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Toward a Laplacian Spectral Determination of Signed ∞-Graphs

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Abstract. A signed graph consists of a (simple) graph $G = (V, E)$ together with a function $\sigma : E \to \{+, -\}$ called signature. Matrices can be associated to signed graphs and the question whether a signed graph is determined by the set of its eigenvalues has gathered the attention of several researchers. In this paper we study the spectral determination with respect to the Laplacian spectrum of signed ∞-graphs. After computing some spectral invariants and obtain some constraints on the cospectral mates, we obtain some non isomorphic signed graphs cospectral to signed ∞-graphs and we study the spectral characterization of the signed ∞-graphs containing a triangle.

1. Introduction

Let $G = (V, E)$ be a simple, finite and undirected graph. We denote the order and the size of G by $|V| = n$ and $|E| = m$. Let $\sigma : E(G) \to \{+, -\}$ be a map defined on the edge set of G. Then $\Gamma = (G, \sigma)$ is called a *signed graph*, the graph G is its *underlying graph* and σ is its *sign function* (or *signature*). An edge e is *positive* (*negative*) if $\sigma(e) = 1$ (resp. $\sigma(e) = -1$). If $\sigma(e) = 1$ (resp. $\sigma(e) = -1$) for all edges in $E(G)$ then we write $(G, +)$ (resp. $(G, -)$). A cycle of Γ is said to be *balanced*, or *positive*, if it contains an even number of negative edges, otherwise the cycle is *unbalanced*, or *negative*. A signed graph is said to be *balanced* if all its cycles are balanced; otherwise, it is *unbalanced*. For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be (signature) *switching equivalent* to Γ. Furthermore, two signed graphs Γ and Λ are said to be *switching isomorphic* if Γ is isomorphic to a switching of Λ (see [5]). Here, switching isomorphic signed graphs are considered to be the same signed graph.

Signed graphs introduced as early as 1953 by Harary [10], to model social relations involving disliking, indifference, and liking [9]. Indeed signed graphs have been used frequently to model affect ties for social actors. A traffic control problem at an intersection can be modeled by a signed graph, where the nodes denote the streams and the edges represent the relation between the streams together with a sign associated with it [3]. Signed graph can be also used as a graph theoretic tool to study transportation problem [4]. Signed graphs are much studied in the literature because of their use in modeling a variety of physical and socio-psychological processes (see [2, 7]) and also because of their interesting connections with many classical mathematical systems (see for example, [1, 6, 8, 11]).

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Signed graphs can be studied by means of matrices associated to them. One of the most important graph matrices is the adjacency matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ whenever vertices i and j are adjacent and $a_{ij} = 0$ otherwise. The Laplacian matrix and singless Laplacian matrix are defined by $L(G) = D(G) - A(G)$, and $Q(G) = A(G) + D(G)$ respectively. Recall that $D(G)$, is the diagonal matrix of the graph G, which is $D(G) = \mathrm{diag}(\deg(v_1), \deg(v_2), \dots, \deg(v_n))$.

Similar definition can be used for signed graphs. Let $\Gamma = (G, \sigma)$ be a signed graph, the matrix $A(\Gamma) = (a^{\sigma}_{ij})$ with $a^{\sigma}_{ij} = \sigma(ij)a_{ij}$ is called the *(signed) adjacency matrix* and $L(\Gamma) = D(G) - A(\Gamma)$ is the corresponding Laplacian matrix. Note that both the adjacency and Laplacian matrices are real and symmetric.

Let $\phi(\Gamma, x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ be the adjacency characteristic polynomial (or A-polynomial) whose roots, namely the adjacency eigenvalues (A-eigenvalues), are denoted by $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$. Similarly, we denote by $\psi(\Gamma, x) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ the Laplacian polynomial (or L-polynomial), and Laplacian eigenvalues are denoted by $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma) \geq 0$. Notably, switching isomorphic signed graphs get the same set of adjacency and Laplacian eigenvalues ([19]).

Let $\Gamma = (G, \sigma)$ be a signed graph. A subgraph whose components are trees or unbalanced unicyclic graphs is called a *signed TU-subgraph*. Let H be a signed TU-subgraph, then $H = \bigcup_{i=1}^{t} T_i \bigcup_{j=1}^{c} U_j$, where, if any, the T_i's are trees and the U_j's are unbalanced unicyclic graphs. The weight of the signed TU-subgraph H is defined as $w(H) = 4^c \prod_{i=1}^{t} |V(T_i)|$, (See [8] for details).

Theorem 1.1. *[6] Let Γ be a signed graph and $\psi(\Gamma, x) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ be the Laplacian polynomial of Γ. Then we have*

$$b_i = (-1)^i \sum_{H \in \mathcal{H}_i} w(H), \quad i = 1, 2, \dots, n. \tag{1}$$

where $\mathcal{H}_i$ denotes the set of signed TU-subgraphs of Γ built on i edges.

Observe that by Formula (1) the signs of bridges has no influence on the polynomial coefficients. Hence, we will assume that bridges always take a positive sign. Also, it is not difficult to see that the above polynomials are invariant under switching isomorphisms.

A natural question is then posed. Given a signed graph, are there non-switching isomorphic signed graphs which get the same eigenvalues? Such problem is called *spectral determination* (or, with some abuse of notation, *spectral characterization*). Spectral determinations have been widely studied for unsigned graphs and recently this kind of study has been taken to signed graph spectra, as well. The spectral determination of (unsigned) ∞-graphs, which is the coalescence between two cycles C_r and C_s and related graphs, has been already considered in [16]. The spectral determination of signed lollipop graph, which is the coalescence between a cycle and a path, has been studied and it is shown that a signed lollipop graph is determined by the spectrum of its Laplacian matrix (or simply DLS) [5]. These works motivate us to consider signed ∞-graphs and study the spectral determination of these signed graphs.

The remainder is as follows. In Section 2, we consider the spectral invariants related to the signed ∞-graphs. In Section 3, we study the triangle-free signed ∞-graphs and we determine a new family of Laplacian cospectral mates. In Section 4, we study signed ∞-graphs with at least on triangle, and we prove that, with a few exceptions of small order, all of them does not have connected Laplacian cospectral mates.

2. Notation and basic results

Let us denote the ∞-graph by $G_{r,s}$ whose order is $r+s-1$. In Figure 1 we depicted an example of a signed ∞-graph. Since there are two cycles in any ∞-graph, these cycles can be both balanced, one balanced and the other unbalanced and both unbalanced. So we denote the sign of an ∞-graph by $\sigma = +, \sigma = \pm$ and $\sigma = -$, respectively. Note that for $\sigma = \pm$ ($\sigma = \mp$), we assume C_r is unbalanced and C_s is balanced (respectively, C_r is balanced and C_s is unbalanced). Also by σ_r and σ_s we denote the sign of C_r and C_s, respectively.

The following corollary is an immediate consequence of Theorem 1.1.

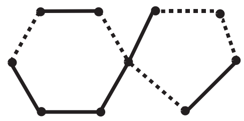

Figure 1: The ∞-graph $(G_{6,5}, \mp)$.

Corollary 2.1. *Let $\Gamma = (G_{r,s}, \sigma)$ be a signed ∞-graph and $\psi(\Gamma, x) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ be its Laplacian polynomial.*

(i) If $\sigma = +$, then $b_n = 0$ and $b_{n-1} = (-1)^{n-1}rs(r + s - 1)$.

(ii) If $\sigma = \pm$, then $b_n = (-1)^n 4s$ and $b_{n-1} = (-1)^{n-1}[rs(r + s - 1) + \frac{2}{3}(s^3 - s)]$. (Note that for $\sigma = \pm$, we let $\sigma_r = -$ and $\sigma_s = +$).

(iii) If $\sigma = -$, then $b_n = (-1)^n 4(r + s)$ and $b_{n-1} = (-1)^{n-1}[rs(r + s - 1) + \frac{2}{3}(r^3 - r + s^3 - s)]$.

The following theorem shows the relation between Laplacian polynomial of a signed graph and the adjacency polynomial of its signed subdivision graph, $\mathcal{S}(\Gamma)$, and its signed line graph, $\mathcal{L}(\Gamma)$ too. For the definition of signed subdivision graph $\mathcal{S}(\Gamma)$ and signed line graph $\mathcal{L}(\Gamma)$ we refer to [5].

Theorem 2.2. *[5, Theorem 2.3] Let Γ be a signed graph of order n and size m, and $\phi(\Gamma)$ and $\psi(\Gamma)$ be its adjacency and Laplacian polynomials, respectively. Then*

(i) $\phi(\mathcal{L}(\Gamma), x) = (x + 2)^{m-n}\psi(\Gamma, x + 2)$,

(ii) $\phi(\mathcal{S}(\Gamma), x) = x^{m-n}\psi(\Gamma, x^2)$.

The following result is the interlacing theorem in the edge variant, that can be deduced from the ordinary vertex variant interlacing theorem for the adjacency matrix combined with Theorem 2.2(*ii*).

Theorem 2.3. *[5, Theorem 2.5] Let $\Gamma = (G, \sigma)$ be a signed graph and $\Gamma - e$ be the signed graph obtained from Γ by deleting the edge e. Then*

$$\mu_1(\Gamma) \geq \mu_1(\Gamma - e) \geq \mu_2(\Gamma) \geq \mu_2(\Gamma - e) \geq \cdots \geq \mu_n(\Gamma) \geq \mu_n(\Gamma - e).$$

In the following lemma, an upper bound for the largest Laplacian eigenvalue of a signed graph is given.

Lemma 2.4. *[5, Lemma 2.8] Let $\Gamma = (G, \sigma)$ be a signed graph with Δ_1 and Δ_2 being the first and second largest vertex degrees in G, and let $\mu(\Gamma)$ be its Laplacian spectral radius. Then $\mu(\Gamma) \leq \Delta_1 + \Delta_2$, with equality if and only if $\Gamma = K_{1,n}$ or $\Gamma = (K_n, -)$.*

Let $T_k = \sum_{i=1}^{n} \mu_i^k$, be the k^{th} spectral moment for the Laplacian spectrum of a signed graph Γ, for a non-negative integer k. In the following theorem the relation between $T_i (i = 0, 1, 2, 3)$ and some known parameters of the graph is given.

Theorem 2.5. *[5, Theorem 3.4] Let $\Gamma = (G, \sigma)$ be a signed graph with n vertices, m edges, t^+ balanced triangles, t^- unbalanced triangles and degree sequence $(d_1, d_2, \ldots, d_n)$. Then*

(i) $T_0 = n$,

(ii) $T_1 = \sum_{i=1}^{n} d_i = 2m$,

(iii) $T_2 = 2m + \sum_{i=1}^{n} d_i^2$,

(iv) $T_3 = 6(t^- - t^+) + 3\sum_{i=1}^n d_i^2 + \sum_{i=1}^n d_i^3$.

In the following theorem some relations between two L-cospectral signed graphs are presented.

Theorem 2.6. *[5, Theorem 3.5] Let $\Gamma = (G, \sigma)$ and $\Lambda = (H, \sigma')$ be two L-cospectral signed graphs. Then,*

(i) Γ *and* Λ *have the same number of vertices and edges;*

(ii) Γ *and* Λ *have the same number of balanced components;*

(iii) Γ *and* Λ *have the same Laplacian spectral moments;*

(iv) Γ *and* Λ *have the same sum of squares of degrees,*
$\sum_{i=1}^n d_G(v_i)^2 = \sum_{i=1}^n d_H(v_i)^2$;

(v) $6(t_\Gamma^- - t_\Gamma^+) + \sum_{i=1}^n d_G(v_i)^3 = 6(t_\Lambda^- - t_\Lambda^+) + \sum_{i=1}^n d_H(v_i)^3$.

Spectral determination of unsigned ∞-graphs are considered in the papers [16] and [12]. The main result of these papers are as follow.

Theorem 2.7. *[16, Theorem 5.1] Any (unsigned) triangle-free ∞-graph is determined by its Laplacian spectrum.*

Theorem 2.8. *[16, Theorem 6.3] Any ∞-graph but $G_{r,r+1}$ ($r \geq 3$) is determined by its signless Laplacian spectrum.*

Theorem 2.9. *[12, Theorem 6.1] All (unsigned) 2-rose graphs, except for $G_{3,4}$ and $G_{3,5}$, are determined by the Laplacian spectrum. Both $G_{3,4}$ and $G_{3,5}$ have one Laplacian cospectral mate.*

In view of Theorems 2.7 and 2.9, we know that the balanced $(G_{r,s}, +)$ has no connected balanced Laplacian cospectral mates, with the exception of $G_{3,4}$ and $G_{3,5}$ which both get at least one Laplacian cospectral mate. Recall that the signless Laplacian of graphs corresponds to the Laplacian of signed graphs in which all edges are taken negative, so from Theorem 2.8 we know that $(G_{r,r+1}, \mp)$ and $(G_{r,r-1}, \mp)$, with r even, also have a Laplacian cospectral mate.

The following theorem gives bounds on the first and second largest eigenvalue of any signed ∞-graph. They were computed for unsigned graphs but they are still valid for signed graphs.

Theorem 2.10. *Let $(G_{r,s}, \sigma)$ be a signed ∞-graph. Then we have,*

(*i*) $5 < \mu_1(G_{r,s}, \sigma) < 6$,

(*ii*) $\mu_2(G_{r,s}, \sigma) < 4$.

Proof. (*i*) The lower bound for $\mu_1(G_{r,s}, \sigma)$ follows by Theorem 2.3 and the fact that $K_{1,4}$ is a subgraph of $G_{r,s}$, while the upper bound is an immediate consequence of Theorem 2.4.

(*ii*) Let v be the common vertex of two cycles in the ∞-graph. By applying the interlacing theorem to the vertex v of subdivision graph $G_{2r,2s}$, we obtain two paths P_{2r-1} and P_{2s-1}. Hence, by using Theorem 2.3, we have $\lambda_2(G_{2r,2s}, \sigma) \leq \lambda_1(P_{2r-1} \cup P_{2s-1}, \sigma) < 2$. From Theorem 2.2 we find that the second largest eigenvalue of signed graph $(G_{r,s}, \sigma)$ is less than $2^2 = 4$. □

The degree sequence of a signed graph is as the same as the degree sequence of its underlying graph. It has been shown in [16, Theorem 4.1], by using the spectral moments T_0, T_1 and T_2, that every L-cospectral mate for an (unsigned) ∞-graph must have degree sequence $(4, 2^{n-1})$ or $(3^3, 2^{n-4}, 1)$. Since those spectral moments do not depend on the signatures, the same restriction of [16, Theorem 4.1] holds for signed ∞-graphs.

Theorem 2.11. *Let $\Lambda = (H, \sigma')$ be a graph L-cospectral to a signed ∞-graph $\Gamma = (G_{r,s}, \sigma)$ of order n. Then* deg(H) *(degree sequence of the graph H) belongs to the set* $\{(4, 2^{n-1}), (3^3, 2^{n-4}, 1)\}$.

From Theorems 2.10 and 2.11 we have an important restriction on disconnected Laplacian cospectral mates. In fact, since the second largest L-eigenvalue is less than 4, then any Laplacian cospectral mate cannot have two components with a vertex of degree 3 each, otherwise two copies of $K_{1,3}$ appear and the second largest Laplacian eigenvalues is at least 4 by interlacing theorem. So paths and cycles can appear as disconnected components, however paths have two vertices of degree 1, so they are discarded as well. In the sequel we will focus only on connected cospectral mates, but the reader should keep in mind that possible Laplacian cospectral mates have one or more signed cycles as additional components.

3. Signed ∞-graphs with $r, s > 3$

In this section, we will prove that an ∞-graph is isomorphic to a connected signed graph $\Lambda = (H, \sigma')$ such that $\deg(H) = (4, 2^{n-1})$ if and only if they are L-cospectral. Next, we will find connected graphs $\Lambda = (H, \sigma')$ which are not cospectral with $(G_{r,s}, \sigma)$ in which $\deg(H) = (3^3, 2^{n-4}, 1)$.

Theorem 3.1. *Let $\Gamma = (G_{r,s}, \sigma)$ be a signed ∞-graph of order n. If a connected graph $\Lambda = (H, \sigma')$ has degree sequence $(3^3, 2^{n-4}, 1)$ and $\psi(\Gamma) = \psi(\Lambda)$, then*

(i) if Γ is triangle-free, then Λ has an unbalanced triangle, Fig. 2.

(ii) if Γ has a balanced triangle, then Λ is either a triangle-free graph or it contains a balanced triangle and an unbalanced one, Fig.s 3 and 4.

(iii) if Γ has an unbalanced triangle, then Λ has two unbalanced triangles, Fig. 4.

Proof. (i) Let $\Gamma = (G_{r,s}, \sigma)$ be a triangle-free ∞-graph. By Lemma 2.6(v) we have $t^-_\Lambda - t^+_\Lambda = 1$. Since $t^-_\Lambda, t^+_\Lambda \in \{0, 1, 2\}$, we find that $t^-_\Lambda = 1$ and $t^+_\Lambda = 0$.
The proof of (ii) and (iii) is straightforward. □

Suppose H is a connected graph with $\deg(H) = (3^3, 2^{n-4}, 1)$. Assume also that H has one triangle and one cycle, say C_k. Then we conclude that H is one of the graphs $H_i (i = 1, 2, \ldots, 5)$ which are shown in Figure 2. Note that triangles of these graphs are unbalanced by Theorem 3.1.

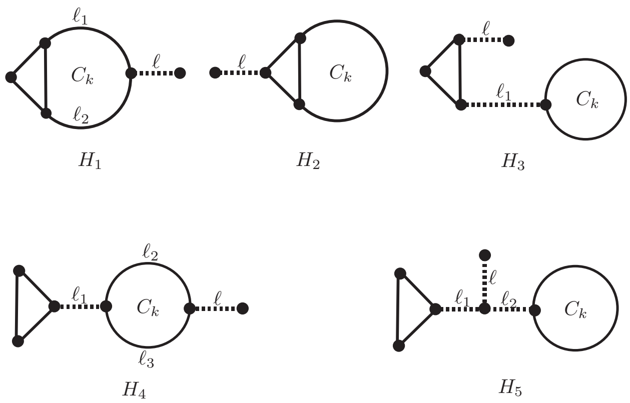

Figure 2: Graphs that can be cospectral with a signed triangle-free graph $(G_{r,s}, \sigma)$ for $r, s > 3$.

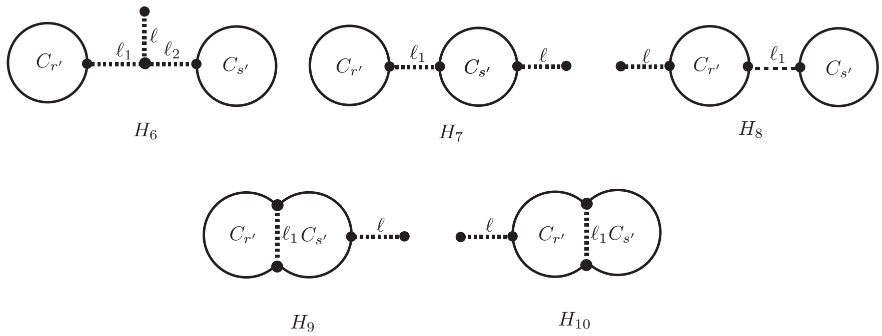

Figure 3: Triangle-free graphs possibly L-cospectral with $(G_{3,s}, \sigma)$.

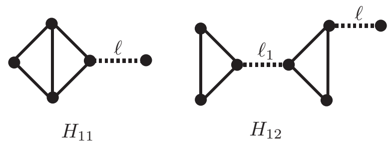

Figure 4: Graphs that can be cospectral with a signed graph $(G_{3,s}, \sigma)$ containing two triangles.

As an immediate consequence of Theorem 1.1 we have the following theorem. Note that we omit $(-1)^{n-i}$ as sign of coefficient of x^{n-i}.

Theorem 3.2. *Let $\Lambda = (H_i, \sigma')$ be one of the graphs in Figures 2, 3 and 4 with Laplacian polynomial $\psi(\Lambda, x) = x^n + b'_1 x^{n-1} + \cdots + b'_{n-1}x + b'_n$. Let ℓ and ℓ_is be the lengths of paths shown in Figures 2, 3 and 4. Then:*

(i) for $i = 1, \ldots, 5$, if $\sigma_k = +$, (σ_k is sign of cycle C_k), $b'_n = 4k$, and if $\sigma_k = -$, for graphs H_1 and H_2, $b'_n = 4(k+1)$, for graphs H_3 and H_4, $b'_n = 4(k + 3 + 4\ell_1)$ and for graph H_5 , $b'_n = 4(k + 3 + 4\ell_1 + 4\ell_2)$,

(ii) for $i = 6, \ldots, 10$, if $\sigma_r = -$ and $\sigma_s = +$, then $b'_n = 4s'$, if $\sigma_r = +$ and $\sigma_s = -$, then $b'_n = 4r'$ and if $\sigma_r = \sigma_s = -$, then for H_6, $b'_n = 4(r' + s' + 4\ell_1 + 4\ell_2)$, for H_7 and H_8, $b'_n = 4(r' + s' + 4\ell_1)$, and for H_9 and H_{10}, $b'_n = 4(r' + s' - 2\ell_1)$.

(iii) for $i = 11, 12, 13$, if the graph H_i has one balanced and one unbalanced cycle, then $b'_n = 12$. If two cycles are unbalanced, then we have $b'_n = 16$, $b'_n = 4(6 + 4\ell_1)$ and $b'_n = 4(6 + 4\ell_1 + 4\ell_2)$ for graphs H_{11}, H_{12} and H_{13}, respectively.

The next theorem deals with Laplacian polynomial of graphs $H_i, (i = 1, \ldots, 5)$.

Theorem 3.3. *Let $\Lambda = (H_i, \sigma')$ be one of the graphs $H_i, (i = 1, \ldots, 5)$ with Laplacian polynomial $\psi(\Lambda, x) = x^n + b'_1 x^{n-1} + \cdots + b'_{n-1}x + b'_n$. Then $(-1)^{n-1}b'_{n-1}$ for graph with a balanced cycle C_k or an unbalanced cycle C_k is given in Tables 1 and 2 respectively.*

Proof. Consider the graph H_1 having an unbalanced triangle, a balanced cycle C_k and a path of length $\ell = n - k - 1$, which is attached to a vertex of degree 2 in C_k. For finding b'_{n-1} we should delete two edges of H_1. Notice that if any deleted edge of H_1 is not from cycle C_k, then $w(H) = 0$. Let at least one deleted edge is from C_k. We consider subset $\mathcal{H}'_{n-1}$ containing all TU-subgraphs with $n-1$ edges obtained by deleting just one edge of C_k not belonging to the triangle. Define similarly $\mathcal{H}''_{n-1}$ as the TU-subgraphs with $n-1$ edges obtained by deleting the edge common to C_k and the triangle, and $\mathcal{H}'''_{n-1}$ as the TU- subgraphs with $n-1$ edges obtained by deleting two edges of C_k. Thus

$$\sum_{H \in \mathcal{H}'_i} w(H) = (k-1)(2n + 4\sum_{i=1}^{\ell} i),$$

$$\sum_{H\in\mathcal{H}_i''} w(H) = 2n + 4\sum_{i=1}^{\ell} i$$

and

$\sum_{H\in\mathcal{H}_i'''} w(H) = n(k-1) + 4\,[\sum_{i=1}^{\ell_1-1}\frac{i(i+1)}{2} + \sum_{i=1}^{\ell_2-1}\frac{i(i+1)}{2} + \sum_{i=1}^{\ell_1}\sum_{j=1}^{\ell_2}(\ell+i+j-1)]$.

Since $(-1)^{n-1}b'_{n-1} = \sum_{H\in\mathcal{H}_i} w(H)$ the proof for H_1 is complete. A similar argument can be used for other graphs. □

Table 1: $(-1)^{n-1}b'_{n-1}$ in graphs H_i with $\sigma_k = +$, $i = 1, 2, \ldots, 5$.

H_i	$(-1)^{n-1}b'_{n-1}$
H_1	$n(3k-1) + \frac{2}{3}k^3 - 2k^2 + k(2\ell^2 + 2\ell + 4\ell\ell_1 + \frac{4}{3}) - 4\ell\ell_1(\ell_1+1)$
H_2	$2kn^2 - (4k^2 - k + 1)n + \frac{4}{3}k(2k^2+1)$
H_3	$2kn^2 - (7k + 4k\ell)n - \frac{4}{3}k^3 + 2k^2 + k(4\ell^2 + 12\ell + \frac{34}{3})$
H_4	$\frac{2}{3}n^3 - (2\ell + 2\ell_1 + 4)n^2 + (2\ell^2 + 2\ell_1^2 + 8\ell + 8\ell_1 + 4\ell\ell_1 + 4\ell\ell_2$ $+3k + \frac{22}{3})n - \frac{2}{3}\ell^3 - (-2k + 2\ell_1 + 4\ell_2 + 4)\ell^2 - (4\ell_1\ell_2 - 4k\ell_1$ $-2k + 2\ell_1^2 + 4\ell_2^2 + 8\ell_1 + 8\ell_2 + \frac{22}{3})\ell - \frac{2}{3}\ell_1^3 + (2k-4)\ell_1^2$ $+(4k^2 - 2k - \frac{22}{3})\ell_1 - 4$
H_5	$2kn^2 - (7k + 4k\ell)n - \frac{4}{3}k^3 + 2k^2 + (4\ell^2 + 4\ell\ell_1 + 12\ell + \frac{34}{3})k$

Now we will prove that an ∞-graph is isomorphic to a connected graph with $\deg(H) = (4, 2^{n-1})$ if and only if they are L-cospectral.

Theorem 3.4. *Let $\Gamma = (G_{r,s}, \sigma)$ be an ∞-graph of order n. Let the connected graph $\Lambda = (H, \sigma')$ with degree sequence $(4, 2^{n-1})$ have the property that $\psi(\Gamma) = \psi(\Lambda)$. Then Λ is switching isomorphic to Γ.*

Proof. Since H is connected, so it is obvious that H is an ∞-graph, say $H = G_{r_1,s_1}$. By Theorem 2.6 (i) and (v), we find that $t_\Gamma^- - t_\Gamma^+ = t_\Lambda^- - t_\Lambda^+$. So graphs Γ and Λ have both the same number of triangles with the same signs.

The case of two triangles, i.e. graph $G_{3,3}$, will be considered in Theorem 4.1. If the graph $\Gamma = (G_{3,s}, \sigma)$ contains one triangle, then $\Lambda = (G_{r_1,s_1}, \sigma')$ must have one triangle. So $r = r_1 = 3$ and $s = s_1$ or $r = s_1 = 3$ and $s = r_1$. Without loss of generality, let $\Lambda = (G_{3,s_1}, \sigma')$. By using Corollary 2.1, we find that the cycles C_s and C_{s_1} have the same signs.

Let $\Gamma = (G_{r,s}, \sigma)$ does not have any triangle. By using [16, Theorem 5.1] we can prove theorem for case $\Gamma = (G_{r,s}, +)$. Now let $\sigma_r = +$ and $\sigma_s = -$. By comparing b_n and b'_n, we conclude that Λ does not have two unbalanced cycles. Hence either $\sigma_{r_1} = +$ and $\sigma_{s_1} = -$ or $\sigma_{r_1} = -$ and $\sigma_{s_1} = +$. So we have $r = r_1$ and $s = s_1$.

Finally, let C_r and C_s are unbalanced. This implies that C_{r_1} and C_{s_1} are unbalanced too. Let $r < r_1$. Since $r + s = r_1 + s_1$, then $s > s_1$. We can consider a positive integer say k so that $r_1 = r + k$ and $s_1 = s - k$. Since we assume that $\Gamma = (G_{r,s}, \sigma)$ and $\Lambda = (G_{r_1,s_1}, \sigma')$ are cospectral, the coefficients of their Laplacian polynomials are the same. By comparing b_{n-1} and b'_{n-1} we have $nk(s - r - k) + 2(r^2k + rk^2 - s^2k + sk^2) = 0$. Without loss of generality, let $r \geq s$. If $r = s$, then we have $(4r - n)k^2 = 0$, and since r is a positive integer, we have $k = 0$, which is a contradiction. If $r > s$, let $r = s + a$, where $a > 0$. Then we have $k(k + a)(2s + a + 1) = 0$ which is not possible as a, k, s are positive integers. □

Table 2: $(-1)^{n-1}b'_{n-1}$ in graphs H_i with $\sigma_k = -, i = 1, \ldots, 5$.

H_i	$(-1)^{n-1}b'_{n-1}$
H_1	$n(3k-1) + \frac{2}{3}k^3 - 2k^2 + k(2\ell^2 + 2\ell + 4\ell\ell_1 + \frac{4}{3}) - 4\ell\ell_1(\ell_1 + 1)$ $+2\ell + 2\ell_2 + 4$
H_2	$(2k+1)n^2 - (4k^2 + 3k - 1)n + \frac{8}{3}k^3 + 2k^2 - \frac{2}{3}k$
H_3	$\frac{8}{3}n^3 - (6k + 8\ell + 10)n^2 + (16\ell^2 + 8k^2 + 40\ell + 13k + 12k\ell$ $+\frac{106}{3})n - \frac{32}{3}\ell^3 - (12k + 40)\ell^2 - (8k^2 + 28k - \frac{184}{3})\ell - 4k^3$ $-8k^2 - 24k - 36$
H_4	$\frac{2}{3}n^3 - (2\ell + 2\ell_1 + 4)n^2 + (2\ell^2 + 2\ell_1^2 + 8\ell + 8\ell_1 + 4\ell\ell_1 + 4\ell\ell_2 + 3k$ $+\frac{22}{3})n - \frac{2}{3}\ell^3 + (2k + 6\ell_1 - 4\ell_2 + 2)\ell^2 - (4\ell_1\ell_2 - 4k\ell_1 - 2k + 2\ell_1^2$ $+4\ell_2 + 8\ell_2 - \frac{4}{3})\ell + 2\ell_1^3 + (2k+2)\ell_1^2 + (4k^2 - 2k + 20)\ell_1 + 12$
H_5	$\frac{8}{3}n^3 - (6k + 8\ell + 10)n^2 + (16\ell^2 + 8k^2 + 40\ell + 13k + 16\ell\ell_1 + 12k\ell$ $+\frac{106}{3})n - \frac{32}{3}\ell^3 - (12k + 16\ell_1 + 40)\ell^2 - (8k^2 + 28k + 44\ell_1 + 12k\ell_1\ell$ $+16\ell_1^2 + \frac{208}{3}) - 4k^3 - 8k^2 - 24k - 36$

For graphs of degree sequence $\deg(H) = (3^3, 2^{n-4}, 1)$ that can be cospectral with ∞-graph, we have the following conjecture. We expect that ∞-graph does not have cospectral mates among the signed graphs H_1, H_3, H_4 and H_5. However, in some special cases it can be cospectral with graph H_2.

Conjecture 3.5. *The signed graphs of type (H_i, σ'), with $i = 1, 3, 4, 5$, can not be cospectral with the graph $(G_{r,s}, \sigma)$, for $r, s > 3$.*

Conjecture 3.5 is stated for signed graphs H_1, H_3, H_4 and H_5. But for the signed graph H_2 it is possible to find some r and s such that (H_2, σ) is cospectral with $(G_{r,s}, \sigma)$. In fact, from Theorem 2.8 in [16], we know that $(G_{r,r+1}, \mp)$ and $(G_{r,r-1}, \mp)$ with r even are L-cospectral with some opportunely signed H_2. In fact, the result of [16] can be given without the restriction on r and by adapting the original proof to signed graphs.

Theorem 3.6. *Graph H_2 with $n = 2(\ell+1)$ vertices and balanced C_k is cospectral with $G_{\ell+1,\ell+2}$, with balanced $C_{\ell+1}$ and unbalanced $C_{\ell+2}$. Indeed graph H_2 with $n = 2(\ell+1)$ vertices and unbalanced C_k is cospectral with $G_{\ell+1,\ell+2}$, with unbalanced $C_{\ell+1}$ and balanced $C_{\ell+2}$.*

Proof. We will use the same strategy of Lemma 6.11 in [16].

Consider for example the pair $(G_{\ell+1,\ell+2}, \mp)$ with ℓ even and (H_2, σ) such that it has a pendant path of length ℓ and the cycle $C_{\ell+1}$ is balanced, which means that we can take a negative edge in the triangle and none on the edges of $C_{\ell+1}$. If we pass to the subdivision signed graphs we get the pair $(G_{2\ell+2,2\ell+4}, -)$ and $S(H_2)$ of order $2\ell+5$ with the cycle $C_{2\ell+4}$ being unbalanced and the C_6 (subdivision of the triangle) being positive. If we now use the decomposition formulas as in [16] and the transformation $\phi(P_n) = \frac{x^{2n}-1}{x^{n+2}-x^n}$, we can check by software that the polynomials are indeed equal.

The remaining cases can be proved similarly. □

4. Signed ∞-graphs with $r = 3$ and $s \geq 3$

In this section we deal with signed infinity graphs with at least one triangle, that is $(G_{3,s}, \sigma)$, and we look for their connected Laplacian cospectral mates.

The case with exactly two triangles is considered in the next theorem.

Theorem 4.1. *Let $\Lambda = (H, \sigma')$ be a connected L-cospectral mate with $\Gamma = (G_{3,3}, \sigma)$. Then H and $G_{3,3}$ are isomorphic and $\sigma = \sigma'$.*

Proof. Since Γ and Λ are L-cospectral, we find that Γ has order 5 and size 6. By Theorem 2.11, $\deg(H) \in \{(4, 2^4), (3^3, 2, 1)\}$. On the other hand, H is a connected graph. So by [13] the graph H is isomorphic to one of the graphs in Figure 5.

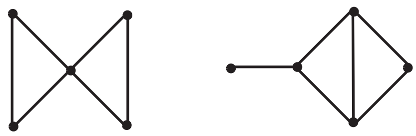

Figure 5: Two graphs corresponding to Theorem 4.1.

By Theorem 2.6(v), we have

$$6(t_\Gamma^- - t_\Gamma^+) + \sum_{i=1}^{n} d_\Gamma(v_i)^3 = 6(t_\Lambda^- - t_\Lambda^+) + \sum_{i=1}^{n} d_H(v_i)^3.$$

There are three different cases for the two triangles in the graph $G_{3,3}$:

(i) $t_\Gamma^- = 0$ and $t_\Gamma^+ = 2$, ($\sigma = +$);

(ii) $t_\Gamma^- = 2$ and $t_\Gamma^+ = 0$, ($\sigma = -$);

(iii) $t_\Gamma^- = t_\Gamma^+ = 1$, ($\sigma = \pm$).

First assume that the degree sequence of H is equal to $(4, 2^4)$. In this case we have

$$6(t_\Gamma^- - t_\Gamma^+) + 96 = 6(t_\Lambda^- - t_\Lambda^+) + 96.$$

Hence for case (i), $t_\Lambda^- - t_\Lambda^+ = -2$. So $t_\Lambda^- = 0$ and $t_\Lambda^+ = 2$, which implies that $\sigma' = +$. A similar argument can be used to prove for $\sigma = +$ and $\sigma = \pm$.

Let the degree sequence of H is equal to $(3^3, 2, 1)$. So we have $6(t_\Gamma^- - t_\Gamma^+) + 96 = 6(t_\Lambda^- - t_\Lambda^+) + 90$. For $\sigma = +$, we have $6(0-2)+96 = 6(t_\Lambda^- - t_\Lambda^+)+90$. Hence $t_\Lambda^- = 0$ and $t_\Lambda^+ = 1$. This is impossible for graph Λ. Similarly, we can prove the two remaining cases. Thus the graph $(G_{3,3}, \sigma)$ is determined by its Laplacian spectrum. $\square$

Now let ∞-graph have exactly one triangle. Without loss of generality we can assume that $s \geq 4$. Recall that such signed ∞-graphs can have connected cospectral mates among the H_i's with $i = 6, 7, \ldots, 13$.

Remark 4.2. *The signed graphs H_{11}, H_{12} and H_{13} with all possible signatures are special cases of graphs $H_1, H_2, \ldots, H_5$. So we can easily compute the coefficient b'_{n-1} for such graphs. The relation between these graphs is given in Table 3. The two cycles in these graphs are denoted by C_3' and C_3'' from left to right. Note that b and ub stand for balanced cycle and unbalanced cycle, respectively.*

Table 3: Relation between graphs H_{11}, H_{12}, H_{13} and graphs $H_1, H_2, \ldots, H_5$.

graph	C_3'	C_3''	comments
H_{11}	b	ub	H_1, C_k balanced, $\ell_1 = \ell_2 = 1$, $\ell = n-4$
H_{11}	ub	b	H_2, C_k balanced, $\ell = n-4$
H_{11}	ub	ub	H_1, C_k unbalanced, $\ell_1 = \ell_2 = 1$, $\ell = n-4$
H_{12}	b	ub	H_4, C_k balanced, $\ell_2 = 1$,$\ell_3 = 2$, $\ell_1 = n-\ell-5$
H_{12}	ub	b	H_3, C_k balanced, $\ell_1 = n-\ell-5$
H_{12}	ub	ub	H_4, C_k unbalanced, $\ell_2 = 1$, $\ell_3 = 2$, $\ell_1 = n-\ell-5$
H_{13}	b	ub	H_5, $C_k$balanced, $\ell = n-\ell_1-\ell_2-5$
H_{13}	ub	b	H_5, C_k balanced, $\ell = n-\ell_1-\ell_2-5$
H_{13}	ub	ub	H_5, C_k unbalanced, $\ell = n-\ell_1-\ell_2-5$

We next consider the three following cases, according to all possible signatures of $(G_{3,s}, \sigma)$. The first case is covered by [12, Theorem 6.1], and the H_9's have two quadrangles each.

Lemma 4.3. *The signed ∞-graph $\Gamma = (G_{3,s}, +)$ with $s \geq 6$ has no connected L-cospectral mates. The signed graphs $(G_{3,4}, +)$ and $(G_{3,5}, +)$ are L-cospectral to $(H_9, +)$ with $\ell = 1$ and $\ell_1 = 2$ and to $(H_9, +)$ with $\ell = 1$ and $\ell_1 = 1$, respectively.*

In the next lemma we consider the signature $\sigma = \pm$.

Lemma 4.4. *The signed graph $\Gamma = (G_{3,s}, \pm)$ has no connected L-cospectral mate, with the exception of $s = 4$.*

Proof. The exception $s = 4$ is a special case of Theorem 3.6. The signed graph $\Gamma = (G_{3,s}, \pm)$ has an unbalanced triangle and a balanced cycle $s = n-2$. By looking to the least Laplacian coefficient we have that $|b_n| = 4s = 4(n-2)$. From Theorem 3.1, the L-cospectral mate has two unbalanced triangles and it is one of the H_i's with $i = 11, 12, 13$ given in Figure 3.

The signed graph $(H_{11}, -)$ has $|b_n| = 16$, which can be equal to that of $\Gamma = (G_{3,s}, \pm)$ for $s = 4$ and $n = 6$, and this leads to the known cospectral mate.

The signed graph $(H_{12}, -)$ has $|b_n| = 4(6 + 4\ell_1)$, and by equating the order and b_n we get that

$$\begin{cases} \ell_1 + \ell + 5 = n \\ 6 + 4\ell_1 = n - 2 \end{cases}$$

From the above system we get $n = 4(\ell_1 + 2)$ and $\ell = 3(\ell_1 + 1)$. However, if $\ell_1 \geq 2$ leads to two copies of $K_{1,3}$ and $\lambda_2(H_{12}, -) \geq 4$, so it is $\ell_1 = 1$. Consequently $\ell = 6$ and $n = 12$. The so obtained signed graph is not a cospectral mate of $\Gamma = (G_{3,10}, \pm)$.

Finally, the signed graph $(H_{13}, -)$ has $|b_n| = 4(6+4\ell_1+4\ell_2)$. Similarly to above, we get that $\ell = 3(\ell_1+\ell_2+1)$ and $n = 4(\ell_1 + \ell_2 + 2)$. Also in this case, $\ell_1 > 1$ or $\ell_2 > 1$ leads to two copies of $K_{1,3}$. Hence it is $\ell_1 = \ell_2 = 1$, consequently we get $\ell = 9$ and $n = 16$, but the so obtained signed graph is not cospectral with $\Gamma = (G_{3,14}, \pm)$.

The proof is completed. □

Lemma 4.5. *The signed graph $\Gamma = (G_{3,s}, -)$ has no connected L-cospectral mate, with the exception of $s = 4$.*

Proof. The exception $s = 4$ is a special case of Theorem 3.6, so we can assume that $s \geq 5$. The signed graph $\Gamma = (G_{3,s}, -)$ has an unbalanced triangle and an unbalanced cycle $s = n-2$. By looking to the least Laplacian coefficient we have that $|b_n| = 4(r + s) = 4(n + 1)$. From Theorem 3.1, the L-cospectral mate has two unbalanced triangles and it is one of the H_i's with $i = 11, 12, 13$ given in Figure 4.

The signed graph $(H_{11}, -)$ has $|b_n| = 16$, which can be equal to that of $\Gamma = (G_{3,s}, -)$ for $n = 3$, and this is impossible.

The signed graph $(H_{12}, -)$ has $|b_n| = 4(6 + 4\ell_1)$, and by equating the order and b_n we get that

$$\begin{cases} \ell_1 + \ell + 5 = n \\ 6 + 4\ell_1 = n + 1 \end{cases}$$

From the above system we get $n = 5 + 4\ell_1$ and $\ell = 3\ell_1$. However, if $\ell_1 > 1$ leads to two copies of $K_{1,3}$ and $\lambda_2(H_{12}, -) \geq 4$, so it is $\ell_1 = 1$. Consequently $\ell = 3$ and $n = 9$. The so obtained signed graph is not a cospectral mate of $\Gamma = (G_{3,7}, -)$.

Finally, the signed graph $(H_{13}, -)$ has $|b_n| = 4(6 + 4\ell_1 + 4\ell_2)$. Similarly to above, we get that $\ell = 3(\ell_1 + \ell_2)$ and $n = 4\ell_1 + 4\ell_2 + 5$. Also in this case, $\ell_1 > 1$ or $\ell_2 > 1$ leads to two copies of $K_{1,3}$. Hence it is $\ell_1 = \ell_2 = 1$, consequently we get $\ell = 6$ and $n = 13$, but the so obtained signed graph is not cospectral with $\Gamma = (G_{3,11}, -)$.

The proof is completed. □

With the next lemma we have considered all possible cases.

Lemma 4.6. *The signed graph $\Gamma = (G_{r,3}, \pm)$ has no connected L-cospectral mate, with the exception of $r = 4$.*

Proof. The exception $r = 4$ is a special case of Theorem 3.6, so we can assume that $r \geq 5$. The signed graph $\Gamma = (G_{r,3}, \pm)$ has a balanced triangle and an unbalanced cycle $r = n - 2$. By looking to the least Laplacian coefficient we have that $|b_n| = 12$. From Theorem 3.1, the L-cospectral mate is either triangle-free (so it is one of the H_i's with $i = 6, 7, \ldots, 10$ given in Figure 3), or it has two triangles of different signs (so it is one of the H_i's with $i = 11, 12, 13$ given in Figure 4).

We consider first the former case, so let $\Lambda = (H, \sigma)$ be a L-cospectral mate of $\Gamma = (G_{r,3}, \pm)$ without triangles. So H is one of the graphs in Figure 3, namely one among H_6, H_7, H_8,H_9 and H_{10}. We shall compare the last coefficients of the L-polynomials; recall that for $\Gamma = (G_{r,3}, \pm)$, we have that $|b_n| = 12$.

Assume that the H_i's ($i = 6, 7, \ldots, 10$) have a balanced cycle. We have that $b_n = 4r'$ or $b_n = 4s'$, with $r', s' \geq 4$, and equality is not possible.

So, we next assume that the H_i's ($i = 6, 7, \ldots, 10$) have just unbalanced cycles. We check the least Laplacian coefficient. For the signed graphs $(H_6, -)$, we get $b_n = 4(r' + s' + 4\ell_1)$, for $(H_7, -)$ and $(H_8, -)$, we get $b_n = 4(r' + s' + 4\ell_1 + 4\ell_2)$. Again, being $r', s' \geq 4$, and equality is not possible. It remains to consider the signed graphs $(H_9, -)$ and $(H_{10}, -)$ for which $b_n = 4(r' + s' - 2\ell_1)$. Also in this case, the equality is impossible, since $r', s' \geq \ell_1 + 2$, that yields $b_n \geq 16$.

So it remains to consider the case that $\Lambda = (H, \sigma)$ is one among $(H_{11}, \pm)$, $(H_{11}, \mp)$, $(H_{12}, \pm)$, $(H_{12}, \mp)$, and $(H_{13}, \pm)$, where, if present, $\ell_1 = 1$ and $\ell_2 = 1$ to avoid $2K_{1,3}$ as subgraph. Note that the least Laplacian coefficient is $|b_n| = 12$ for all signed graphs considered in this part of the proof. Therefore, we compare che second least Laplacian coefficient $|b_{n-1}|$. For the graphs $\Gamma = (G_{n-2,3}, \pm)$ we get $|b_{n-1}| = 3n^2 - 6n + 16$. For the H_i's we have the following values in terms of the order n, obtained from Table 2 and Table 3:

$$b_{n-1}(H_{11}, \pm) = 6n^2 - 30n + 60; \quad b_{n-1}(H_{11}, \mp) = 6n^2 - 34n + 76;$$

$$b_{n-1}(H_{12}, \pm) = 6n^2 - 37n + 112; \quad b_{n-1}(H_{12}, \mp) = 6n^2 - 57n + 232;$$

$$b_{n-1}(H_{13}, \pm) = 6n^2 - 57n + 268.$$

By equating the above expressions we get as valid (positive integer) solutions: $n = 6$ for $(H_{11}, \mp)$ which leads to the known cospectral graph, and $n = 8, 9$ for $(H_{12}, \mp)$ which lead to non cospectral graphs.

This ends the proof. □

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