)^\perp$, we get

$$g_M(\nabla_X BY, \phi V) = g_N(F_*\nabla_X BY, F_*\phi V), \;\; g_M(\nabla_Y BX, \phi V) = g_N(F_*\nabla_Y BX, F_*\phi V).$$

From (2.11) and (4.39), we get

$$\begin{aligned} g_M([X, Y], V) &= -g_N((\nabla F_*)(BY, X), F_*\phi V) + g_M(CY, (\nabla_V\phi)X) \\ &- g_M(CX, (\nabla_V\phi)Y) + 2g_M((\nabla_Y\phi)X, \phi V) \\ &- g_M(CY, \phi\mathcal{A}_X V) + g_M(CX, \phi\mathcal{A}_Y V) \\ &+ g_N((\nabla F_*)(BX, Y), F_*\phi V) \end{aligned}$$

which proves (1) $\Leftrightarrow$ (2). On the other hand, by (2.11), we have

$$(\nabla F_*)(Y, BX) - (\nabla F_*)(X, BY) = -F_*(\nabla_Y BX - \nabla_X BY)$$

Then, according to (2.7), we get

$$(\nabla F_*)(Y, BX) - (\nabla F_*)(X, BY) = -F_*(\mathcal{A}_Y BX - \mathcal{A}_X BY).$$

Notice $\mathcal{A}_Y BX - \mathcal{A}_X BY \in \Gamma((kerF_*)^\perp$, this implies that (2) $\Leftrightarrow$ (3). □

Remark 4.18. *If $(kerF_*)^\perp = \phi(kerF_*) \oplus \{\xi\}$, then we can get $CX = 0$ for $X \in \Gamma((kerF_*)^\perp)$.*

Corollary 4.19. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) such that $(kerF_*)^\perp = \phi(kerF_*) \oplus \{\xi\}$. Then the following assertions are equivalent to each other:*

1. $(kerF_*)^\perp$ *is integrable,*
2. $(\nabla F_*)(Y, \phi X) - (\nabla F_*)(X, \phi Y) = -2F_*((\nabla_Y \phi)X)$,
3. $\mathcal{A}_X \phi Y - \mathcal{A}_Y \phi X = -2\mathcal{H}((\nabla_Y \phi)X)$,

for $X, Y \in \Gamma((kerF_*)^\perp)$.

Theorem 4.20. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold* $M(\phi, \xi, \eta, g_M)$ *to a Riemannian manifold* (N, g_N)*. Then the following criteria are equivalent:*

1. $(kerF_*)^\perp$ *defines a totally geodesic foliation on M,*
2. $g_M(\mathcal{A}_X BY, \phi V) = g_M((\nabla_X \phi)Y, \phi V) - g_M(CY, (\nabla_V \phi)X)) + g_M(CY, \phi \mathcal{A}_X V)$,
3. $g_N(\nabla F_*(X, \phi Y), F_* \phi V) = -g_M((\nabla_X \phi)Y, \phi V) + g_M(CY, (\nabla_V \phi)X)) - g_M(CY, \phi \mathcal{A}_X V)$,

for $X, Y \in \Gamma((kerF_*)^\perp)$; $V \in \Gamma(kerF_*)$.

Proof. For $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(kerF_*)$, by (3.15), we get

$$g_M(\nabla_X Y, V) = g_M(\nabla_X \phi Y, \phi V) - g_M((\nabla_X \phi)Y, \phi V).$$

And using (2.7), (4.33) and (4.39), we have

$$\begin{aligned} g_M(\nabla_X Y, V) &= g_M(\mathcal{A}_X BY + \mathcal{V}\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - g_M((\nabla_X \phi)Y, \phi V) \\ &= g_M(\mathcal{A}_X BY, \phi V) + g_M(CY, (\nabla_V \phi)X) - g_M((\nabla_X \phi)Y, \phi V) - g_M(CY, \phi \mathcal{A}_X V). \end{aligned}$$

The above equation shows (1) ⇔ (2). Since F is a Riemannian submersion and $\phi V \in \Gamma((kerF_*)^\perp$, we have

$$\begin{aligned} g_M(\mathcal{A}_X BY, \phi V) &= g_M(\nabla_X BY, \phi V) \\ &= g_N(F_* \nabla_X BY, F_* \phi V). \end{aligned}$$

Using (2.11) and (2.12), we get

$$\begin{aligned} g_M(\mathcal{A}_X BY, \phi V) &= -g_N((\nabla F_*)(X, BY), F_* \phi V) \\ &= -g_N((\nabla F_*)(X, \phi Y), F_* \phi V), \end{aligned}$$

which shows that (2) ⇔ (3). □

Corollary 4.21. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold* $M(\phi, \xi, \eta, g_M)$ *to a Riemannian manifold* (N, g_N) *such that* $(kerF_*)^\perp = \phi(kerF_*) \oplus \{\xi\}$*. Then the following assertions are equivalent to each other:*

1. $(kerF_*)^\perp$ *defines a totally geodesic foliation on M,*
2. $\mathcal{A}_X \phi Y = \mathcal{H}((\nabla_X \phi)Y)$,
3. $(\nabla F_*)(X, \phi Y) = -F_*((\nabla_X \phi)Y)$,

for $X, Y \in \Gamma((kerF_*)^\perp)$.

For the vertical distribution $kerF_*$, we have:

Theorem 4.22. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold* $M(\phi, \xi, \eta, g_M)$ *to a Riemannian manifold* (N, g_N)*. Then the following assertions are equivalent to each other:*

1. $(kerF_*)$ *defines a totally geodesic foliation on M,*
2. $g_N((\nabla F_*)(V, \phi X), F_* \phi W) + g_M(\phi W, (\nabla_V \phi)X) = 0$,
3. $\mathcal{H}((\nabla_V \phi)X) - \mathcal{T}_V BX - \mathcal{A}_{CX} V \in \Gamma(\mu)$,

for $X \in \Gamma((kerF_*)^\perp)$; $V, W \in \Gamma(kerF_*)$.

Proof. For $X \in \Gamma((kerF_*)^\perp)$, $V, W \in \Gamma(kerF_*)$, since $\xi \in \Gamma((kerF_*)^\perp)$, we have $g_M(W, \xi) = 0$. Then by $g_M(W, X) = 0$, we have $g_M(\nabla_V W, X) = -g_M(W, \nabla_V X)$. By (2.6) and (3.15), we obtain

$$\begin{aligned} g_M(\nabla_V W, X) &= -g_M(W, \nabla_V X) \\ &= -g_M(\phi W, \phi \nabla_V X) \\ &= -g_M(\phi W, \mathcal{H}\nabla_V \phi X) + g_M(\phi W, (\nabla_V \phi)X). \end{aligned}$$

Since $[V, \phi X] \in \Gamma(kerF_*)$, $\phi W \in \Gamma((kerF_*)^\perp)$, then $g_M([V, \phi X], \phi W) = 0$. By (2.11), we have

$$\begin{aligned} g_M(\nabla_V W, X) &= -g_N(F_*\phi W, F_*\mathcal{H}\nabla_V \phi X) + g_M(\phi W, (\nabla_V \phi)X) \\ &= g_N((\nabla F_*)(V, \phi X), F_*\phi W) + g_M(\phi W, (\nabla_V \phi)X), \end{aligned}$$

which shows (1) ⇔ (2). Next, by some calculation, we get

$$g_N((\nabla F_*)(V, \phi X), F_*\phi W) = -g_M(\phi W, \nabla_V \phi X).$$

Using (4.33), we have

$$g_N((\nabla F_*)(V, \phi X), F_*\phi W) = -g_M(\phi W, \nabla_V BX + \nabla_V CX).$$

Hence, we have

$$g_N((\nabla F_*)(V, \phi X), F_*\phi W) = -g_M(\phi W, \nabla_V BX + [V, CX] + \nabla_{CX} V).$$

Since $[V, CX] \in \Gamma((kerF_*)$, using (2.5) and (2.7), we get

$$g_N((\nabla F_*)(V, \phi X), F_*\phi W) = -g_M(\phi W, \mathcal{T}_V BX + \mathcal{A}_{CX} V).$$

This shows (2) ⇔ (3). □

Corollary 4.23. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) such that $(kerF_*)^\perp = \phi(kerF_*) \oplus \{\xi\}$. Then the following assertions are equivalent to each other:*

1. *$(kerF_*)$ defines a totally geodesic foliation on M,*
2. $(\nabla F_*)(V, \phi X) + F_*((\nabla_V \phi)X) = 0$,
3. $\mathcal{H}((\nabla_V \phi)X) = \mathcal{T}_V \phi X$,

for $X \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(kerF_*)$.

Theorem 4.24. *Let F be an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) such that $(kerF_*)^\perp = \phi(kerF_*) \oplus \{\xi\}$. Then F is a totally geodesic map if and only if*

$$\phi \mathcal{T}_W \phi V + \mathcal{H}((\nabla_W \phi)\phi V) = 0, \tag{40}$$

and

$$\phi \mathcal{A}_X \phi W + \mathcal{H}((\nabla_X \phi)\phi W) = 0, \tag{41}$$

for $V, W \in \Gamma(kerF_*)$, $X \in \Gamma((kerF_*)^\perp)$.

Proof. $\forall X \in \Gamma((kerF_*)^\perp)$, put $X = \phi X_1 + a\xi$, $X_1 \in kerF_*$, $a \in R$, then we have

$$F_*\phi(X) = F_*\phi(\phi X_1 + a\xi) = F_*(X_1 - \eta(X_1)\xi) = 0.$$

Thus

$$F_*\phi(X) = 0, \forall X \in \Gamma((kerF_*)^\perp). \tag{42}$$

For $V, W \in \Gamma(kerF_*)$, $X \in \Gamma((kerF_*)^\perp)$, by (2.6) and (3.18), it is easy to obtain

$$(\nabla F_*)(W, V) = F_*(\phi\mathcal{T}_W\phi V) + F_*((\nabla_W\phi)\phi V). \tag{43}$$

One the other hand, by (3.14) and (2.11), we have

$$F_*(\phi\nabla_X\phi W) = (\nabla F_*)(X, W) - F_*((\nabla_X\phi)\phi W).$$

Then, by (2.8) and (4.42), we get

$$(\nabla F_*)(X, W) = F_*((\nabla_X\phi)\phi W) + F_*(\phi\mathcal{A}_X\phi W). \tag{44}$$

Hence, proof comes from (2.12) (4.43) and (4.44). □

Finally,we study the necessary and sufficient condition for anti-invariant Riemannian submersion such that $(kerF_*)^\perp = \phi(kerF_*) \oplus \{\xi\}$ to be harmonic.

Theorem 4.25. *Let F is an anti-invariant Riemannian submersion from a nearly-K-cosymplectic manifold $M(\phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) such that $(kerF_*)^\perp = \phi(kerF_*)\oplus\{\xi\}$. Then F is harmonic if and only if trace$(\phi\mathcal{T}_V) = 0$, for $V \in \Gamma(kerF_*)$.*

Proof. From [6] we know that F is harmonic if and only if F has minimal fibres. Thus F is harmonic if and only if $\sum_{i=1}^{k} \mathcal{T}_{e_i}e_i = 0$, where k denotes the dimension of $kerF_*$. On the other hand, by (3,15), we get

$$\mathcal{H}(\phi\nabla_V W) = \phi(\mathcal{V}\nabla_V W), \tag{45}$$

for $V, W \in \Gamma(kerF_*)$. By (4.45) and some calculations, we obtain

$$\mathcal{T}_V\phi W - \phi\mathcal{T}_V W = \mathcal{V}((\nabla_V\phi)W).$$

Then, by (3.17), we have

$$\begin{aligned} \sum_{i=1}^{k} g_M(\mathcal{T}_{e_i}\phi e_i, V) &= \sum_{i=1}^{k} g_M(\phi\mathcal{T}_{e_i}e_i, V) + \sum_{i=1}^{k} g_M((\nabla_{e_i}\phi)e_i, V) \\ &= -\sum_{i=1}^{k} g_M(\mathcal{T}_{e_i}e_i, \phi V) \end{aligned}$$

for any $V \in \Gamma(kerF_*)$. And by (2.9), we get

$$\sum_{i=1}^{k} g_M(\phi e_i, \mathcal{T}_{e_i}V) = \sum_{i=1}^{k} g_M(\mathcal{T}_{e_i}e_i, \phi V).$$

By (2.3) and (3.14), we have

$$\sum_{i=1}^{k} g_M(e_i, \phi\mathcal{T}_V e_i) = -\sum_{i=1}^{k} g_M(\mathcal{T}_{e_i}e_i, \phi V).$$

This completes the proof. □

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