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Note on the Uniqueness Holomorphic Function on the Unit Disk

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Abstract. Let f be an holomorphic function the unit disk to itself. We provide conditions on the local behavior of f along boundary near a finite set of the boundary points that requires f to be a finite Blaschke product.

1. Introduction

In 1994, Daniel M. Burns and Steven G. Krantz ([1]) proved that if the holomorphic function $f : D \to D$ satisfies the condition

$$f(z) = z + O\left((z-1)^4\right) \qquad z \to 1,\ z \in D, \tag{1.1}$$

then $f(z) \equiv z$ on the unit disk.

The example

$$f(z) = z + \frac{1}{10}(z-1)^3$$

shows that the exponent 4 in (1.1) can not be replaced by 3. In fact, the proof shows that $O\left((z-1)^4\right)$ can be replaced by $o\left((z-1)^3\right)$.

In 2001, Dov Chelst ([2]), in turn, established the following generalization of this result.

Theorem 1.1. *Let $f : D \to D$ be a holomorphic function from the disk to itself. In addition, let $\phi : D \to D$ be a finite Blaschke product which equals $\tau \in \partial D$ on a finite set $A_f \subset \partial D$. If*

(i) for a given $\gamma_0 \in A_f$,

$$f(z) = \phi(z) + o\left((z-\gamma_0)^3\right), \text{ as } z \to \gamma_0,$$

(ii) for all $\gamma \in A_f - \{\gamma_0\}$,

$$f(z) = \phi(z) + O\left((z-\gamma)^{k_\gamma}\right), \text{ for some } k_\gamma \geq 2 \text{ as } z \to \gamma,$$

then $f(z) \equiv \phi(z)$ on the disk.

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It was shown that the above condition $k_\gamma \geq 2$ can not be replaced by $k_\gamma \geq 1$.

In ([3]) and ([4]), this problem was generalized in the following aspects:

a) more general majorant was taken instead of the usual power majorant in (*i*) and (*ii*);

b) in (*i*) and (*ii*), the conditions $z \to \gamma$, which usually stated approaching from inside of the disk before, were taken as the behavior of the function f along the boundary.

In 2015, M.Mateljević proved Theorem 1 in ([5]), where instead of Blaschke product was taken inner function and in (*i*) and (*ii*), the behavior of the function f along the boundary was considered.

Recently similar problems were investigated in ([6]) and ([7]). For more detail literature and the other types of the results, we refer to ([8]), ([9]), ([5]), ([10]) and references therein.

In the present study, we refined the results in ([4]). In particular, from our proofs it is followed that $O(z-\gamma)^{k_\gamma}$ in Theorem 1.1 can be replaced by $o(z-\gamma)$.

We propose the following assertion for the proofs of our results.

(A) Let $u = u(z)$ be a positive harmonic function on the open disk $\mathbb{U}(z, r_0)$, $r_0 > 0$. Suppose that for $\theta_0 \in [0, 2\pi)$, $\lim\limits_{r \to r_0} u(re^{i\theta_0}) = 0$ is satisfied. Then

$$\liminf_{r \to r_0} \frac{u(re^{i\theta_0})}{r_0 - r} > 0.$$

This assertion follows from Harnack inequality. For more general results and related estimates, see also ([11, Theorem 1.1]), ([12]), ([13]).

(B) Let the function u be a subharmonic function in the unit disk, E is the finite subset of the unit circle ∂D such that

$$\limsup_{z \to \varsigma,\ z \in D} u(z) \leq 0, \quad \forall \varsigma \in \partial D \backslash E,$$

and

$$u(z) = o(|\varsigma - z|^{-1}) \ \textit{as}\ z \to \varsigma \text{ for each } \ \varsigma \in E,$$

then $u(z) \leq 0$ for all $z \in D$.

The basic exposition for this version of Phragmen-Lindelöf Princible can be found in ([14, pp. 79-90]), ([15, pp. 176-186]) and ([16, Chapter 4, section 8 and Chapter 5, section 9]).

Let $\mathfrak{M}$ be a class of functions $\mu : (0, +\infty) \to (0, +\infty)$ for each of which $\log \mu(x)$ is concave with respect to $\log x$. For each function $\mu \in \mathfrak{M}$ the limit

$$\mu_0 = \lim_{x \to 0} \frac{\log \mu(x)}{\log x}$$

exists, and $-\infty < \mu_0 \leq +\infty$. Here, the function $\mu \in \mathfrak{M}$ is called bilogaritmic concave majorant ([17]).

$\mathfrak{N}$ be the class of sets with zero inner capacity ([18, p.210]).

2. Main Results

Let $d(z, A)$ be the distance from the point z to the set A.

Theorem 2.1. *Let $\phi : D \to D$ be a finite Blaschke product which equals $\tau \in \partial D$ on a finite set $A_f \subset \partial D$ and $f : D \to D$ be a holomorphic function that is continuous on $\overline{D} \cap \left\{z : d(z, A_f) < \delta_0\right\}$ for some δ_0, μ^1, $\mu^2 \in \mathfrak{M}$, $\mu_0^1 > 3$, $\mu_0^2 > 1$. Suppose that the following conditions are satisfied*

(*i*) *for a given $\gamma_0 \in A_f$*

$$f(z) = \phi(z) + O(\mu^1(\left|z - \gamma_0\right|)),\ z \in \partial D,\ z \to \gamma_0,$$

(ii) for all $\gamma \in A_f \setminus \{\gamma_0\}$,

$$f(z) = \phi(z) + O(\mu^2(|z-\gamma|)),\ z \in \partial D,\ z \to \gamma.$$

Then $f(z) \equiv \phi(z)$ *on* D.

Following result is generalization of Theorem 2.1.

Theorem 2.2. *Let* $\phi : D \to D$ *be a finite Blaschke product which equals* $\tau \in \partial D$ *on a finite set* $A_f \subset \partial D$ *and* $f : D \to D$ *be a holomorphic function,* $Q \in \mathfrak{N}$, μ^1, $\mu^2 \in \mathfrak{M}$, $\mu_0^1 > 3$, $\mu_0^2 > 1$. *Let the following conditions are satisfied*

(i) for a given $\gamma_0 \in A_f$,

$$\limsup_{z\to\zeta, z\in D} |f(z) - \phi(z)| = O(\mu^1(|\zeta - \gamma_0|)),\ \zeta \in \partial D \setminus Q,\ \zeta \to \gamma_0, \tag{2.1}$$

(ii) for all $\gamma \in A_f \setminus \{\gamma_0\}$,

$$\limsup_{z\to\zeta, z\in D} |f(z) - \phi(z)| = O(\mu^2(|\zeta - \gamma|)),\ \zeta \in \partial D \setminus Q,\ \zeta \to \gamma \tag{2.2}$$

Then $f(z) \equiv \phi(z)$ *on* D.

Proof. Let the assumptions of Theorem 2.1 are satisfied. By the condition (2.1), there exist a number $C_1 > 0$ and $\delta_0 \in (0, 1)$ such that

$$\limsup_{z\to\zeta, z\in D} |f(z) - \phi(z)| = C_1\mu^1(|\zeta - \gamma_0|)),\ \zeta \in \partial D \setminus Q,\ |\zeta - \gamma_0| \le \delta_0.$$

Let us denote k and C_2 as follows

$$k := \sup_{|z-\gamma_0|=\delta_0,\ z\in D} |f(z) - \phi(z)|,$$

$$C_2 := \max\left\{\frac{k}{\mu^1(\delta_0)}, C_1\right\}.$$

It can be easily seen that for all points of the set $\partial(D \cap U(\gamma_0, \delta_0)) \setminus Q$, the inequality

$$\limsup_{z\to\zeta, z\in D} |f(z) - \phi(z)| = C_2\mu^1(|\zeta - \gamma_0|))$$

is satisfied.

Applying Theorem 3 in ([17]) (see also ([19]), ([20])) to the set $D \cap U(\gamma_0, \delta_0)$ and to the function $f(z) - \phi(z)$, we get

$$|f(z) - \phi(z))| \le C_2\mu^1(|z - \gamma_0|)),\ \forall z \in D \cap U(\gamma_0, \delta_0). \tag{2.3}$$

From $\mu_0^1 > 3$ there are some positive constants ε and $\sigma < \min(\delta_0, 1)$ such that

$$\frac{\log \mu^1(x)}{\log x} \ge 3 + \varepsilon \quad \forall x \in (0, \sigma)$$

and

$$\log \mu^1(x) \le (3 + \varepsilon) \log x, \quad \forall x \in (0, \sigma)$$

In other words,

$$\mu^1(x) \leq x^{3+\varepsilon}, \quad \forall x \in (0, \sigma). \tag{2.4}$$

From the inequalities (2.3) and (2.4) we take the inequality

$$\left|f(z) - \phi(z))\right| \leq C_2 \left|z - \gamma_0\right|^{3+\varepsilon}, \ \forall z \in D \cap U(\gamma_0, \sigma). \tag{2.5}$$

Similarly, for any point $\gamma \in A_f \setminus \{\gamma_0\}$, from the condition $\mu_0^2 > 1$ and (2.2) we have

$$\left|f(z) - \phi(z))\right| \leq C_3 \left|z - \gamma\right|^{1+\varepsilon}, \ \forall z \in D \cap U(\gamma, \sigma_1) \tag{2.6}$$

with some constants C_3 and σ_1.

Consider the following harmonic function in the unit disk

$$\psi(z) = \Re\left(\frac{1 + f(z)}{1 - f(z)}\right) - \Re\left(\frac{1 + \phi(z)}{1 - \phi(z)}\right).$$

Since a finite Blaschke Product ϕ is holomorphic on $\overline{D}$ and and $\left|\phi(z)\right| = 1$ on ∂D, we have the second term of ψ is zero on $\partial D \setminus A_f$ and also the first term of ψ is nonnegative. Consequently, after taking limitinfs to any boundary point in $(\partial D \setminus Q) \setminus A_f$, one always reaches the nonnegative value (infinity is also possible).

Now, let us examine the behaviour of the function ψ at points of set A_f. Let us represent $\psi(z)$ in the form

$$\psi(z) = \Re\left(\frac{2\left(f(z) - \phi(z)\right)}{(1 - f(z))\left(1 - \phi(z)\right)}\right).$$

Now, let us take any point $\gamma \in A_f \setminus \{\gamma_0\}$. It can be easily seen that for any z, $|z| = 1$, $\left|\phi'(z)\right| > 0$. If $\left|\phi'(\gamma)\right| = c_\gamma$, then there exists a constant $\sigma_\gamma \in (0, \sigma_1)$ such that

$$\left|1 - \phi(z)\right| \geq \frac{c_\gamma}{2}\left|\gamma - z\right|, \ \ \forall z \in D \cap \mathbb{U}\left(\gamma, \sigma_\gamma\right). \tag{2.7}$$

From (2.6)

$$\lim_{z \to \gamma} \frac{1 - f(z)}{\gamma - z} = c_\gamma$$

and there exists $\sigma_\gamma' \in (0, \sigma_\gamma)$ such that

$$\left|1 - f(z)\right| \geq \frac{c_\gamma}{2}\left|\gamma - z\right|, \ \forall z \in D \cap \mathbb{U}\left(\gamma, \sigma_\gamma'\right). \tag{2.8}$$

Then, from (2.6) , (2.7) and (2.8)

$$\left|\frac{2\left(f(z) - \phi(z)\right)}{(1 - f(z))\left(1 - \phi(z)\right)}\right| \leq \frac{8C_3}{c_\gamma^2} \frac{1}{\left|\gamma - z\right|^{1-\varepsilon}} \quad \forall z \in D \cap \mathbb{U}\left(\gamma, \sigma_\gamma'\right).$$

Thus, the function $\psi(z)$ satisfies the following relation

$$\lim_{z \to \gamma} \left|z - \gamma\right| \psi(z) = 0 \tag{2.9}$$

on every point $\gamma \in A_f \setminus \{\gamma_0\}$.

Similarly, for the point γ_0, using (2.5), we have

$$\left|\psi(z)\right| \leq C_4 \left|z-\gamma_0\right|^{1+\varepsilon} \qquad \forall z \in D \cap \mathbb{U}\left(\gamma_0, \sigma'_\gamma\right) \tag{2.10}$$

for some positive constants C_4 and σ'. In particular,

$$\lim_{z\to\gamma_0} \psi(z) = 0. \tag{2.11}$$

From also here

$$\lim_{z\to\gamma_0} \left|z-\gamma_0\right| \psi(z) = 0.$$

So, the function $\psi(z)$ satisfies the relation (2.9) on every point of finite set A_f. From the assertion (B) we have either $\psi(z) > 0$, $z \in D$ or $\psi(z) \equiv 0$. If $\psi(z) \equiv 0$, then the proof is finished. Assume that the relation $\psi(z) \equiv 0$ is not satisfied. If we take $z = r\gamma_0$ in (2.10), we obtain

$$\lim_{r\to 1} \frac{\psi(r\gamma_0)}{1-r} = 0. \tag{2.12}$$

If ψ is not constant, (2.11) and (2.12) contradict with assertion (A) statement. Hence, $\psi \equiv 0$. This implies that $f(z) = \phi(z)$ on the disk. □

Theorem 2.2 and Theorem 2.3 generalize the results in ([4]), where instead of the condition $\mu_0^2 > 1$ were taken $\mu_0^2 > 2$. Moreover, the part of the proof of Theorem 2.3 which is after (2.4) shows that $O(z-\gamma)^{k_\gamma}$, $k_\gamma \geq 2$ in Theorem 1.1 can be replaced by $o(z-\gamma)$.

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