Filomat 32:4 (2018), 1359–1378
https://doi.org/10.2298/FIL1804359M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Approximation Properties of Baskakov-Schurer-Szász-Stancu Operators Based on q-integers

M. Mursaleen[a], **A.A.H. AL-Abied**[a], **Khursheed J. Ansari**[b]

[a] *Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India*
[b] *Department of Mathematics, College of Science, King Khalid University, 61413, Abha, Saudi Arabia*

Abstract. In the present paper, we introduce Stancu type generalization of Baskakov-Schurer-Szász operators based on the q-integers and investigate their approximation properties. We obtain rate of convergence, weighted approximation and Voronovskaya type theorem for new operators. Then we obtain a point-wise estimate using the Lipschitz type maximal function. Furthermore, we study A-statistical convergence of these operators and also, in order to obtain a better approximation.

1. Introduction and Preliminaries

The q -calculus has played an important role in the field of approximation theory since last three decades. In the year of 1987, Lupas was the first to apply the q -calculus in approximation theory. He introduced the q -analogue of the well known Bernstein polynomials [11]. Another remarkable application of the q -calculus advented in the year of 1997 by Phillips [12]. He used the q -calculus to define another interesting q -analogue of the classical Bernstein polynomials. Ostrovska [19] obtained more results on the q - Bernstein polynomials. In recent years, many studies have been done related to this subject [2], [3], [8], [9]. For $f \in C[0,\infty)$, a new type of Baskakov-Szász type operators proposed by Gupta and Srivastava [10] which is defined as

$$S_n(f;x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)f(t)dt, \tag{1}$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},\quad b_{n,k}(t) = \frac{(nt)^k}{k!}e^{-nt}.$$

It is observed from [10] that these operators reproduce only the constant functions. In the last decade lots of work has been done on q -operators and approximation by different types of summability operators. We

2010 *Mathematics Subject Classification*. 41A10, 41A25, 41A36.
Keywords. Baskakov-Szász type operators; q-type operators; rate of convergence; weighted approximation; point-wise estimates; Lipschitz type maximal function; statistical convergence.
Received: 28 April 2017; Accepted: 10 August 2017
Communicated by Dragan S. Djordjević
Email addresses: mursaleenm@gmail.com (M. Mursaleen), abied1979@gmail.com (A.A.H. AL-Abied), ansari.jkhursheed@gmail.com (Khursheed J. Ansari)

refer the recent work in this direction due to Aral [1] and Mursaleen et al [16], [17], [18].

The q -integer $[n]_q$, the q -factorial $[n]_q!$ and the q -binomial coefficients are defined by (see [11])

$$[n]_q := \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\} \\ nn, & \text{if } q = 1, \end{cases} \quad \text{for } n \in \mathbb{N} \text{ and } [0]_q = 0,$$

$$[n]_q! := \begin{cases} [n]_q[n-1]_q \cdots [1]_q, & n \geq 1, \\ 1, & n = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

There are two q -analogues of the exponential function e^z, defined as (see also [11]):

For $| z |< \frac{1}{1-q}$ and $| q |< 1$,

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!} = \frac{1}{1-((1-q)z)_q^{\infty}},$$

and for $| q |< 1$,

where $(1-x)_q^{\infty} = \prod_{j=0}^{\infty}(1-q^j x)$.

$$E_q(z) = \prod_{j=0}^{\infty} \left(1+(1-q)q^j z\right)_q^{\infty} = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{z^k}{[k]_q!} = (1+(1-q)z)_q^{\infty} .$$

The q -improper integral is defined as

$$\int_0^{\infty/A} f(x) d_q t = (1-q) \sum_{n\in\mathbb{Z}} f(\frac{q^n}{A})\frac{q^n}{A}, \quad A > 0.$$

The q -Gamma function is given by

$$\Gamma_q(u) = K(A,u) \int_0^{\infty/A(1-q)} t^{u-1} e_q^{-t} d_q t,$$

where

$$K(A,u) = \frac{A^u}{1+A}\left(1+\frac{1}{A}\right)_q^u (1+A)_q^{1-u}.$$

In particular, for $u \in Z$, $K(A,u) = q^{\frac{u(u-1)}{2}}$ and $K(A,0) = 1$.

In [21], Yüksel introduced the q -Baskakov-Schurer-Szász type operators $S_{n,p}^q(f;x)$, which was generalization of (1).

$$S_{n,p}^q(f;x) = [n+p]_q \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} s_{n,p,k}(t) f(t) d_q t \tag{2}$$

where

$$b_{n,p,k}(x) = \begin{bmatrix} n+p+k-1 \\ k \end{bmatrix}_q q^{k^2} \frac{x^k}{(1+x)^{n+p+k}}$$

$$s_{n,p,k}(t) = \frac{([n+p]_q t)^k}{[k]_q!} e_q^{-[n+p]_q t}.$$

2. Operator and some auxiliary results

Let $p,k\in\mathbb{N}$, $n\in\mathbb{N}\setminus\{0\}$, $A>0$ and f be a real valued continuous function on the interval $[0,\infty)$. For every $n\in\mathbb{N}, 0<q<1$ and $f\in C_\gamma[0,\infty):=\Big\{f\in C[0,\infty): f(t)=O(t^\gamma)$ as $t\to\infty$ for some $\gamma>0\Big\}$.

We introduce the Stancu type generalization of q -Baskakov-Schurer-Szász type linear positive operators as

$$S^{(\alpha,\beta)}_{n,q,p}(f;x)=[n+p]_q\sum_{k=0}^{\infty}b_{n,p,k}(x)\int_0^{\infty/A(1-q)}s_{n,p,k}(t)f\Big(\frac{[n+p]_qt+\alpha}{[n+p]_q+\beta}\Big)d_qt. \tag{3}$$

In case $q=1$, $\alpha=0$, $\beta=0$ and $p=0$, the above operators reduce to the operators (1). For $\alpha=0$, $\beta=0$ the operators (3) reduce to the operators (2).

Lemma 2.1. *(see [21]). For $S^q_{n,p}(t^m;x)$, $m=0,1,2$, one has*

$$\begin{aligned}
(i)\quad S^q_{n,p}(1;x) &= 1,\\
(ii)\quad S^q_{n,p}(t;x) &= \frac{1}{q^2}x+\frac{1}{q[n+p]_q},\\
(iii)\quad S^q_{n,p}(t^2;x) &= \frac{[n+p+1]_q}{q^6[n+p]_q}x^2+\frac{1+2q+q^2}{q^5[n+p]_q}x+\frac{1+q}{q^3[n+p]_q^2},\\
(iv)\quad S^q_{n,p}(t^3;x) &= \frac{[n+p+1]_q[n+p+2]_q}{q^{12}[n+p]_q^2}x^3+\frac{(1+2q+3q^2+2q^3+q^4)[n+p+1]_q}{q^{11}[n+p]_q^2}x^2\\
&\quad+\frac{1+3q+5q^2+5q^3+3q^4+q^5}{q^9[n+p]_q^2}x+\frac{1+2q+2q^2+q^3}{q^6[n+p]_q^3},\\
(v)\quad S^q_{n,p}(t^4;x) &= \frac{[n+p+1]_q[n+p+2]_q[n+p+3]_q}{q^{20}[n+p]_q^3}x^4\\
&\quad+\frac{(1+2q+3q^2+4q^3+3q^4+2q^5+q^6)[n+p+1]_q[n+p+2]_q}{q^{19}[n+p]_q^3}x^3\\
&\quad+\frac{(1+7q^2+11q^3+14q^4+14q^5+11q^6+7q^7+3q^8+q^9)[n+p+1]_q}{q^{17}[n+p]_q^3}x^2\\
&\quad+\frac{1+4q+9q^2+15q^3+19q^4+19q^5+15q^6+9q^7+4q^8+q^9}{q^{14}[n+p]_q^3}x\\
&\quad+\frac{1+3q+5q^2+6q^3+5q^4+3q^5+q^6}{q^{10}[n+p]_q^4}.
\end{aligned}$$

Lemma 2.2. *Let $S^{(\alpha,\beta)}_{n,q,p}(f;x)$ be given by* (3). *Then the followings hold:*

$$\begin{aligned}
(i)\quad S^{(\alpha,\beta)}_{n,q,p}(1;x) &= 1,\\
(ii)\quad S^{(\alpha,\beta)}_{n,q,p}(t;x) &= \frac{[n+p]_q}{q^2([n+p]_q+\beta)}x+\frac{1+q\alpha}{q([n+p]_q+\beta)},
\end{aligned}$$

$$(iii)\quad S_{n,q,p}^{(\alpha,\beta)}(t^2;x) = \frac{[n+p]_q[n+p+1]_q}{q^6([n+p]_q+\beta)^2}x^2 + \frac{(1+2q+q^2+2q^3\alpha)[n+p]_q}{q^5([n+p]_q+\beta)^2}x + \frac{(1+q+2q^2\alpha+q^3\alpha^2)}{q^3([n+p]_q+\beta)^2},$$

$$\begin{aligned}(iv)\quad S_{n,q,p}^{(\alpha,\beta)}(t^3;x) &= \frac{[n+p]_q[n+p+1]_q[n+p+2]_q}{q^{12}([n+p]_q+\beta)^3}x^3 \\ &+\frac{(1+2q+3q^2+2q^3+q^4+3q^5\alpha)[n+p]_q[n+p+1]_q}{q^{11}([n+p]_q+\beta)^3}x^2 \\ &+\frac{(1+3q+5q^2+5q^3+3q^4+q^5+3q^4\alpha(1+2q+q^2)+3q^7\alpha^2)[n+p]_q}{q^9([n+p]_q+\beta)^3}x \\ &+\frac{(1+2q+2q^2+q^3+3q^3\alpha(1+q)+3q^5\alpha^2+q^6\alpha^3}{q^6([n+p]_q+\beta)^3},\end{aligned}$$

$$\begin{aligned}(v)\quad S_{n,q,p}^{(\alpha,\beta)}(t^4;x) &= \frac{[n+p]_q[n+p+1]_q[n+p+2]_q[n+p+3]_q}{q^{20}([n+p]_q+\beta)^4}x^4 \\ &+\frac{[n+p]_q[n+p+1]_q[n+p+2]_q}{q^{19}([n+p]_q+\beta)^4}\Big(1+2q+3q^2+4q^3+3q^4+2q^5+q^6 \\ &+4q^{19}\alpha\Big)x^3 + \frac{[n+p]_q[n+p+1]_q}{q^{17}([n+p]_q+\beta)^4}\Big(1+7q^2+11q^3+14q^4+14q^5 \\ &+11q^6+7q^7+3q^8+q^9+4q^6\alpha(1+2q+3q^2+2q^3+q^4)+6q^{11}\alpha^2\Big)x^2 \\ &+\frac{[n+p]_q}{q^{14}([n+p]_q+\beta)^4}\Big(1+4q+9q^2+15q^3+19q^4+19q^5+15q^6 \\ &+9q^7+4q^8+q^9+4q^5\alpha(1+3q+5q^2+5q^3+3q^4+q^5)+6q^9\alpha^2(1+2q+q^2) \\ &+4q^{12}\alpha^3\Big)x + \frac{1}{q^{10}([n+p]_q+\beta)^4}\Big(1+3q+5q^2+6q^3+5q^4+3q^5 \\ &+q^6+4q^4\alpha(1+2q+2q^2+q^3)+6q^7\alpha^2(1+q)+4q^9\alpha^3+q^{10}\alpha^4\Big).\end{aligned}$$

Proof. (i)

$$\begin{aligned}S_{n,q,p}^{(\alpha,\beta)}(1;x) &= [n+p]_q\sum_{k=0}^{\infty}b_{n,p,k}(x)\int_0^{\infty/A(1-q)}s_{n,p,k}(t)d_qt \\ &= [n+p]_q\sum_{k=0}^{\infty}b_{n,p,k}(x)\frac{\Gamma(k+1)}{[k]!K(A,k+1)[n+p]_q} \\ &= 1.\end{aligned}$$

(ii)

$$\begin{aligned}S_{n,q,p}^{(\alpha,\beta)}(t;x) &= [n+p]_q\sum_{k=0}^{\infty}b_{n,p,k}(x)\int_0^{\infty/A(1-q)}s_{n,p,k}(t)\Big(\frac{[n+p]_qt+\alpha}{[n+p]_q+\beta}\Big)d_qt \\ &= \frac{[n+p]_q}{([n+p]_q+\beta)}\sum_{k=0}^{\infty}b_{n,p,k}(x)\int_0^{\infty/A(1-q)}\frac{([n+p]_qt)^{k+1}}{[k]_q!}e_q^{-[n+p]_qt}d_qt \\ &+\frac{\alpha[n+p]_q}{([n+p]_q+\beta)}\sum_{k=0}^{\infty}b_{n,p,k}(x)\int_0^{\infty/A(1-q)}\frac{([n+p]_qt)^k}{[k]_q!}e_q^{-[n+p]_qt}d_qt\end{aligned}$$

$$
\begin{aligned}
&= \frac{[n+p]_q}{([n+p]_q+\beta)}(S^q_{n,p}(t;x)) + \frac{\alpha}{([n+p]_q+\beta)}(S^q_{n,p}(1;x)) \\
&= \frac{[n+p]_q}{([n+p]_q+\beta)}(\frac{1}{q^2}x + \frac{1}{q[n+p]_q}) + \frac{\alpha}{([n+p]_q+\beta)} \\
&= \frac{[n+p]_q}{q^2([n+p]_q+\beta)}x + \frac{1+q\alpha}{q([n+p]_q+\beta)}.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
S^{(\alpha,\beta)}_{n,q,p}(t^2;x) &= [n+p]_q \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} s_{n,p,k}(t)\Big(\frac{[n+p]_q t+\alpha}{[n+p]_q+\beta}\Big)^2 d_qt \\
&= \frac{[n+p]_q}{([n+p]_q+\beta)^2} \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} \frac{([n+p]_q t)^{k+2}}{[k]_q!} e_q^{-[n+p]_q t} d_qt \\
&+ \frac{2\alpha[n+p]_q}{([n+p]_q+\beta)^2} \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} \frac{([n+p]_q t)^{k+1}}{[k]_q!} e_q^{-[n+p]_q t} d_qt \\
&+ \frac{\alpha^2[n+p]_q}{([n+p]_q+\beta)^2} \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} \frac{([n+p]_q t)^{k}}{[k]_q!} e_q^{-[n+p]_q t} d_qt \\
&= \frac{[n+p]_q^2}{([n+p]_q+\beta)^2}(S^q_{n,p}(t^2;x)) + \frac{2\alpha[n+p]_q}{([n+p]_q+\beta)^2}(S^q_{n,p}(t;x)) + \frac{\alpha^2}{([n+p]_q+\beta)^2}(S^q_{n,p}(1;x)) \\
&= \frac{[n+p]_q^2}{([n+p]_q+\beta)^2}\Big(\frac{[n+p+1]_q}{q^6[n+p]_q}x^2 + \frac{1+2q+q^2}{q^5[n+p]_q}x + \frac{1+q}{q^3([n+p]_q)^2}\Big) \\
&+ \frac{2\alpha[n+p]_q}{([n+p]_q+\beta)^2}\Big(\frac{1}{q^2}x + \frac{1}{q[n+p]_q}\Big) + \frac{\alpha^2}{([n+p]_q+\beta)^2} \\
&= \frac{[n+p]_q[n+p+1]_q}{q^6([n+p]_q+\beta)^2}x^2 + \frac{(1+2q+q^2+2q^3\alpha)[n+p]_q}{q^5([n+p]_q+\beta)^2}x + \frac{(1+q+2q^2\alpha+q^3\alpha^2)}{q^3([n+p]_q+\beta)^2}.
\end{aligned}
$$

(v)

$$
\begin{aligned}
S^{(\alpha,\beta)}_{n,q,p}(t^4;x) &= [n+p]_q \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} s_{n,p,k}(t)\Big(\frac{[n+p]_q t+\alpha}{[n+p]_q+\beta}\Big)^4 d_qt \\
&= \frac{[n+p]_q}{([n+p]_q+\beta)^4} \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} \frac{([n+p]_q t)^{k+4}}{[k]_q!} e_q^{-[n+p]_q t} d_qt \\
&+ \frac{4\alpha[n+p]_q}{([n+p]_q+\beta)^4} \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} \frac{([n+p]_q t)^{k+3}}{[k]_q!} e_q^{-[n+p]_q t} d_qt \\
&+ \frac{6\alpha^2[n+p]_q}{([n+p]_q+\beta)^4} \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} \frac{([n+p]_q t)^{k+2}}{[k]_q!} e_q^{-[n+p]_q t} d_qt \\
&+ \frac{4\alpha^3[n+p]_q}{([n+p]_q+\beta)^4} \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} \frac{([n+p]_q t)^{k+1}}{[k]_q!} e_q^{-[n+p]_q t} d_qt \\
&+ \frac{\alpha^4[n+p]_q}{([n+p]_q+\beta)^4} \sum_{k=0}^{\infty} b_{n,p,k}(x) \int_0^{\infty/A(1-q)} \frac{([n+p]_q t)^{k}}{[k]_q!} e_q^{-[n+p]_q t} d_qt
\end{aligned}
$$

$$= \frac{[n+p]_q^4}{([n+p]_q+\beta)^4}(S_{n,p}^q(t^4;x)) + \frac{4\alpha[n+p]_q^3}{([n+p]_q+\beta)^4}(S_{n,p}^q(t^3;x))$$
$$+\frac{6\alpha^2[n+p]_q^2}{([n+p]_q+\beta)^4}(S_{n,p}^q(t^2;x)) + \frac{4\alpha^3[n+p]_q}{([n+p]_q+\beta)^4}(S_{n,p}^q(t;x))$$
$$+\frac{\alpha^4}{([n+p]_q+\beta)^4}(S_{n,p}^q(1;x))$$
$$= \frac{[n+p]_q[n+p+1]_q[n+p+2]_q[n+p+3]_q}{q^{20}([n+p]_q+\beta)^4}x^4$$
$$+\frac{[n+p]_q[n+p+1]_q[n+p+2]_q}{q^{19}([n+p]_q+\beta)^4}\Big(1+2q+3q^2+4q^3+3q^4+2q^5+q^6$$
$$+4q^{19}\alpha\Big)x^3 + \frac{[n+p]_q[n+p+1]_q}{q^{17}([n+p]_q+\beta)^4}\Big(1+7q^2+11q^3+14q^4+14q^5$$
$$+11q^6+7q^7+3q^8+q^9+4q^6\alpha(1+2q+3q^2+2q^3+q^4)+6q^{11}\alpha^2\Big)x^2$$
$$+\frac{[n+p]_q}{q^{14}([n+p]_q+\beta)^4}\Big(1+4q+9q^2+15q^3+19q^4+19q^5+15q^6$$
$$+9q^7+4q^8+q^9+4q^5\alpha(1+3q+5q^2+5q^3+3q^4+q^5)+6q^9\alpha^2(1+2q+q^2)$$
$$+4q^{12}\alpha^3\Big)x + \frac{1}{q^{10}([n+p]_q+\beta)^4}\Big(1+3q+5q^2+6q^3+5q^4+3q^5$$
$$+q^6+4q^4\alpha(1+2q+2q^2+q^3)+6q^7\alpha^2(1+q)+4q^9\alpha^3+q^{10}\alpha^4\Big).$$

□

Lemma 2.3. *For every $q \in (0,1)$, and $p \in N$, we have*

$$(i)S_{n,q,p}^{(\alpha,\beta)}((t-x);x) = \Big(\frac{[n+p]_q}{q^2([n+p]_q+\beta)}-1\Big)x + \frac{1+q\alpha}{q([n+p]_q+\beta)},$$
$$(ii)S_{n,q,p}^{(\alpha,\beta)}((t-x)^2;x) = \Big(\frac{[n+p]_q[n+p+1]_q}{q^6([n+p]_q+\beta)^2} - \frac{2[n+p]_q}{q^2([n+p]_q+\beta)}+1\Big)x^2$$
$$+\Big(\frac{(1+2q+q^2+2q^3\alpha)[n+p]_q}{q^5([n+p]_q+\beta)^2} - \frac{2+2q\alpha}{q([n+p]_q+\beta)}\Big)x$$
$$+\frac{(1+q+2q^2\alpha+q^3\alpha^2)}{q^3([n+p]_q+\beta)^2}$$
$$=: \beta_{n,q}(x),$$
$$(iii)S_{n,q,p}^{(\alpha,\beta)}((t-x)^4;x) = \Big\{\frac{[n+p]_q[n+p+1]_q[n+p+2]_q}{q^{20}([n+p]_q+\beta)^3}\Big(\frac{[n+p+3]_q}{([n+p]_q+\beta)}-4q^{20}\Big)$$
$$+\frac{[n+p]_q}{([n+p]_q+\beta)}\Big(\frac{3[n+p+1]_q}{([n+p]_q+\beta)}-4q^6\Big)+1\Big\}x^4$$

$$+\Big\{\frac{[n+p]_q[n+p+1]_q}{q^{19}([n+p]_q+\beta)^3}\Big(\Big[(1+2q+3q^2+4q^3+3q^4+2q^5+q^6)$$
$$+4q^{19}\alpha\Big]\frac{[n+p+2]_q}{([n+p]_q+\beta)}-4\Big[q^8(1+2q+3q^2+2q^3+q^4)+3q^{13}\alpha\Big]\Big)$$
$$+\frac{1}{q^5([n+p]_q+\beta)}\Big[\frac{3(1+2q+q^2+2q^3\alpha)}{([n+p]_q+\beta)}-4q^5(1+q\alpha)\Big]\Big\}x^3$$
$$+\Big\{\frac{[n+p]_q}{q^{17}([n+p]_q+\beta)^3}\Big(\Big[(1+7q^2+11q^3+14q^4+14q^5+11q^6+7q^7$$
$$+3q^8+q^9)+4q^6\alpha(1+2q+3q^2+2q^3+q^4)+6q^{11}\alpha^2\Big]\frac{[n+p+1]_q}{([n+p]_q+\beta)}$$
$$-4\Big[q^8(1+3q+5q^2+5q^3+3q^4+q^5)+3q^{12}\alpha(1+2q+q^2)+3q^{15}\alpha^2\Big]\Big)$$
$$+\frac{3(1+2q+2q^2\alpha+q^3\alpha^2)}{q^3([n+p]_q+\beta)^2}\Big\}x^2+\Big\{\frac{1}{q^{14}([n+p]_q+\beta)^3}\Big(\Big[(1+4q+9q^2$$
$$+15q^3+19q^4+19q^5+15q^6+9q^7+4q^8+q^9)+4q^4\alpha(1+3q+5q^2$$
$$+5q^3+3q^4+q^5)+6q^9\alpha^2(1+2q+q^2)+4q^{12}\alpha^3\Big]\frac{[n+p]_q}{([n+p]_q+\beta)}$$
$$-4\Big[q^8(1+2q+2q^2+q^3)+3q^{11}\alpha(1+q)+3q^{13}\alpha^2+q^{14}\alpha^3\Big]\Big)\Big\}x$$
$$+\frac{1}{q^{10}([n+p]_q+\beta)^4}\Big((1+3q+5q^2+6q^3+5q^4+3q^5+q^6)$$
$$+4q^4\alpha(1+2q+2q^2+q^3)+6q^7\alpha^2(1+q)+4q^9\alpha^3+q^{10}\alpha^4\Big).$$

Lemma 2.4. *For every $x\in[0,\infty)$, we have*

$$(i)\ \lim_{n\to\infty}[n]_{q_n}S^{(\alpha,\beta)}_{n,q_n,p}((t-x);x) = 1+\alpha,$$
$$(ii)\ \lim_{n\to\infty}[n]_{q_n}S^{(\alpha,\beta)}_{n,q_n,p}((t-x)^2;x) = x(2+x).$$

Lemma 2.5. *For $f\in C_B[0,\infty)$ (the space of all bounded and uniform continuous functions on $[0,\infty)$ endowed with norm $\| f\|=\sup\{| f(x)| : x\in[0,\infty)\}$), one has*

$$\| S^{(\alpha,\beta)}_{n,q,p}(f;x)\|\le\| f\| .$$

3. Direct Theorems

In this section, we prove some direct theorems for the operators $S^{(\alpha,\beta)}_{n,q,p}(f;x)$.

Theorem 3.1. *Let $0<q_n<1$ and $A>0$. Then for each $f\in C_\gamma[0,\infty)$, the sequence of operators $S^{(\alpha,\beta)}_{n,q_n,p}(f;x)$ converges uniformly to f on $[0,A]$ if and only if $\lim_{n\to\infty}q_n=1$.*

Proof. First, we suppose that $\lim_{n\to\infty}q_n=1$. Then we will show that $S^{(\alpha,\beta)}_{n,q_n,p}(f;x)$ converges to f uniformly on $[0,A]$. Note that for $0<q_n<1$ and $q_n\to1$ for $n\to\infty$, we get $[n+p]_{q_n}\to\infty$ as $n\to\infty$. Now it can be easily seen that $\frac{[n+p]_{q_n}}{[n+p]_{q_n}+\beta}=1+q_n^n\frac{([p]_{q_n}-1)}{[n+p]_{q_n}+\beta}-\frac{\beta}{[n+p]_{q_n}+\beta}$. So when $n\to\infty$, $\frac{[n+p]_{q_n}}{[n+p]_{q_n}+\beta}\to1$ and $\frac{[n+p]_{q_n}}{([n+p]_{q_n}+\beta)^2}\to0$. Using this and the Lemma 2.2, we find that

$$S^{(\alpha,\beta)}_{n,q_n,p}(1;x)\to1,\qquad S^{(\alpha,\beta)}_{n,q_n,p}(t;x)\to x,\qquad S^{(\alpha,\beta)}_{n,q_n,p}(t^2;x)\to x^2,$$

uniformly on $[0,A]$ as $n\to\infty$.

Therefore, the Korovkin's theorem proves that the sequence $S^{(\alpha,\beta)}_{n,q_n,p}(f;x)$ converges uniformly to f on $[0,A]$ provided $f\in C_\gamma[0,\infty)$.

We shall prove the converse by the method of contradiction. Suppose that the sequence (q_n) does not converge to 1 . Then there must exist a subsequence (q_{n_i}) of the sequence (q_n) such that $q_{n_i}\in(0,1)$, $q_{n_i}\to\delta\in[0,1)$ as $i\to\infty$. Then $\frac{1}{[n+p]_{q_{n_i}}}=\frac{1-q_{n_i}}{1-(q_{n_i})^{n_i+p}}\to 1-\delta$ as $i\to\infty$ because $(q_{n_i})^{n_i}\to 0$ as $i\to\infty$.

Now if we choose $n=n_i, q=q_{n_i}$ in $S^{(\alpha,\beta)}_{n,q_n,p}(t;x)$ from Lemma 2.2, then we get $S^{(\alpha,\beta)}_{n,q_n,p}(t;x)=\frac{1}{\delta^2(1+\beta(1-\delta))}x+\frac{1-\delta}{\delta(1+\beta(1-\delta))}+\frac{\alpha(1-\delta)}{(1+\beta(1-\delta))}$, which is different from x when $i\to\infty$, which contradicts our supposition. Hence, $\lim_{n\to\infty} q_n=1$. Thus the theorem is completely proved. □

Theorem 3.2. *(Voronovskaja type theorem) Let $f\in C_\gamma[0,\infty)$ and $q_n\in(0,1)$ be a sequence such that $q_n\to 1$ and $q_n^n\to 0$ as $n\to\infty$ such that $f',f''\in C_\gamma[0,\infty)$ and $x\in[0,\infty)$, then we have*

$$\lim_{n\to\infty}[n]_{q_n}(S^{(\alpha,\beta)}_{n,q_n,p}(f;x)-f(x))=(1+\alpha)f'(x)+\frac{x(2+x)}{2}f''(x).$$

Proof. By Taylor's formula, we have

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2, \tag{4}$$

where $r(t,x)$ is the Peano form of the remainder and $\lim_{t\to x} r(t,x)=0$. Applying $S^{(\alpha,\beta)}_{n,q,p}(f;x)$ to the both sides of (4), we have

$$\begin{aligned}[n]_{q_n}(S^{(\alpha,\beta)}_{n,q_n,p}(f;x)-f(x)) &= [n]_{q_n}f'(x)S^{(\alpha,\beta)}_{n,q_n,p}((t-x);x)+\frac{1}{2}[n]_{q_n}f''(x)S^{(\alpha,\beta)}_{n,q_n,p}((t-x)^2;x)\\ &\quad+[n]_{q_n}S^{(\alpha,\beta)}_{n,q_n,p}((t-x)^2r(t,x);x).\end{aligned}$$

Now, we will show that $[n]_{q_n}S^{(\alpha,\beta)}_{n,q_n,p}((t-x)^2r(t,x);x)\to\infty$ when $n\to\infty$. By using the Cauchy-Schwarz inequality, we have

$$S^{(\alpha,\beta)}_{n,q_n,p}((t-x)^2r(t,x);x)\le\sqrt{S^{(\alpha,\beta)}_{n,q_n,p}(r^2(t,x);x)}\sqrt{S^{(\alpha,\beta)}_{n,q_n,p}((t-x)^4;x)}. \tag{5}$$

We observe that $r^2(x,x)=0$ and $r^2(\cdot,x)\in C_\gamma[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty}[n]_{q_n}S^{(\alpha,\beta)}_{n,q_n,p}(r^2(t,x);x)=r^2(x,x)=0, \tag{6}$$

uniformly with respect to $x\in[0,A]$, where $A>0$. Now from (5), (6) and Lemma 2.4 , we obtain

$$\lim_{n\to\infty}[n]_{q_n}S^{(\alpha,\beta)}_{n,q_n,p}(r(t,x)(t-x)^2;x)=0. \tag{7}$$

Now from (7) and Lemma 2.4, we get the required result. □

4. Local approximation

For $C_B[0,\infty)$, let us consider the following K -functional:

$$K_2(f,\delta)=\inf_{g\in W^2}\{\| f-g\| +\delta\| g''\|\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty): g',g''\in C_B[0,\infty)\}$. By Theorem (2.4) of [4], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta}) \tag{8}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}}\ \sup_{x\in[0,\infty)} \mid f(x+2h)-2f(x+h)+f(x)\mid$$

is the second order modulus of smoothness of $f\in C_B[0,\infty)$. The usual modulus of continuity of $f\in C_B[0,\infty)$ is defined by

$$w(f,\delta) = \sup_{0<h\leq\delta}\ \sup_{x\in[0,\infty)} \mid f(x+h)-f(x)\mid .$$

Theorem 4.1. *Let $f\in C_B[0,\infty)$, and $q\in(0,1)$. Then for every $x\in[0,\infty)$, we have*

$$\mid S_{n,q,p}^{(\alpha,\beta)}(f;x)-f(x)\mid\leq C\omega_2(f,\delta_n(x))+\omega\Big(f,\Big|\frac{1+q\alpha}{q([n+p]_q+\beta)}+\frac{[n+p]_q}{q^2([n+p]_q+\beta)}x-x\Big|\Big), \tag{9}$$

where

$$\delta_n(x)=\sqrt{S_{n,q,p}^{(\alpha,\beta)}((t-x)^2;x)+\Big(\frac{1+q\alpha}{q([n+p]_q+\beta)}+\frac{[n+p]_q}{q^2([n+p]_q+\beta)}x-x\Big)^2}.$$

Proof. Introduce auxiliary operators as follows:

$$\overline{S}_{n,q,p}^{(\alpha,\beta)}(f;x)=S_{n,q,p}^{(\alpha,\beta)}(f;x)-f\Big(\frac{1+q\alpha}{q([n+p]_q+\beta)}+\frac{[n+p]_q}{q^2([n+p]_q+\beta)}x\Big)+f(x).$$

In the light of the Lemma 2.2, it can be easily seen that $\overline{S}_{n,q,p}^{(\alpha,\beta)}(1;x)=1$ and $\overline{S}_{n,q,p}^{(\alpha,\beta)}(t;x)=x$. Now from the Taylor's formula, for $g\in W^2$, we can write

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)\mathrm{d}u.$$

Applying the operators $\overline{S}_{n,q,p}^{(\alpha,\beta)}$ to both sides of the above equation, we get

$$\begin{aligned}
&\overline{S}_{n,q,p}^{(\alpha,\beta)}(g;x)-g(x)\\
&=g'(x)\overline{S}_{n,q,p}^{(\alpha,\beta)}(t-x;x)+\overline{S}_{n,q,p}^{(\alpha,\beta)}\left(\int_x^t(t-u)g''(u)\mathrm{d}u\right)\\
&=\overline{S}_{n,q,p}^{(\alpha,\beta)}\left(\int_x^t(t-u)g''(u)\mathrm{d}u;x\right)\\
&=S_{n,q,p}^{(\alpha,\beta)}\left(\int_x^t(t-u)g''(u)\mathrm{d}u;x\right)-\int_x^{\frac{1+q\alpha}{q([n+p]_q+\beta)}+\frac{[n+p]_q}{q^2([n+p]_q+\beta)}x}\Big(\frac{1+q\alpha}{q([n+p]_q+\beta)}+\frac{[n+p]_q}{q^2([n+p]_q+\beta)}x-u\Big)g''(u)\mathrm{d}u.
\end{aligned}$$

Therefore, we will have

$$\begin{aligned}
&\left|\overline{S}_{n,q,p}^{(\alpha,\beta)}(g;x)-g(x)\right|\\
&\leq S_{n,q,p}^{(\alpha,\beta)}\Big((t-x)^2;x\Big)\parallel g''\parallel+\Big(\frac{1+q\alpha}{q([n+p]_q+\beta)}+\frac{[n+p]_q}{q^2([n+p]_q+\beta)}x-x\Big)^2\parallel g''\parallel\\
&=\delta_n^2(x)\parallel g''\parallel .
\end{aligned}$$

In view of (9), we obtain

$$
\begin{aligned}
\mid S_{n,q,p}^{(\alpha,\beta)}(f;x)-f(x)\mid \quad &\leq \quad \mid \overline{S}_{n,q,p}^{(\alpha,\beta)}(f-g;x)-g(x)\mid + \mid \overline{S}_{n,q,p}^{(\alpha,\beta)}(g;x)-g(x)\mid \\
&+\left|f\left(\frac{1+q\alpha}{q([n+p]_q+\beta)}+\frac{[n+p]_q}{q^2([n+p]_q+\beta)}x\right)-f(x)\right|.
\end{aligned}
$$

By Lemma 2.5

$$
\| \overline{S}_{n,q,p}^{(\alpha,\beta)}(f;x) \| \leq 3 \| f \|,
$$

so, we get

$$
\mid S_{n,q,p}^{(\alpha,\beta)}(f;x)-f(x)\mid \leq 4\| f-g\| +\delta_n^2(x)\| g''\| +\omega\left(f;\left|\frac{1+q\alpha}{q([n+p]_q+\beta)}+\frac{[n+p]_q}{q^2([n+p]_q+\beta)}x-x\right|\right).
$$

On taking the infimum of the right hand side running over all $g\in W^2$ and using the definition of the Peetre's functional, we get

$$
\mid S_{n,q,p}^{(\alpha,\beta)}(f;x)-f(x)\mid \leq 4K_2(f,\delta)+\omega\left(f;\left|\frac{1+q\alpha}{q([n+p]_q+\beta)}+\frac{[n+p]_q}{q^2([n+p]_q+\beta)}x-x\right|\right).
$$

Now in view of (8), we obtain

$$
\mid S_{n,q,p}^{(\alpha,\beta)}(f;x)-f(x)\mid \leq C\omega_2(f,\delta_n(x))+\omega\left(f,\left|\frac{1+q\alpha}{q([n+p]_q+\beta)}+\frac{[n+p]_q}{q^2([n+p]_q+\beta)}x-x\right|\right),
$$

and this completes the proof of the theorem. □

Theorem 4.2. *Let $f\in C_\gamma[0,\infty)$, $q_n\in(0,1)$, such that $q_n\to 1$, as $n\to\infty$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1]\subset[0,\infty)$. Then, we have*

$$
\mid S_{n,q_n,p}^{(\alpha,\beta)}(f;x)-f(x)\mid \leq 4M_f(1+a^2)\beta_{n,q_n}(x)+2\omega_{a+1}\left(f,\ \sqrt{\beta_{n,q_n}(x)}\right)
$$

where $\beta_{n,q_n}(x)$ is defined in Lemma 2.3.

Proof. For $x\in[0,a]$ and $t>a+1$, since $t-x>1$, we have

$$
\mid f(t)-f(x)\mid \leq 4M_f(1+a^2)(t-x)^2+\left(1+\frac{\mid t-x\mid}{\delta}\right)\omega_{a+1}(f,\delta),\ \delta>0.
$$

Thus, by applying the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\mid S_{n,q_n,p}^{(\alpha,\beta)}(f;x)-f(x)\mid \\
&\leq 4M_f(1+a^2)(S_{n,q_n,p}^{(\alpha,\beta)}((t-x)^2;x))+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}(S_{n,q_n,p}^{(\alpha,\beta)}((t-x)^2;x))^{\frac{1}{2}}\right) \\
&=4M_f(1+a^2)\beta_{n,q_n}(x)+2\omega_{a+1}\left(f,\ \sqrt{\beta_{n,q_n}(x)}\right).
\end{aligned}
$$

Choosing $\delta=\sqrt{\beta_{n,q_n}(x)}$ we have desired result. □

5. Weighted approximation

The weighted Korovkin-type theorems were proved by Gadzhiev [7]. A real function $\gamma(x) = 1 + x^2$ is called a weight function if it is continuous on $\mathbb{R}$ and $\lim_{|x|\to\infty} \gamma(x) = \infty, \gamma(x) \geq 1$ for all $x \in \mathbb{R}$.

Let $B_\gamma[0,\infty)$ be the set of all functions f defined on the positive real axis satisfying growth condition $| f(x) | \leq M_f \gamma(x)$, where M_f is a constant depending only on f. $B_\gamma[0,\infty)$ is a normed space with the norm $\| f \|_\gamma = \sup\{| f(x) | / \gamma(x) : x \geq 0\}$, for any $f \in B_\gamma[0,\infty)$. $C_\gamma[0,\infty)$ denotes the subspace of all continuous functions in $B_\gamma[0,\infty)$ and $C^*_\gamma[0,\infty)$ denotes the subspace of all functions $f \in C_\gamma[0,\infty)$ for which $\lim_{|x|\to\infty}(f(x)/\gamma(x))$ exists finitely.

Theorem 5.1. *Let $q_n \in (0,1)$ be a sequence such that $q_n \to 1$ as $n \to \infty$. Then, for every $f \in C^*_\gamma[0,\infty)$, we have*

$$\lim_{n\to\infty} \| S^{(\alpha,\beta)}_{n,q_n,p}(f;x) - f(x) \|_\gamma = 0.$$

Proof. Using theorem in [6], we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \| S^{(\alpha,\beta)}_{n,q_n,p}(t^r;x) - x^r \|_\gamma = 0, \qquad r = 0,1,2. \tag{10}$$

Since, $S^{(\alpha,\beta)}_{n,q_n,p}(1;x) = 1$, the first condition of (10) is satisfied for $r = 0$. Now,

$$\begin{aligned}
\| S^{(\alpha,\beta)}_{n,q_n,p}(t;x) - x \|_\gamma &= \sup_{x\in[0,\infty)} \frac{| S^{(\alpha,\beta)}_{n,q_n,p}(t;x) - x |}{1+x^2} \\
&\leq \left| \frac{[n+p]_{q_n}}{q_n^2([n+p]_{q_n}+\beta)} - 1 \right| \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{1+q_n\alpha}{q_n([n+p]_{q_n}+\beta)} \\
&\leq \left| \frac{[n+p]_{q_n}}{q_n^2([n+p]_{q_n}+\beta)} - 1 \right| + \frac{1+q_n\alpha}{q_n([n+p]_{q_n}+\beta)}
\end{aligned}$$

which implies that the condition in (10) holds for $r = 1$. Similarly, we can write

$$\begin{aligned}
&\| S^{(\alpha,\beta)}_{n,q_n,p}(t;x) - x^2 \|_\gamma \\
&= \sup_{x\in[0,\infty)} \frac{| S^{(\alpha,\beta)}_{n,q_n,p}(t;x) - x^2 |}{1+x^2} \\
&\leq \left| \frac{[n+p]_{q_n}[n+p+1]_{q_n}}{q_n^6([n+p]_{q_n}+\beta)^2} - 1 \right| \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\
&+ \left| \frac{(1+2q_n+q_n^2+2q_n^3\alpha)[n+p]_{q_n}}{q_n^5([n+p]_{q_n}+\beta)^2} \right| \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{(1+q_n+2q_n^2\alpha+q_n^3\alpha^2)}{q_n^3([n+p]_{q_n}+\beta)^2} \\
&\leq \left| \frac{[n+p]_{q_n}[n+p+1]_{q_n}}{q_n^6([n+p]_{q_n}+\beta)^2} - 1 \right| + \left| \frac{(1+2q_n+q_n^2+2q_n^3\alpha)[n+p]_{q_n}}{q_n^5([n+p]_{q_n}+\beta)^2} \right| + \frac{(1+q_n+2q_n^2\alpha+q_n^3\alpha^2)}{q_n^3([n+p]_{q_n}+\beta)^2}.
\end{aligned}$$

Which implies that

$$\lim_{n\to\infty} \| S^{(\alpha,\beta)}_{n,q_n,p}(t^2;x) - x^2 \|_\gamma = 0,$$

(10) holds for $r = 2$. □

Theorem 5.2. *Let $\alpha > 0$, $q_n \in (0,1)$ sequence such that $q_n \to 1$ as $n \to \infty$ and $f \in C^*_\gamma[0,\infty)$. Then, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{| S^{(\alpha,\beta)}_{n,q_n,p}(f;x) - f(x) |}{(1+x^2)^{1+\alpha}} = 0.$$

Proof. Let $x_0 \in [0,\infty)$ be arbitrary but fixed. Then

$$\sup_{x\in[0,\infty)} \frac{| S^{(\alpha,\beta)}_{n,q_n,p}(f;x) - f(x) |}{(1+x^2)^{1+\alpha}}$$

$$\leq \sup_{x\leq x_0} \frac{| S^{(\alpha,\beta)}_{n,q_n,p}(f;x) - f(x) |}{(1+x^2)^{1+\alpha}} + \sup_{x>x_0} \frac{| S^{(\alpha,\beta)}_{n,q_n,p}(f;x) - f(x) |}{(1+x^2)^{1+\alpha}}$$

$$\leq \| S^{(\alpha,\beta)}_{n,q_n,p}(f) - f \|_{C[0,x_0]} + \| f \|_\gamma \sup_{x>x_0} \frac{| S^{(\alpha,\beta)}_{n,q_n,p}(1+t^2;x) |}{(1+x^2)^{1+\alpha}} + \sup_{x>x_0} \frac{| f(x) |}{(1+x^2)^{1+\alpha}}. \tag{11}$$

Since $| f(x) | \leq \| f \|_\gamma (1+x^2)$, we have $\sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}} \leq \frac{\|f\|_\gamma}{(1+x_0^2)^\alpha}$. Let $\varepsilon > 0$ be arbitrary. We can choose x_0 to be so large that

$$\frac{\| f \|_\gamma}{(1+x_0^2)^\alpha} < \frac{\varepsilon}{3}. \tag{12}$$

In view of Theorem 3.1, we obtain

$$\| f \|_\gamma \lim_{n\to\infty} \frac{| S^{(\alpha,\beta)}_{n,q_n,p}(1+t^2;x) |}{(1+x^2)^{1+\alpha}} = \frac{1+x^2}{(1+x^2)^{1+\alpha}} \| f \|_\gamma = \frac{\| f \|_\gamma}{(1+x^2)^\alpha}$$

$$\leq \frac{\| f \|_\gamma}{(1+x_0^2)^\alpha} < \frac{\varepsilon}{3}. \tag{13}$$

Using Theorem 4.2, we can see that the first term of the inequality (11), implies that

$$\| S^{(\alpha,\beta)}_{n,q_n,p}(f) - f \|_{C[0,x_0]} < \frac{\varepsilon}{3}, \qquad as \qquad n\to\infty. \tag{14}$$

Combining (12) and (14), we get the desired result. □

For $f \in C^*_\gamma[0,\infty)$, the weighted modulus of continuity is defined as

$$\Omega_\gamma(f;\delta) = \sup_{x\geq 0, 0\leq h\leq\delta} \frac{| f(x+h) - f(x) |}{1+(x+h)^2}.$$

Lemma 5.3. *For every $f \in C^*_\gamma[0,\infty)$, then*

(*i*) *$\Omega_\gamma(f;\delta)$ is monotone increasing function of δ ,*
(*ii*) $\lim_{\delta\to 0} \Omega_\gamma(f;\delta) = 0$,
(*iii*) *for any $\lambda \in [0,\infty)$, $\Omega_\gamma(f;\lambda\delta) \leq (1+\lambda)\Omega_\gamma(f;\delta)$.*

Theorem 5.4. *If $f \in C^*_\gamma[0,\infty)$, then for sufficiently large n , we have*

$$| S^{(\alpha,\beta)}_{n,q_n,p}(f;x) - f(x) | \leq K(1+x^{2+\lambda})\Omega_\gamma(f;\delta_n), \quad x\in[0,\infty),$$

where $\lambda \geq 1$, $\delta_n = \max\{\alpha_n,\beta_n,\gamma_n\}$ and K is a positive constant independent of f and n .

Proof. From the definition of $\Omega_\gamma(f;\delta)$ and Lemma 5.3, we have

$$\begin{aligned} | f(t) - f(x) | &\leq (1+(x+| t-x |)^2)\Big(1+\frac{| t-x |}{\delta}\Big)\Omega_\gamma(f;\delta) \\ &\leq (1+(2x+t)^2)\Big(1+\frac{| t-x |}{\delta}\Big)\Omega_\gamma(f;\delta) \\ &:= \Phi_x(t)\Big(1+\frac{1}{\delta}\Psi_x(t)\Big)\Omega_\gamma(f;\delta), \end{aligned}$$

where $\Phi_x(t) = (1 + (2x + t)^2)$ and $\Psi_x(t) =| t - x |$. Then, we obtain

$$| S^{(\alpha,\beta)}_{n,q_n,p}(f; x) - f(x) |\leq \Big(S^{(\alpha,\beta)}_{n,q_n,p}(\Phi_x; x) + \frac{1}{\delta_n} S^{(\alpha,\beta)}_{n,q_n,p}(\Phi_x\Psi_x; x)\Big)\Omega_\gamma(f; \delta_n).$$

Now, applying the Cauchy-Schwarz inequality to the second term on the right hand side, we get

$$| S^{(\alpha,\beta)}_{n,q_n,p}(f; x) - f(x) |\leq \Big(S^{(\alpha,\beta)}_{n,q_n,p}(\Phi_x; x) + \frac{1}{\delta_n} \sqrt{S^{(\alpha,\beta)}_{n,q_n,p}(\Psi^2_x; x)}\sqrt{S^{(\alpha,\beta)}_{n,q_n,p}(\Phi^2_x; x)}\Big)\Omega_\gamma(f; \delta_n). \quad (15)$$

From Lemma 2.2

$$\begin{aligned}\frac{1}{1+x^2} S^{(\alpha,\beta)}_{n,q_n,p}(1 + t^2; x) &= \frac{1}{1+x^2} + \Big(\frac{[n + p]_{q_n}[n + p + 1]_{q_n}}{q_n^6([n + p]_{q_n} + \beta)^2}\Big)\frac{x^2}{1+x^2} \\ &\quad + \Big(\frac{(1 + 2q_n + q_n^2 + 2q_n^3\alpha)[n + p]_{q_n}}{q_n^5([n + p]_{q_n} + \beta)^2}\Big)\frac{x}{1+x^2} \\ &\quad + \Big(\frac{(1 + q_n + 2q_n^2\alpha + q_n^3\alpha^2)}{q_n^3([n + p]_{q_n} + \beta)^2}\Big)\frac{1}{1+x^2} \\ &\leq 1 + C_1, \qquad \textit{for sufficiently large } n, \end{aligned} \quad (16)$$

where C_1 is a positive constant. From (16) , there exists a positive constant K_1 such that $S^{(\alpha,\beta)}_{n,q_n,p}(\Phi_x; x) \leq K_1(1 + x^2)$, for sufficiently large n . Proceeding similarly, $\frac{1}{1+x^4} S^{(\alpha,\beta)}_{n,q_n,p}(1 + t^4; x) \leq 1 + C_2$, for sufficiently large n , where C_2 is a positive constant. So there exists a positive constant K_2 such that $\sqrt{S^{(\alpha,\beta)}_{n,q_n,p}(\Phi^2_x; x)} \leq K_2(1 + x^2)$, where $x \in [0, \infty)$ and n is large enough. Also, we get

$$\begin{aligned} S^{(\alpha,\beta)}_{n,q_n,p}(\Psi^2_x; x) &= \Big(\frac{[n + p]_{q_n}[n + p + 1]_{q_n}}{q_n^6([n + p]_{q_n} + \beta)^2} - \frac{2[n + p]_{q_n}}{q_n^2([n + p]_{q_n} + \beta)} + 1\Big)x^2 \\ &\quad + \Big(\frac{(1 + 2q_n + q_n^2 + 2q_n^3\alpha)[n + p]_{q_n}}{q_n^5([n + p]_{q_n} + \beta)^2} - \frac{2 + 2q_n\alpha}{q_n([n + p]_{q_n} + \beta)}\Big)x \\ &\quad + \frac{(1 + q_n + 2q_n^2\alpha + q_n^3\alpha^2)}{q_n^3([n + p]_{q_n} + \beta)^2} \\ &\leq \alpha_n x^2 + \beta_n x + \gamma_n. \end{aligned}$$

Hence, from (15) , we have

$$| S^{(\alpha,\beta)}_{n,q_n,p}(f; x) - f(x) |\leq (1 + x^2)\Big(K_1 + \frac{1}{\delta_n} K_2 \sqrt{\alpha_n x^2 + \beta_n x + \gamma_n}\Big)\Omega_\gamma(f; \delta_n).$$

If we take $\delta_n = \max\{\alpha_n, \beta_n, \gamma_n\}$, then we get

$$\begin{aligned} | S^{(\alpha,\beta)}_{n,q_n,p}(f; x) - f(x) | &\leq (1 + x^2)\Big(K_1 + K_2\sqrt{x^2 + x + 1}\Big)\Omega_\gamma(f; \delta_n) \\ &\leq K_3(1 + x^{2+\lambda})\Omega_\gamma(f; \delta_n), \end{aligned}$$

for sufficiently large n and $x \in [0, \infty)$. Hence, the proof is completed. □

6. Point-wise estimates

In this section, we establish some point-wise estimates of the rate of convergence of the operators $S_{n,q,p}^{(\alpha,\beta)}$. First, we give the relationship between the local smoothness of f and local approximation. We know that a function $f \in C[0,\infty)$ is in $Lip_M(\eta)$ on E, $\eta \in (0,1]$, E be any bounded subset of the interval $[0,\infty)$ if it satisfies the condition

$$| f(t) - f(x) |\leq M \mid t - x \mid^{\eta}, \quad t \in [0,\infty) \quad and \quad x \in E,$$

where M is a constant depending only on η and f.

Theorem 6.1. *Let $f \in Lip_M(\eta)$, $E \in [0,\infty)$ and $\eta \in (0,1]$. Then, we have*

$$| S_{n,q,p}^{(\alpha,\beta)}(f;x) - f(x) |\leq M\Big(\delta_n^{\frac{\eta}{2}}(x) + 2d^{\eta}(x,E)\Big), \qquad x \in [0,\infty),$$

where

$$\begin{aligned}\delta_n^{\frac{\eta}{2}}(x) &= \Big(\frac{[n+p]_{q_n}[n+p+1]_{q_n}}{q_n^6([n+p]_{q_n}+\beta)^2} - \frac{2[n+p]_{q_n}}{q_n^2([n+p]_{q_n}+\beta)} + 1\Big)x^2 \\ &+\Big(\frac{(1+2q_n+q_n^2+2q_n^3\alpha)[n+p]_{q_n}}{q_n^5([n+p]_{q_n}+\beta)^2} - \frac{2+2q_n\alpha}{q_n([n+p]_{q_n}+\beta)}\Big)x \\ &+\frac{(1+q_n+2q_n^2\alpha+q_n^3\alpha^2)}{q_n^3([n+p]_{q_n}+\beta)^2},\end{aligned}$$

where M is a constant depending on η and f, and $d(x,E)$ is the distance between x and E defined as $d(x,E) = \inf\{| t - x |: t \in E\}$.

Proof. Let $\overline{E}$ be the closure of E in $[0,\infty)$. Then there exists at least one point $x_0 \in \overline{E}$ such that $d(x,E) =| x - x_0 |$,

$$| f(t) - f(x) | \leq| f(t) - f(x_0) | + | f(x_0) - f(x) | .$$

By our hypothesis and the monotonicity of $S_{n,q,p}^{(\alpha,\beta)}$, we get

$$\begin{aligned}| S_{n,q,p}^{(\alpha,\beta)}(f;x) - f(x) | &\leq S_{n,q,p}^{(\alpha,\beta)}(| f(t) - f(x_0) |;x) + S_{n,q,p}^{(\alpha,\beta)}(| f(x) - f(x_0) |;x) \\ &\leq M\{S_{n,q,p}^{(\alpha,\beta)}(| t - x_0 |^{\eta};x)+ | x - x_0 |^{\eta}\} \\ &\leq M\{S_{n,q,p}^{(\alpha,\beta)}(| t - x |^{\eta};x) + 2 | x - x_0 |^{\eta}\}.\end{aligned}$$

Now, applying Hölder's inequality with $p = \frac{2}{\eta}$ and $\frac{1}{q} = 1 - \frac{1}{p}$, we obtain

$$| S_{n,q,p}^{(\alpha,\beta)}(f;x) - f(x) |\leq M\Big\{\Big[S_{n,q,p}^{(\alpha,\beta)}(| t - x |^2;x)\Big]^{\frac{\eta}{2}} +2d^{\eta}(x,E)\Big\},$$

from which the desired result immediate. □

Next, we obtain the local direct estimate of the operators defined in (3), using the Lipschitz-type maximal function of order η introduced by Lenze [14] as

$$\widetilde{\omega}_{\eta}(f;x) = \sup_{t\neq x, t\in[0,\infty)} \frac{| f(t) - f(x) |}{| t - x |^{\eta}}, \quad x \in [0,\infty) \quad and \quad \eta \in (0,1]. \tag{17}$$

Theorem 6.2. *Let $f \in C_B[0,\infty)$, $0 < \eta \leq 1$ and $q_n(0,1)$ such that $q_n \to 1$ as $n \to \infty$. Then for every $x \in [0,\infty)$, we have*

$$| S^{(\alpha,\beta)}_{n,q_n,p}(f;x) - f(x) | \leq \widetilde{\omega}_\eta(f;x)\delta_n^{\frac{\eta}{2}}(x),$$

where $\delta_n(x) = S^{(\alpha,\beta)}_{n,q_n,p}((t-x)^2;x)$.

Proof. In the light of the Lemma 2.2 , we have

$$\begin{aligned} | S^{(\alpha,\beta)}_{n,q_n,p}(f;x) - f(x) | &\leq S^{(\alpha,\beta)}_{n,q_n,p}(| f(t) - f(x) |;x) \\ &\leq \widetilde{\omega}_\eta(f;x) S^{(\alpha,\beta)}_{n,q_n,p}(| t-x |^\eta;x) \end{aligned}$$

and in view of (17) , we have

$$| f(t) - f(x) | \leq \widetilde{\omega}_\eta(f;x) | t-x |^\eta .$$

When we use the Hölder's inequality with $p = \frac{2}{\eta}$ and $\frac{1}{q} = 1 - \frac{1}{p}$, we obtain

$$\begin{aligned} | S^{(\alpha,\beta)}_{n,q_n,p}(f;x) - f(x) | &\leq \widetilde{\omega}_\eta(f;x) S^{(\alpha,\beta)}_{n,q_n,p}((| t-x |^2;x))^{\frac{\eta}{2}} \\ &\leq \widetilde{\omega}_\eta(f;x)\delta_n^{\frac{\eta}{2}}(x). \end{aligned}$$

Thus, the proof is completed. □

7. Statistical convergence

Kolk [13] introduced the notion of A -statistical convergence by taking an arbitrary nonnegative regular matrix A . Let $A = (a_{nk})$ be a non-negative infinite summability matrix. For a given sequence $x := (x)_n$, the A -transform of x denoted by $Ax : (Ax)_n$ is defined as

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k,$$

provided the series converges for each n . A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim_n (x)_n = L$. Then $x = (x)_n$ is said to be A -statistically convergent to L , i.e., $st_A - \lim_n (x)_n = L$ if for every $\varepsilon > 0$, $\lim_n \sum_{k:|x_k - L| \geq \varepsilon} a_{nk} = 0$. If we replace A by C_1 then A is a Cesaro matrix of order one and A -statistical convergence is reduced to the statistical convergence [5]. Similarly, if $A = I$, the identity matrix, then A -statistical convergence is called ordinary convergence. Now we take a sequence q_n such that $q_n \in (0,1)$ satisfying the following:

$$st_A - \lim_n q_n = 1, \quad st_A - \lim_n q_n^n = a \in (0,1), \quad st_A - \lim_n \frac{1}{[n]_{q_n}} = 0. \tag{18}$$

Theorem 7.1. *Let $A = (a_{nk})$ be a non-negative regular summability matrix and q_n be a sequence satisfying the above conditions. Then for any $f \in C_\gamma[0,\infty)$, we have*

$$st_A - \lim_n \| S^{(\alpha,\beta)}_{n,q_n,p}(f;\cdot) - f \| = 0.$$

Proof. Let $e_i(x) = x^i$, where $i = 0,1,2$. Then, from Lemma 2.2 (i), we have

$$st_A - \lim_n \| S^{(\alpha,\beta)}_{n,q_n,p}(e_0;\cdot) - e_0 \| = 0.$$

Next, again from Lemma 2.2 (ii), we have

$$\lim_n \| S^{(\alpha,\beta)}_{n,q_n,p}(e_1;\cdot) - e_1 \| \leq \left| \frac{[n+p]_{q_n}}{q_n^2([n+p]_{q_n}+\beta)} - 1 \right| + \left| \frac{1+q_n\alpha}{q_n([n+p]_{q_n}+\beta)} \right|$$

and in view of (18) , we have

$$st_A - \lim_n \left(\frac{1}{q_n^2} \frac{1-q_n^{n+p}}{1-q_n^{n+p}+\beta(1-q_n)} - 1 \right) = 0$$

and

$$st_A - \lim_n \left(\frac{1+q_n\alpha}{q_n([n+p]_{q_n}+\beta)} \right) = 0.$$

Now, for a given $\varepsilon > 0$, let us define the following sets:

$$\begin{aligned} U &= \left\{ n \in N : \| S^{(\alpha,\beta)}_{n,q_n,p}(e_1;\cdot) - e_1 \| \geq \varepsilon \right\}, \\ U_1 &= \left\{ n \in N : \frac{1}{q_n^2} \frac{1-q_n^{n+p}}{1-q_n^{n+p}+\beta(1-q_n)} - 1 \geq \frac{\varepsilon}{2} \right\}, \\ U_2 &= \left\{ n \in N : \frac{1+q_n\alpha}{q_n([n+p]_{q_n}+\beta)} \geq \frac{\varepsilon}{2} \right\}. \end{aligned}$$

The containment $U \subseteq U_1 \cup U_2$ is obvious which in turn implies that

$$\sum_{n\in U} a_{nk} \leq \sum_{n\in U_1} a_{nk} + \sum_{n\in U_2} a_{nk},$$

and hence, we have

$$st_A - \lim_n \| S^{(\alpha,\beta)}_{n,q_n,p}(e_1;\cdot) - e_1 \| = 0.$$

Further, using Lemma 2.2 (iii), we have

$$\begin{aligned} &\| S^{(\alpha,\beta)}_{n,q_n,p}(e_2;\cdot) - e_2 \| \\ &\leq \left| \frac{(1-q_n^{n+p})(1-q_n^{n+p+1})}{q_n^6(1-q_n^{n+p}+\beta(1-q_n))^2} - 1 \right| + \left| \frac{(1+2q_n+q_n^2+2q_n^3\alpha)(1-q_n^{n+p})(1-q_n)}{q_n^5(1-q_n^{n+p}+\beta(1-q_n))^2} \right| + \left| \frac{(1+q_n+2q_n^2\alpha+q_n^3\alpha^2)}{q_n^3([n+p]_{q_n}+\beta)^2} \right|. \end{aligned}$$

From (18) , we have

$$\begin{aligned} st_A - \lim_n \left(\frac{(1-q_n^{n+p})(1-q_n^{n+p+1})}{q_n^6(1-q_n^{n+p}+\beta(1-q_n))^2} - 1 \right) &= 0, \\ st_A - \lim_n \left(\frac{(1+2q_n+q_n^2+2q_n^3\alpha)(1-q_n^{n+p})(1-q_n)}{q_n^5(1-q_n^{n+p}+\beta(1-q_n))^2} \right) &= 0, \\ st_A - \lim_n \left(\frac{(1+q_n+2q_n^2\alpha+q_n^3\alpha^2)}{q_n^3([n+p]_{q_n}+\beta)^2} \right) &= 0. \end{aligned}$$

Now, for each $\varepsilon > 0$, we define the following sets:

$$\begin{aligned} V &= \left\{ n \in N : \| S^{(\alpha,\beta)}_{n,q_n,p}(e_2;\cdot) - e_2 \| \geq \varepsilon \right\}, \\ V_1 &= \left\{ n \in N : \frac{(1-q_n^{n+p})(1-q_n^{n+p+1})}{q_n^6(1-q_n^{n+p}+\beta(1-q_n))^2} - 1 \geq \frac{\varepsilon}{3} \right\}, \\ V_2 &= \left\{ n \in N : \frac{(1+2q_n+q_n^2+2q_n^3\alpha)(1-q_n^{n+p})(1-q_n)}{q_n^5(1-q_n^{n+p}+\beta(1-q_n))^2} \geq \frac{\varepsilon}{3} \right\}, \\ V_3 &= \left\{ n \in N : \frac{(1+q_n+2q_n^2\alpha+q_n^3\alpha^2)}{q_n^3([n+p]_{q_n}+\beta)^2} \geq \frac{\varepsilon}{3} \right\}. \end{aligned}$$

The containment $V \subseteq V_1 \cup V_2 \cup V_3$ is obvious which in turn implies that

$$\sum_{n\in V} a_{nk} \leq \sum_{n\in V_1} a_{nk} + \sum_{n\in V_2} a_{nk} + \sum_{n\in V_3} a_{nk}.$$

Therefore, we get

$$st_A - \lim_n \| S^{(\alpha,\beta)}_{n,q_n,p}(e_2;\cdot) - e_2 \| = 0.$$

This completes the proof. □

Theorem 7.2. *Let $A = (a_{nk})$ be a non-negative regular summability matrix and q_n be a sequence satisfying* (18). *Let the operators $S^{(\alpha,\beta)}_{n,q_n,p}$, $n \in N$ be defined as in* (3). *Then, for each function $f \in C_\gamma[0,\infty)$, we have*

$$st_A - \lim_n \| S^{(\alpha,\beta)}_{n,q_n,p}(f;\cdot) - f \|_\gamma = 0,$$

where $\gamma(x) = 1 + x^{2+\lambda}$, $\lambda > 0$.

Proof. It is sufficient to prove that $st_A - \lim_n \| S^{(\alpha,\beta)}_{n,q_n,p}(e_i;\cdot) - e_i \|_\gamma = 0$, where $e_i(x) = x^i$, $i = 0, 1, 2$. Then, from the Lemma 2.2 (i), we have

$$st_A - \lim_n \| S^{(\alpha,\beta)}_{n,q_n,p}(e_0;\cdot) - e_0 \|_\gamma = 0.$$

Next, again from the Lemma 2.2 (ii), we have

$$\begin{aligned} \| S^{(\alpha,\beta)}_{n,q_n,p}(e_1;\cdot) - e_1 \|_\gamma &\leq \sup_{x\in[0,\infty)} \left\{ \frac{x}{1+x^2} \left| \frac{[n+p]_{q_n}}{q_n^2([n+p]_{q_n}+\beta)} - 1 \right| + \frac{1}{1+x^2} \frac{1+q_n\alpha}{q_n([n+p]_{q_n}+\beta)} \right\} \\ &\leq \left| \frac{[n+p]_{q_n}}{q_n^2([n+p]_{q_n}+\beta)} - 1 \right| + \frac{1+q_n\alpha}{q_n([n+p]_{q_n}+\beta)}. \end{aligned}$$

For each $\varepsilon > 0$, let us define the following sets:

$$\begin{aligned} G &= \left\{ k : \| S^{(\alpha,\beta)}_{n,q_k,p}(e_1;\cdot) - e_1 \| \geq \varepsilon \right\}, \\ G_1 &= \left\{ k : \frac{1}{q_k^2} \frac{1-q_k^{k+p}}{1-q_k^{k+p}+\beta(1-q_k)} - 1 \geq \frac{\varepsilon}{2} \right\}, \\ G_2 &= \left\{ k : \frac{1+q_k\alpha}{q_k([k+p]_{q_k}+\beta)} \geq \frac{\varepsilon}{2} \right\}. \end{aligned}$$

The containment $G \subseteq G_1 \cup G_2$ is obvious which in turn implies that

$$\sum_{k\in G} a_{nk} \leq \sum_{k\in G_1} a_{nk} + \sum_{k\in G_2} a_{nk}.$$

Hence, on taking the limit as $n \to \infty$,

$$st_A - \lim_n \parallel S^{(\alpha,\beta)}_{n,q_n,p}(e_1;\cdot) - e_1 \parallel_\gamma = 0.$$

Proceeding similarly,

$$\begin{aligned}&\parallel S^{(\alpha,\beta)}_{n,q_n,p}(e_2;\cdot) - e_2 \parallel_\gamma \\ &\leq \left|\frac{(1-q_n^{n+p})(1-q_n^{n+p+1})}{q_n^6(1-q_n^{n+p}+\beta(1-q_n))^2} - 1\right| + \left|\frac{(1+2q_n+q_n^2+2q_n^3\alpha)(1-q_n^{n+p})(1-q_n)}{q_n^5(1-q_n^{n+p}+\beta(1-q_n))^2}\right| \\ &+ \left|\frac{(1+q_n+2q_n^2\alpha+q_n^3\alpha^2)}{q_n^3([n+p]_{q_n}+\beta)^2}\right|.\end{aligned}$$

Now, let us define the following sets:

$$\begin{aligned} M &= \left\{k : \parallel S^{(\alpha,\beta)}_{n,q_k,p}(e_2;\cdot) - e_2 \parallel \geq \varepsilon\right\}, \\ M_1 &= \left\{k : \frac{(1-q_k^{k+p})(1-q_k^{k+p+1})}{q_k^6(1-q_k^{k+p}+\beta(1-q_k))^2} - 1 \geq \frac{\varepsilon}{3}\right\}, \\ M_2 &= \left\{k : \frac{(1+2q_k+q_k^2+2q_k^3\alpha)(1-q_k^{k+p})(1-q_k)}{q_k^5(1-q_k^{k+p}+\beta(1-q_k))^2} \geq \frac{\varepsilon}{3}\right\}, \\ M_3 &= \left\{k : \frac{(1+q_k+2q_k^2\alpha+q_k^3\alpha^2)}{q_k^3([k+p]_{q_k}+\beta)^2} \geq \frac{\varepsilon}{3}\right\}. \end{aligned}$$

Then we obtain $M \subseteq M_1 \cup M_2 \cup M_3$ which implies that

$$\sum_{k\in M} a_{nk} \leq \sum_{k\in M_1} a_{nk} + \sum_{k\in M_2} a_{nk} + \sum_{k\in M_3} a_{nk}.$$

Hence, on taking the limit as $n \to \infty$,

$$st_A - \lim_n \parallel S^{(\alpha,\beta)}_{n,q_n,p}(e_2;\cdot) - e_2 \parallel_\gamma = 0.$$

This completes the proof of the theorem. □

8. Better estimates

It is well known that the classical Bernstein polynomials preserve constant as well as linear functions. To make the convergence faster, King [12] proposed an approach to modify the Bernstein polynomials, so that the sequence preserves test functions e_0 and e_2 , where $e_i(t) = t^i$, $i = 0, 1, 2$. As the operator $S^{(\alpha,\beta)}_{n,q,p}(f;x)$ defined in (3) reproduces only constant functions, this motivated us to propose the modification of this operator, so that it can preserve constant as well as linear functions. The modification of the operators given in (3) is defined as

$$\widetilde{S}^{(\alpha,\beta)}_{n,q,p}(f;x) = [n+p]_q \sum_{k=0}^{\infty} b_n\Big(r^q_{n,p}(x)\Big) \int_0^{\infty/A(1-q)} s_{n,p,k}(t) f\Big(\frac{[n+p]_q t+\alpha}{[n+p]_q+\beta}\Big) d_q t$$

where $r^q_{n,p}(x) = \frac{q^2([n+p]_q+\beta)x - q(1+q\alpha)}{[n+p]_q}$ for $x \in I_n = [\frac{\alpha}{[n+p]_q+\beta}, \infty)$.

Lemma 8.1. *For each $x \in I_n$, by simple computations, we have*

$$(i) \qquad \widetilde{S}_{n,q,p}^{(\alpha,\beta)}(1;x) = 1,$$

$$(ii) \qquad \widetilde{S}_{n,q,p}^{(\alpha,\beta)}(t;x) = x,$$

$$(iii) \qquad \widetilde{S}_{n,q,p}^{(\alpha,\beta)}(t^2;x)$$

$$= \frac{[n+p+1]_q}{q^2[n+p]_q}x^2 + \frac{(1+2q+q^2+2q^3\alpha)[n+p]_q - 2(1+q\alpha)[n+p+1]_q}{q^3([n+p]_q+\beta)[n+p]_q}x$$
$$+\frac{(1+q\alpha)^2[n+p+1]_q - [n+p]_q((1+q\alpha)(1+2q+q^2+2q^3\alpha)+q(1+q+2q^2\alpha+q^3\alpha^2))}{q^4([n+p]_q+\beta)^2[n+p]_q}.$$

Consequently, for each $x \in I_n$, we have the following equalities:

$$\widetilde{S}_{n,q,p}^{(\alpha,\beta)}(t-x;x) = 0,$$

$$\widetilde{S}_{n,q,p}^{(\alpha,\beta)}((t-x)^2;x) = \left(\frac{[n+p+1]_q}{q^2[n+p]_q} - 1\right)x^2 + \frac{(1+2q+q^2+2q^3\alpha)[n+p]_q - 2(1+q\alpha)[n+p+1]_q}{q^3([n+p]_q+\beta)[n+p]_q}x$$
$$+ \frac{(1+q\alpha)^2[n+p+1]_q - [n+p]_q((1+q\alpha)(1+2q+q^2+2q^3\alpha)+q(1+q+2q^2\alpha+q^3\alpha^2))}{q^4([n+p]_q+\beta)^2[n+p]_q},$$
$$=: \xi_n(x). \tag{19}$$

Theorem 8.2. *Let $f \in C_B[0,\infty)$, and $q \in (0,1)$. Then for every $x \in I_n$, there exists a positive constant C such that*

$$| \widetilde{S}_{n,q,p}^{(\alpha,\beta)}(f;x) - f(x) | \leq C\omega_2(f, \sqrt{\xi_n(x)}))$$

where $\xi_n(x)$ is given by (19).

Proof. Now from the Taylor's formula, for $g \in W^2$, $x \in I_n$ and $t \in [0,\infty)$, we can write

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u.$$

Applying the operators $\widetilde{S}_{n,q,p}^{(\alpha,\beta)}$ to both sides of the above equation we get

$$\widetilde{S}_{n,q,p}^{(\alpha,\beta)}(g;x) - g(x) = g'(x)\widetilde{S}_{n,q,p}^{(\alpha,\beta)}(t-x;x) + \widetilde{S}_{n,q,p}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right).$$

Obviously, we have

$$| \int_x^t (t-u)g''(u)\mathrm{d}u | \leq (t-x)^2 \parallel g'' \parallel .$$

Therefore

$$| \widetilde{S}_{n,q,p}^{(\alpha,\beta)}(g;x) - g(x) | \leq \widetilde{S}_{n,q,p}^{(\alpha,\beta)}((t-x)^2;x) \parallel g'' \parallel = \xi_n(x) \parallel g'' \parallel .$$

Since

$$\widetilde{S}_{n,q,p}^{(\alpha,\beta)}(f;x) \leq \parallel f \parallel,$$

we get

$$\begin{aligned} | \widetilde{S}_{n,q,p}^{(\alpha,\beta)}(f;x) - f(x) | &\leq | \widetilde{S}_{n,q,p}^{(\alpha,\beta)}(f-g;x) | + | (f-g)(x) | + | \widetilde{S}_{n,q,p}^{(\alpha,\beta)}(g;x) - g(x) | \\ &\leq 2 \parallel f-g \parallel + \xi_n(x) \parallel g'' \parallel . \end{aligned}$$

Finally, taking the infimum over all $g \in W^2$, and using (8) we obtain

$$| \widetilde{S}_{n,q,p}^{(\alpha,\beta)}(f;x) - f(x) | \leq C\omega_2(f, \sqrt{\xi_n(x)})),$$

which proves the theorem. □

9. Acknowledgement

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research groups program under grant number R.G.P. 1/13/38.

References

[1] A. Aral, A generalization of Szász-Mirakyan operators based on q -integers, Mathematical and Computer Modelling 47 (2008) 1052-1062.
[2] A. Aral, T. Acar, Voronovskaya type result for q -derivative of q -Baskakov operators, J. Appl. Funct. Anal. 7 (2012) 321-331.
[3] A. Aral, V. Gupta, On the Durrmeyer type modification of the q -Baskakov type operators, Nonlinear Anal. 72 (2010) 1171-1180.
[4] R.A. Devore, G.G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.
[5] H. Fast, Sur la convergence statistique. Colloq. Math. 2 (1951) 241–244.
[6] A.D. Gadzhiev, Theorems of the type of P.P. Korovkin's theorems (Russian), presented at the international conference on the theory of approximation of functions (Kaluga, 1975). Mat. Zametki 20(5), (1976) 781-786.
[7] A.D. Gadzhiev, A problem on the convergence of a sequence of positive linear operators on unbounded sets, and theorems that are analogous to P.P. Korovkin's theorem. (Russian) Dokl. Akad. Nauk SSSR 218 (1974) 1001-1004.
[8] V. Gupta, A note on q -Baskakov-Szász operators, Lobachevskii J. Math. 31 (2010) 359-366.
[9] V. Gupta, A. Aral, Convergence of the q -analogue of Szász-beta operators, Appl. Math. Comput. 216 (2010) 374-380.
[10] V. Gupta, G.S. Srivastava, Simultaneous approximation by Baskakov-Szász type operators, Bull. Math. Dela Soc. Sci. Math. de Roumanie (N.S.) 37 (1993).
[11] V. Kac, P. Cheung, Quantum Calculus. Springer-Verlag New York, 2002.
[12] J.P. King, Positive linear operators which preserve x^2 . Acta Mathematica Hungarica 99.3 (2003): 203-208.
[13] E. Kolk, Matrix summability of statistically convergent sequences. Anal. (Munich) 13 (1993) 77–83.
[14] B. Lenze, On Lipschitz type maximal functions and their smoothness spaces. Indag. Math. 50(1), (1988) 53-63 .
[15] A. Lupaş, A q -analogue of the Bernstein operator, Rocky Mountain J. Math. 36(5)(2006) 1615-1629.
[16] M. Mursaleen, K.J. Ansari, Asif Khan, Approximation by a Kantorovich type q -Bernstein-Stancu operators, Complex Analysis and Operator Theory, 11(1) (2017) 85–107.
[17] M. Mursaleen, K.J. Ansari, Approximation of q -Stancu-Beta operators which preserve x^2 , Bull. Malaysian Math. Sci. Soc., DOI: 10.1007/s40840-015-0146-9.
[18] M. Mursaleen, Khursheed J. Ansari, Md. Nasiruzzaman, Approximation by q -analogue of Jakimovski-Leviatan operators involving q -Appell polynomials, Iranian Jour. Sci. Tech. Transaction A: Science (Accepted).
[19] S. Ostrovska, q -Bernstein polynomials and their iterates, J. Approx. Theory 123(2)(2003) 232-255.
[20] G. M. Phillips, Bernstein polynomials based on the q -integers, Ann. Numer. Math. 4(1997) 511-518.
[21] İ. Yüksel . Approximation by q -Baskakov-Schurer-Szász type operators. In Simos T, Psihoyios G, Tsitouras C, editors. AIP Conference Proceedings 2013 Oct 17 (Vol. 1558, No. 1, pp. 1136-1139).