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Three-Point Fractional h-sum Boundary Value Problems for Sequential Caputo Fractional h-sum-difference Equations

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Abstract. In this article, we study an existence and uniqueness results for a sequential nonlinear Caputo fractional h-sum-difference equation with three-point fractional h-sum boundary conditions, by using the Banach contraction principle and the Schauder's fixed point theorem. Our problem contains different orders in three fractional difference operators and three fractional sums. Finally, we provide an example to displays the importance of these results.

1. Introduction

Fractional difference equations can be used for describing many problems in the real-world phenomena such as physics, mechanics, chemistry, control systems, electrical networks, and flow in porous media. In particular, fractional calculus appears in the studied in biology, ecology and other areas (see [1]-[2]). Mathematicians have used this fractional calculus to model and solve various related problems. Boundary value problems for fractional difference equations, which have helped to build up some of the basic theory of this field can be seen in the textbooks [3] and the papers [4]-[36] and references cited therein.

There is a development of boundary value problems for sequential fractional difference equations which shows an operation of the examinative function. The study may also have another function which is related to our interested one. These creations are incorporating with nonlocal conditions which are both extensive and more complex. For example, Goodrich [15] considered a discrete fractional boundary value problem for a sequential fractional difference equations of the forms

$$\begin{cases} -\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}y(t) = f(t+\mu_1+\mu_2+\mu_3-1, y(t+\mu_1+\mu_2+\mu_3-1)), \\ y(0) = 0 = y(b+2), \end{cases} \tag{1}$$

where $t \in \mathbb{N}_{2-\mu_1-\mu_2-\mu_3, b+2-\mu_1-\mu_2-\mu_3}$, $0 < \mu_1, \mu_2, \mu_3 < 1$, $1 < \mu_2+\mu_3 < 2$, $1 < \mu_1+\mu_2+\mu_3 < 2$ and $f : \mathbb{N}_0 \times \mathbb{R} \to [0, +\infty)$ is a continuous function.

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Weidong [22] examined the sequential Caputo fractional boundary value problem with a p-Laplacian

$$\begin{cases} \Delta_C^{\beta}[\phi_p(\Delta_C^{\alpha}x)](t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)), \quad t \in \mathbb{N}_{0,b}, \\ \Delta_C^{\beta}x(\beta-1) + \Delta_C^{\beta}x(\beta+b) = 0, \\ x(\alpha+\beta-2) + x(\alpha+\beta+b) = 0, \end{cases} \tag{2}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha+\beta \leq 2$, $f : \mathbb{N}_{\alpha+\beta-1,\alpha+\beta+T-1} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ϕ_p is the p-Laplacian operator.

Recently, Sitthiwirattham [29] investigated three-point fractional sum boundary value problems for sequential Caputo fractional difference equations of the forms

$$\begin{cases} \Delta_{\alpha}^{\alpha}(\Delta_{\alpha+\beta-1}^{\beta} + \lambda E_{\beta})x(t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)), \\ x(\alpha+\beta-2) = 0, \quad x(\alpha+\beta+T) = \rho\Delta_{\alpha+\beta-1}^{-\gamma}x(\eta+\gamma), \end{cases} \tag{3}$$

where $t \in \mathbb{N}_{0,T}$, $0 < \alpha, \beta \leq 1, 1 < \alpha+\beta \leq 2, 0 < \gamma \leq 1, \eta \in \mathbb{N}_{\alpha+\beta-1,\alpha+\beta+T-1}$, ρ is a constant, $f : \mathbb{N}_{\alpha+\beta-2,\alpha+\beta+T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $E_{\beta}x(t) = x(t+\beta-1)$.

Presently, we focus on the difference operator with step 1. Our knowledge, there is a gap in the literature about the details of this operation. To make it more general and flexible in the sense that it has the freedom to choose the different step h. However, not much research has involved the development of h-sum and h-difference operators (see [37]-[43]).

We find that the boundary value problem for h-difference equations has not been studied. The results mentioned above are the motivation for this research. In this paper, we study a sequential Caputo fractional h-sum-difference equation with three point fractional h-sum boundary value conditions of the form

$$\begin{aligned} {}_C\Delta_h^{\alpha}\Big[{}_C\Delta_h^{\beta}\Big(\Delta_C^{\omega} + (e^{\lambda}-1)\Delta_C^{\omega-1}E\Big)\Big]u(t) &= F\Big[t + \Big(\alpha+\beta-1+\frac{\omega}{h}\Big)h, u\Big(t + \Big(\alpha+\beta-1+\frac{\omega}{h}\Big)h\Big), \\ &\qquad (\Psi_h^{\gamma}u)\Big(t + \Big(\alpha+\beta+\gamma-1+\frac{\omega}{h}\Big)h\Big)\Big], \qquad t \in (h\mathbb{N})_{0,Th} \\ u\Big(\Big[\alpha+\beta-2+\frac{\omega}{h}\Big]h\Big) &= u(\eta) = 0, \\ \Delta_h^{-\theta}g\Big(\Big[T+\alpha+\beta+\theta+\frac{\omega}{h}\Big]h\Big)u\Big(\Big[T+\alpha+\beta+\theta+\frac{\omega}{h}\Big]h\Big) &= 0, \end{aligned} \tag{4}$$

where $\alpha, \beta, \omega, \gamma, \theta \in (0,1]$, $2 < \alpha+\beta+\omega \leq 3$, $(h\mathbb{N})_{0,Th} := \{0, h, 2h, \ldots, Th\}$, $\eta \in (h\mathbb{N})_{[\alpha+\beta-1+\frac{\omega}{h}]h,[T+\alpha+\beta-1+\frac{\omega}{h}]h}$, $Eu(t) = u(t+1)$, $F \in C\Big((h\mathbb{N})_{[\alpha+\beta-2+\frac{\omega-1}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h} \times \mathbb{R}^2, \mathbb{R}\Big)$, $g \in C\Big((h\mathbb{N})_{[\alpha+\beta-2+\frac{\omega}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h}, \mathbb{R}^+\Big)$, and for $\varphi \in C\Big((h\mathbb{N})_{[\alpha+\beta-2+\frac{\omega}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h} \times (h\mathbb{N})_{[\alpha+\beta-2+\frac{\omega}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h}, [0,\infty)\Big)$,

$$\Psi_h^{\gamma}u)(t) := [\Delta_h^{-\gamma}\varphi\, u](t+\gamma) = \frac{h}{\Gamma_h(\gamma)} \sum_{s=\alpha+\beta-2+\frac{\omega}{h}}^{\frac{t}{h}} (t-\sigma(hs))_h^{\underline{\gamma-1}}\, \varphi\Big(t+\gamma h, s\Big)u(hs).$$

Noting from our problem, there are three sequential of two Caputo fractional h-difference operators of order α, β and a ω-order Caputo fractional difference operator, moreover, there exist two fractional h-sum of order γ, θ and a $(1-\omega)$-order fractional sum. So, this problem is complicated more than the previous works.

The rest of paper is organized as follows. In Section 2 we provide some definitions and basic lemmas. Next, we convert the problem to an equivalent summation equation in order to derive a representation for the solution of (4). In Section 3, we prove existence and uniqueness results of the problem (4) by employing the Banach contraction principle and the Schauder's theorem. Furthermore, we also show the existence of a positive solution to (4). Finally an illustrative example is given in Section 4.

2. Preliminaries

In the following, there are basic notations, definitions, and lemmas which are used to get the main results.

Definition 2.1. *For any $t, \alpha \in \mathbb{R}$ and $h > 0$, the h-falling function is defined by*

$$t_h^{\underline{\alpha}} := h^{\alpha} \frac{\Gamma\left(\frac{t}{h} + 1\right)}{\Gamma\left(\frac{t}{h} + 1 - \alpha\right)},$$

where $\frac{t}{h} + 1 \notin \mathbb{Z}^- \cup \{0\}$, and we use the convention that division at a pole yields zero. If $h = 1$, then $t_h^{\underline{\alpha}} = t^{\underline{\alpha}}$.

Definition 2.2. *For $\alpha, h > 0$ and f define on $(h\mathbb{N})_a := \{a, a+h, a+2h, \ldots\}$, the α-order fractional h-sum of f is defined by*

$$\Delta_h^{-\alpha} f(t) := \frac{h}{\Gamma(\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\alpha} (t - \sigma(hs))_h^{\underline{\alpha-1}} f(hs),$$

where $t \in \mathbb{N}_{a+\alpha h} := \{a + \alpha h, a + (\alpha+1)h, a + (\alpha+2)h, ...\}$ and $\sigma(hs) = (s+1)h$. If $h = 1$, then $\Delta_h^{-\alpha} f(t) = \Delta^{-\alpha} f(t)$.

Definition 2.3. *For $\alpha > 0$ and f define on $(h\mathbb{N})_a$, the α-order Caputo fractional h-difference of f is defined by*

$${}_C\Delta_h^{\alpha} f(t) := \Delta_h^{-(N-\alpha)} \Delta_h^N f(t) = \frac{h}{\Gamma(N-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-(N-\alpha)} (t - \sigma(hs))_h^{\underline{N-\alpha-1}} \Delta_h^N f(sh),$$

where $t \in \mathbb{N}_{a+(N-\alpha)h}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N - 1 < \alpha < N$.
If $\alpha = N$ then ${}_C\Delta_h^{\alpha} f(t) = \Delta_h^N f(t)$, and if $h = 1$ then ${}_C\Delta_h^{\alpha} f(t) = \Delta_C^{\alpha} f(t)$.

To present the solution of the boundary value problem (4), we need the following lemma that deals with a linear variant of the boundary value problem (4).

Lemma 2.4. *Let $\alpha, \beta, \omega, \theta \in (0,1]$, $2 < \alpha+\beta+\omega \le 3$, $\eta \in (h\mathbb{N})_{[\alpha+\beta-1+\frac{\omega}{h}]h, [T+\alpha+\beta-1+\frac{\omega}{h}]h}$, $g \in C\Big((h\mathbb{N})_{[\alpha+\beta-2+\frac{\omega}{h}]h, [T+\alpha+\beta+\frac{\omega}{h}]h}, \mathbb{R}^+\Big)$ and $f \in C\Big((h\mathbb{N})_{[\alpha+\beta-1+\frac{\omega}{h}]h, [T+\alpha+\beta-1+\frac{\omega}{h}]h}, \mathbb{R}\Big)$ be given. Then for $t \in (h\mathbb{N})_{0,Th}$, the problem*

$$ {}_C\Delta_h^{\alpha}\Big[{}_C\Delta_h^{\beta}\Big(\Delta_C^{\omega} + (e^{\lambda} - 1)\Delta_C^{\omega-1}E\Big)\Big]u(t) = f\left(\left[t + \alpha + \beta - 1 + \frac{\omega}{h}\right]h\right), \tag{5}$$

$$\begin{cases} u\left(\left[\alpha + \beta - 2 + \frac{\omega}{h}\right]h\right) = u(\eta) = 0, \\ \Delta_h^{-\theta} g\left(\left[T + \alpha + \beta + \theta + \frac{\omega}{h}\right]h\right) u\left(\left[T + \alpha + \beta + \theta + \frac{\omega}{h}\right]h\right) = 0, \end{cases} \tag{6}$$

has the unique solution

$$\begin{aligned} e^{\lambda t}u(t) = {} & \frac{\mathcal{P}[h]}{\Lambda\Gamma(\omega-1)} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} e^{\lambda z}(z - \sigma(s))^{\underline{\omega-2}} \\ & - \frac{h\mathcal{Q}[h]}{\Lambda\Gamma(\omega-1)\Gamma(\beta)} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} e^{\lambda z}(z - \sigma(s))^{\underline{\omega-2}}(s - \sigma(hv))_h^{\underline{\beta-1}} \\ & + \frac{h^2}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \sum_{x=0}^{\frac{v}{h}-\alpha} e^{\lambda z}(z - \sigma(s))^{\underline{\omega-2}}(s - \sigma(hv))_h^{\underline{\beta-1}}(v - \sigma(hx))_h^{\underline{\alpha-1}} \times \\ & f\left(\left[x + \alpha + \beta - 1 + \frac{\omega}{h}\right]h\right) \end{aligned} \tag{7}$$

where the functionals $\mathcal{P}[f]$ and $\mathcal{Q}[f]$ are defined as

$$\mathcal{P}[f] = \Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{he^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}}{\Lambda\Gamma(\omega-1)\Gamma(\beta)}\Bigg]\times$$
$$\Bigg[\sum_{r=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^3e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}\times$$
$$(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}g\Big(\Big[T+\alpha+\beta+\theta+\frac{\omega}{h}\Big]h\Big)f\Big(\Big[x+\alpha+\beta-1+\frac{\omega}{h}\Big]h\Big)\Bigg]$$
$$-\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}e^{\lambda(z-r)}h^2(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\theta)\Gamma(\omega-1)}\times$$
$$\frac{(s-\sigma(hv))_h^{\underline{\beta-1}}}{\Gamma(\beta)}g\Big(\Big[T+\alpha+\beta+\theta+\frac{\omega}{h}\Big]h\Big)\Bigg]\cdot\Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)}\times$$
$$\frac{h^2e^{\lambda(z-\eta)}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}}{\Gamma(\beta)\Gamma(\alpha)}f\Big(\Big[x+\alpha+\beta-1+\frac{\omega}{h}\Big]h\Big)\Bigg], \tag{8}$$

$$\mathcal{Q}[f] = \Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\frac{e^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)}\Bigg]\cdot\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}h^3e^{\lambda(z-r)}\times$$
$$\frac{\left(\left[T+\alpha+\beta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}g\Big(\Big[T+\alpha+\beta+\theta+\frac{\omega}{h}\Big]h\Big)\times$$
$$f\Big(\Big[x+\alpha+\beta-1+\frac{\omega}{h}\Big]h\Big)\Bigg]-\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\frac{\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}}{\Gamma(\theta)\Gamma(\beta)\Gamma(\omega-1)}\times$$
$$h^2e^{\lambda(z-r)}(z-\sigma(s))^{\underline{\omega-2}}g\Big(\Big[T+\alpha+\beta+\theta+\frac{\omega}{h}\Big]h\Big)\Bigg]\cdot\Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}h^2e^{\lambda(z-\eta)}\times$$
$$\frac{(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}f\Big(\Big[x+\alpha+\beta-1+\frac{\omega}{h}\Big]h\Big)\Bigg], \tag{9}$$

and

$$\Lambda = \Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{h^2e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)}\times$$
$$(s-\sigma(hv))_h^{\underline{\beta-1}}g\Big(\Big[T+\alpha+\beta+\theta+\frac{\omega}{h}\Big]h\Big)\Bigg]\cdot\Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\frac{e^{-\lambda z}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)}\Bigg]$$
$$-\Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{he^{-\lambda z}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}}{\Lambda\Gamma(\omega-1)\Gamma(\beta)}\Bigg]\times$$

$$\Big[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\frac{h^2e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\theta)\Gamma(\beta)\Gamma(\omega-1)}\times$$
$$g\Big(\Big[T+\alpha+\beta+\theta+\frac{\omega}{h}\Big]h\Big)\Big]. \tag{10}$$

Proof. We use the fractional h-sum of order α for (5) to obtain

$${}_C\Delta_h^{\beta}\Big[\Delta_C^{\omega}+(e^{\lambda}-1)\Delta_C^{\omega-1}E\Big]u(t)=C_0+h\sum_{s=0}^{\frac{t}{h}-\alpha}\frac{(t-\sigma(hs))_h^{\underline{\alpha-1}}}{\Gamma(\alpha)}f\left(\left[s+\alpha+\beta-1+\frac{\omega}{h}\right]h\right), \tag{11}$$

for $t\in\mathbb{N}_{(\alpha-1)h,(T+\alpha)h}$. In addition using the fractional h-sum of order β for (11), we get

$$\begin{aligned}&\Delta_C^{\omega}u(t)+(e^{\lambda}-1)\Delta_C^{\omega-1}Eu(t)\\&= C_1+C_0h\sum_{s=0}^{\frac{t}{h}-\beta}\frac{(t-\sigma(hs))_h^{\underline{\beta-1}}}{\Gamma(\beta)}+h^2\sum_{s=0}^{\frac{t}{h}-\beta}\sum_{v=0}^{\frac{s}{h}-\alpha}\frac{(t-\sigma(hs))_h^{\underline{\beta-1}}(s-\sigma(hv))_h^{\underline{\alpha-1}}}{\Gamma(\beta)\Gamma(\alpha)}f\left(\left[v+\alpha+\beta-1+\frac{\omega}{h}\right]h\right),\end{aligned} \tag{12}$$

for $t\in\mathbb{N}_{(\alpha+\beta-2)h,(T+\alpha+\beta)h}$. Taking the fractional sum of order ω for (12), we get

$$\begin{aligned}&u(t)+(e^{\lambda}-1)\Delta^{-1}u(t+1)\\&= C_2+C_1\sum_{s=\alpha+\beta-2}^{t-\omega}\frac{(t-\sigma(s))^{\underline{\omega-1}}}{\Gamma(\omega)}+C_0h\sum_{s=\alpha+\beta-2}^{t-\omega}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{(t-\sigma(s))^{\underline{\omega-1}}(s-\sigma(vh))_h^{\underline{\beta-1}}}{\Gamma(\omega)\Gamma(\beta)}\\&\quad+\sum_{s=\alpha+\beta-2}^{t-\omega}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^2(t-\sigma(s))^{\underline{\omega-1}}(s-\sigma(vh))_h^{\underline{\beta-1}}(v-\sigma(xh))_h^{\underline{\alpha-1}}}{\Gamma(\omega)\Gamma(\beta)\Gamma(\alpha)}f\left(\left[x+\alpha+\beta-1+\frac{\omega}{h}\right]h\right),\end{aligned} \tag{13}$$

for $t\in\mathbb{N}_{[\alpha+\beta-2+\frac{\omega}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h}$. Next, taking the forward difference Δ for (13), we have

$$\begin{aligned}&\Delta u(t)+(e^{\lambda}-1)u(t+1)\\&= C_1\sum_{s=\alpha+\beta-2}^{t-\omega+1}\frac{(t-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)}+C_0\sum_{s=\alpha+\beta-2}^{t-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{h(t-\sigma(s))^{\underline{\omega-2}}(s-\sigma(vh))_h^{\underline{\beta-1}}}{\Gamma(\omega-1)\Gamma(\beta)}\\&\quad+\sum_{s=\alpha+\beta-2}^{t-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^2(t-\sigma(s))^{\underline{\omega-2}}(s-\sigma(vh))_h^{\underline{\beta-1}}(v-\sigma(xh))_h^{\underline{\alpha-1}}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}f\left(\left[x+\alpha+\beta-1+\frac{\omega}{h}\right]h\right),\end{aligned} \tag{14}$$

multiply the equation (14) by $e^{\lambda t}$, we obtain

$$\begin{aligned}&\Delta\Big[e^{\lambda t}u(t)\Big]\\&= e^{\lambda t}\Delta u(t)+(e^{\lambda}-1)e^{\lambda t}u(t+1)\\&= e^{\lambda t}\Big\{C_1\sum_{s=\alpha+\beta-2}^{t-\omega+1}\frac{(t-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)}+C_0\sum_{s=\alpha+\beta-2}^{t-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{h(t-\sigma(s))^{\underline{\omega-2}}(s-\sigma(vh))_h^{\underline{\beta-1}}}{\Gamma(\omega-1)\Gamma(\beta)}\\&\quad+\sum_{s=\alpha+\beta-2}^{t-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^2(t-\sigma(s))^{\underline{\omega-2}}(s-\sigma(vh))_h^{\underline{\beta-1}}(v-\sigma(xh))_h^{\underline{\alpha-1}}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}f\left(\left[x+\alpha+\beta-1+\frac{\omega}{h}\right]h\right)\Big\},\end{aligned} \tag{15}$$

for $t \in \mathbb{N}_{[\alpha+\beta-2+\frac{\omega}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h}$. Using the sum Δ^{-1} for (15),

$$\begin{aligned} &e^{\lambda t}u(t) \\ &= C_2 + C_1 \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \frac{e^{\lambda z}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)} \qquad (16) \\ &+C_0 \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \frac{he^{\lambda z}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(vh))_h^{\underline{\beta-1}}}{\Gamma(\omega-1)\Gamma(\beta)} \\ &+ \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \sum_{x=0}^{\frac{v}{h}-\alpha} \frac{h^2 e^{\lambda z}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(vh))_h^{\underline{\beta-1}}(v-\sigma(xh))_h^{\underline{\alpha-1}}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)} f\left(\left[x+\alpha+\beta-1+\frac{\omega}{h}\right]h\right), \end{aligned}$$

for $t \in \mathbb{N}_{[\alpha+\beta-2+\frac{\omega}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h}$. By substituting $t = \left[\alpha+\beta-2+\frac{\omega}{h}\right]h$ and $t=\eta$ into (16), and employing the first and second conditions of (6), we find that $C_2 = 0$ and

$$\begin{aligned} &C_1 \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{\eta-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \frac{e^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)} \qquad (17) \\ &+C_0 \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{\eta-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \frac{he^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}}{\Lambda\Gamma(\omega-1)\Gamma(\beta)} \\ &= -\sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{\eta-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \sum_{x=0}^{\frac{v}{h}-\alpha} \frac{h^2 e^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}} f\left(\left[x+\alpha+\beta-1+\frac{\omega}{h}\right]h\right). \end{aligned}$$

Using the fractional h-sum of order θ for (16), and substituting $t = \left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h$, and employing the third condition of (6), this implies

$$\begin{aligned} &C_1 \sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{r-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \frac{h^2 e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\theta)\Gamma(\beta)\Gamma(\omega-1)} \times \\ &g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right) \qquad (18) \\ &+C_0 \sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{r-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \frac{h^2 e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)} \times \\ &(s-\sigma(hv))_h^{\underline{\beta-1}} g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right) \\ &= -\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{r-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \sum_{x=0}^{\frac{v}{h}-\alpha} \frac{h^3 e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)} \times \\ &(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}} g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right) f\left(\left[x+\alpha+\beta-1+\frac{\omega}{h}\right]h\right). \end{aligned}$$

Solving the system of equations (17) and (18), we obtain

$$C_1 = \mathcal{P}[f] \quad \text{and} \quad C_0 = -\mathcal{Q}[f]$$

where $\mathcal{P}[f], \mathcal{Q}[f]$ are defined by (8), (9), respectively. Substituting the constants C_0, C_1 and C_2 into (15), we obtain the solution (7). □

3. Main Results

In this section, we wish to prove the existence solution to the problem (4). To accomplish this, let $C = C\Big((h\mathbb{N})_{[\alpha+\beta-2+\frac{\omega}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h}, \mathbb{R}\Big)$ be a Banach space of all continuous functions u from $(h\mathbb{N})_{[\alpha+\beta-2+\frac{\omega}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h}$ to $\mathbb{R}$, with the norm defined by

$$\|u\|_C = \max_{t\in(h\mathbb{N})_{[\alpha+\beta-2+\frac{\omega}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h}} |u(t)|.$$

We consider the operator $\mathcal{A} : C \to C$ by

$$\begin{aligned}(\mathcal{A}u)(t) \;=\; & \frac{e^{-\lambda t}\mathcal{P}[F_u]}{\Lambda\Gamma(\omega-1)} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} e^{\lambda z}(z-\sigma(s))^{\underline{\omega-2}} \\ & -\frac{he^{-\lambda t}\mathcal{Q}[F_u]}{\Lambda\Gamma(\omega-1)\Gamma(\beta)} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} e^{\lambda z}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}} \\ & +\frac{h^2e^{-\lambda t}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \sum_{x=0}^{\frac{v}{h}-\alpha} e^{\lambda z}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}} \times \\ & (v-\sigma(hx))_h^{\underline{\alpha-1}} F\Big(x+\Big(\alpha+\beta-1+\frac{\omega}{h}\Big)h, u\Big(x+\Big(\alpha+\beta-1+\frac{\omega}{h}\Big)h\Big), \\ & (\Psi_h^{\gamma}u)\Big(x+\Big(\alpha+\beta+\gamma-1+\frac{\omega}{h}\Big)h\Big)\Big), \end{aligned} \tag{19}$$

where the functionals $\mathcal{P}[F_u]$ and $\mathcal{Q}[F_u]$ are defined by

$$\begin{aligned}\mathcal{P}[F_u] \;=\; & \Bigg[\sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{\eta-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \frac{he^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}}{\Lambda\Gamma(\omega-1)\Gamma(\beta)}\Bigg] \times \\ & \Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{r-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \sum_{x=0}^{\frac{v}{h}-\alpha} \frac{\Big(\big[T+\alpha+\beta+\theta+\frac{\omega}{h}\big]h-\sigma(hr)\Big)_h^{\underline{\theta-1}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)} \times \\ & h^3e^{\lambda(z-r)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}\, g\Big(\Big[T+\alpha+\beta+\theta+\frac{\omega}{h}\Big]h\Big) \times \\ & F\Big(x+\Big(\alpha+\beta-1+\frac{\omega}{h}\Big)h, u\Big(x+\Big(\alpha+\beta-1+\frac{\omega}{h}\Big)h\Big), (\Psi_h^{\gamma}u)\Big(x+\Big(\alpha+\beta+\gamma-1+\frac{\omega}{h}\Big)h\Big)\Big)\Bigg] \\ & -\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}} \sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{r-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \frac{\Big(\big[T+\alpha+\beta+\theta+\frac{\omega}{h}\big]h-\sigma(hr)\Big)_h^{\underline{\theta-1}} e^{\lambda(z-r)}}{\Gamma(\theta)} \times \\ & \frac{h^2(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}}{\Gamma(\omega-1)\Gamma_h(\beta)}\, g\Big(\Big[T+\alpha+\beta+\theta+\frac{\omega}{h}\Big]h\Big)\Bigg] \times \\ & \Bigg[\sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{\eta-1} \sum_{s=\alpha+\beta-2}^{z-\omega+1} \sum_{v=\alpha-1}^{\frac{s}{h}-\beta} \sum_{x=0}^{\frac{v}{h}-\alpha} \frac{h^2e^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)} \times \\ & F\Big(x+\Big(\alpha+\beta-1+\frac{\omega}{h}\Big)h, u\Big(x+\Big(\alpha+\beta-1+\frac{\omega}{h}\Big)h\Big), (\Psi_h^{\gamma}u)\Big(x+\Big(\alpha+\beta+\gamma-1+\frac{\omega}{h}\Big)h\Big)\Big)\Bigg], \end{aligned} \tag{20}$$

$$\begin{aligned}
\mathcal{Q}[F_u] = & \Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\frac{e^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)}\Bigg]\times\\
& \Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}\times\\
& h^3e^{\lambda(z-r)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}\,g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right)\times\\
& F\left(x+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h,u\left(x+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h\right),(\Psi_h^{\gamma}u)\left(x+\left(\alpha+\beta+\gamma-1+\frac{\omega}{h}\right)h\right)\right)\Bigg]\\
& -\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\frac{h^2\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}e^{\lambda(z-r)}}{\Gamma(\theta)\Gamma(\beta)}\times\\
& \frac{(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)}\,g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right)\Bigg]\times\\
& \Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^2e^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}\times\\
& F\left(x+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h,u\left(x+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h\right),(\Psi_h^{\gamma}u)\left(x+\left(\alpha+\beta+\gamma-1+\frac{\omega}{h}\right)h\right)\right)\Bigg], \quad (21)
\end{aligned}$$

and Λ is defined by (10).

We find that the problem (4) has solutions if and only if the operator $\mathcal{A}$ has fixed points. Next, we present the existence and uniqueness of a solution to the problem (4), by using the Banach contraction principle.

Theorem 3.1. *Assume that* $F:(h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega}{h}\right]h,\left[T+\alpha+\beta+\frac{\omega}{h}\right]h}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are continuous,* $\varphi:(h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega}{h}\right]h,\left[T+\alpha+\beta+\frac{\omega}{h}\right]h}\times(h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega}{h}\right]h,\left[T+\alpha+\beta+\frac{\omega}{h}\right]h}\to[0,\infty)$ *is continuous with* $\varphi_0=\max\left\{\varphi(t+\gamma h,s):(t,s)\in(h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega}{h}\right]h,\left[T+\alpha+\beta+\frac{\omega}{h}\right]h}\times(h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega}{h}\right]h,\left[T+\alpha+\beta+\frac{\omega}{h}\right]h}\right\}$. *In addition, suppose that:*

(H_1) *There exist constants* $L_1,L_2>0$ *such that for each* $t\in(h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega}{h}\right]h,\left[T+\alpha+\beta+\frac{\omega}{h}\right]h}$ *and* $u,v\in\mathbb{R}$

$$\left|F(t,u,\Psi_h^{\gamma}u)-F(t,v,\Psi_h^{\gamma}v)\right|\le L_1|u-v|+L_2|(\Psi^{\gamma}u)-(\Psi^{\gamma}v)|.$$

(H_2) $0<g(t)<\mathcal{G}$ *for each* $t\in(h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega}{h}\right]h,\left[T+\alpha+\beta+\frac{\omega}{h}\right]h}$.

(H_3) $\left(L_1+L_2\frac{\varphi_0\left((T+2)h\right)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\chi<1.$

Then the problem (4) *has a unique solution on* $(h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega}{h}\right]h,\left[T+\alpha+\beta+\frac{\omega}{h}\right]h}$.

where

$$\Omega_1 = \left[\frac{e^{-\lambda}(\eta-\alpha-\beta)^{\underline{\omega-1}}(\eta-\omega+(1-\alpha)h)_h^{\underline{\beta}}}{\Gamma(\omega)\Gamma(\beta+1)} \cdot \frac{e^{-\lambda}\mathcal{G}\Big(h(T+\theta+2)\Big)_h^{\underline{\theta}}\Big(T+\frac{\omega}{h}\Big)^{\underline{\omega-1}}\Big(T+\beta+h+(1-h)\left(\frac{\omega}{h}+\alpha\right)\Big)_h^{\underline{\beta}}}{\Gamma(\theta+1)\Gamma(\omega)\Gamma(\beta+1)}\right]\times$$
$$\left[\frac{\left(\frac{1}{h}\left[T+\alpha+(1-h)\left(\frac{\omega}{h}+\beta\right)\right]\right)_h^{\underline{\alpha}}}{|\Lambda|\Gamma(\alpha+1)} + \frac{\left(\frac{\eta-\omega}{h}-\beta\right)_h^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right] \tag{22}$$

$$\Omega_2 = \left[\frac{e^{-\lambda}(\eta-\alpha-\beta)^{\underline{\omega-1}}}{\Gamma(\omega)} \cdot \frac{e^{-\lambda}\mathcal{G}\Big(h(T+\theta+2)\Big)_h^{\underline{\theta}}\Big(T+\frac{\omega}{h}\Big)^{\underline{\omega-1}}}{\Gamma(\theta+1)\Gamma(\omega)}\right]\times \tag{23}$$
$$\left[\frac{\Big(T+\beta+h+(1-h)\left(\frac{\omega}{h}+\alpha\right)\Big)_h^{\underline{\beta}}}{\Gamma(\beta+1)} + \frac{(\eta-\omega+(1-\alpha)h)_h^{\underline{\beta}}\left(\frac{\eta-\omega}{h}-\beta\right)_h^{\underline{\alpha}}}{\Gamma(\beta+1)\Gamma(\alpha+1)}\right]$$

$$\Omega_3 = \frac{e^{-\lambda}\Big(Th+\omega+(h-1)(\alpha+\beta)\Big)^{\underline{\omega-1}}\Big((T+\beta+1)h\Big)_h^{\underline{\beta}}(T+\alpha)_h^{\underline{\alpha}}}{\Gamma(\omega)\Gamma(\beta+1)\Gamma(\alpha+1)}, \tag{24}$$

and

$$\chi := \frac{e^{-\lambda}\Big(Th+\omega+(h-1)(\alpha+\beta)\Big)^{\underline{\omega-1}}}{|\Lambda|\Gamma(\omega)}\left[\Omega_1 + \Omega_2\frac{\Big((T+\beta+1)h\Big)_h^{\underline{\beta}}}{\Gamma(\beta+1)}\right] + \Omega_3. \tag{25}$$

Proof. We show that $\mathcal{A}$ is a contraction. For any $u, v \in C$, we have

$$\Big|\mathcal{P}[F_u] - \mathcal{P}[F_v]\Big|$$
$$= \Bigg\|\Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{he^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}}{\Lambda\Gamma(\omega-1)\Gamma(\beta)}\Bigg]\times$$
$$\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta+\theta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^3e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}\times$$
$$(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}\,g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right)\Big|F\Big(x,u,(\Psi_h^{\gamma}u)\Big)-F\Big(x,v,(\Psi_h^{\gamma}v)\Big)\Big|\Bigg]$$
$$-\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{h^2e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)}\times$$
$$(s-\sigma(hv))_h^{\underline{\beta-1}}\,g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right)\Bigg]\Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^2e^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)}\times$$
$$\frac{(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}}{\Gamma(\beta)\Gamma(\alpha)}\Big|F\Big(x,u,(\Psi_h^{\gamma}u)\Big)-F\Big(x,v,(\Psi_h^{\gamma}v)\Big)\Big|\Bigg]\Bigg\|$$
$$\leq \left(L_1 + L_2\frac{\varphi_0\Big(t-\omega-(\alpha+\beta-2)h\Big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u-v\|_C\left[\frac{e^{-\lambda}(\eta-\alpha-\beta)^{\underline{\omega-1}}(\eta-\omega+(1-\alpha)h)_h^{\underline{\beta}}}{|\Lambda|\Gamma(\omega)\Gamma(\beta+1)}\right]\times$$
$$e^{-\lambda}\mathcal{G}\Big(h(T+\theta+2)\Big)_h^{\underline{\theta}}\left[\frac{\Big(T+\frac{\omega}{h}\Big)^{\underline{\omega-1}}\Big(T+\beta+h+(1-h)\left(\frac{\omega}{h}+\alpha\right)\Big)_h^{\underline{\beta}}\left(\frac{1}{h}\left[T+\alpha+(1-h)\left(\frac{\omega}{h}+\beta\right)\right]\right)_h^{\underline{\alpha}}}{\Gamma(\theta+1)\Gamma(\omega)\Gamma(\beta+1)\Gamma(\alpha+1)}\right]$$

$$
\begin{aligned}
&+\left(L_1+L_2\frac{\varphi_0\Big(t-\omega-(\alpha+\beta-2)h\Big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u-v\|_C\times\\
&\left[\frac{e^{-\lambda}\mathcal{G}\Big(h(T+\theta+2)\Big)_h^{\underline{\theta}}\left(T+\frac{\omega}{h}\right)^{\underline{\omega-1}}\left(T+\beta+h+(1-h)\left(\frac{\omega}{h}+\alpha\right)\right)_h^{\underline{\beta}}}{\Gamma(\theta+1)\Gamma(\omega)\Gamma(\beta+1)}\right]\times\\
&\left[\frac{e^{-\lambda}(\eta-\alpha-\beta)^{\underline{\omega-1}}(\eta-\omega+(1-\alpha)h)_h^{\underline{\beta}}\left(\frac{\eta-\omega}{h}-\beta\right)_h^{\underline{\alpha}}}{\Gamma(\omega)\Gamma(\beta+1)\Gamma(\alpha+1)}\right]\\
=\ &\|u-v\|_C\,\Omega_1\left(L_1+L_2\frac{\varphi_0\Big(t-\omega-(\alpha+\beta-2)h\Big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\mathcal{Q}[F_u]-\mathcal{Q}[F_v]\Big|\\
=\ &\Bigg|\Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\frac{e^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\omega-1)}\Bigg]\times\\
&\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^3e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}\times\\
&(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}\,g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right)\Big|F\Big(x,u,(\Psi_h^{\gamma}u)\Big)-F\Big(x,v,(\Psi_h^{\gamma}v)\Big)\Big|\Bigg]\\
&-\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\frac{h^2e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}(z-\sigma(s))^{\underline{\omega-2}}}{\Gamma(\theta)\Gamma(\beta)\Gamma(\omega-1)}\times\\
&g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right)\Bigg]\cdot\Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^2e^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}\times\\
&(v-\sigma(hx))_h^{\underline{\alpha-1}}\Big|F\Big(x,u,(\Psi_h^{\gamma}u)\Big)-F\Big(x,v,(\Psi_h^{\gamma}v)\Big)\Big|\Bigg]\Bigg|\\
\le\ &\left(L_1+L_2\frac{\varphi_0\Big(t-\omega-(\alpha+\beta-2)h\Big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u-v\|_C\times\\
&\left[\frac{e^{-\lambda}(\eta-\alpha-\beta)^{\underline{\omega-1}}}{\Gamma(\omega)}\right]\left[\frac{e^{-\lambda}\mathcal{G}\Big(h(T+\theta+2)\Big)_h^{\underline{\theta}}\left(T+\frac{\omega}{h}\right)^{\underline{\omega-1}}\left(T+\beta+h+(1-h)\left(\frac{\omega}{h}+\alpha\right)\right)_h^{\underline{\beta}}}{\Gamma(\theta+1)\Gamma(\omega)\Gamma(\beta+1)}\right]\\
&+\left(L_1+L_2\frac{\varphi_0\Big(t-\omega-(\alpha+\beta-2)h\Big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u-v\|_C\left[\frac{e^{-\lambda}\mathcal{G}\Big(h(T+\theta+2)\Big)_h^{\underline{\theta}}\left(T+\frac{\omega}{h}\right)^{\underline{\omega-1}}}{\Gamma(\theta+1)\Gamma(\omega)}\right]\times\\
&\left[\frac{e^{-\lambda}(\eta-\alpha-\beta)^{\underline{\omega-1}}(\eta-\omega+(1-\alpha)h)_h^{\underline{\beta}}\left(\frac{\eta-\omega}{h}-\beta\right)_h^{\underline{\alpha}}}{\Gamma(\omega)\Gamma(\beta+1)\Gamma(\alpha+1)}\right]\\
=\ &\|u-v\|_C\,\Omega_2\left(L_1+L_2\frac{\varphi_0\Big(t-\omega-(\alpha+\beta-2)h\Big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right).
\end{aligned}
$$

For each $t \in (h\mathbb{N})_{[\alpha+\beta-2+\frac{\omega-1}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h}$, we have

$$\begin{aligned}
&\Big|(\mathcal{A}u)(t)-(\mathcal{A}v)(t)\Big| \\
&= \Bigg|\frac{e^{-\lambda t}\Big|\mathcal{P}[F_u]-\mathcal{P}[F_v]\Big|}{\Lambda\Gamma(\omega-1)}\sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}e^{\lambda z}(z-\sigma(s))^{\underline{\omega-2}} \\
&\quad -\frac{he^{-\lambda t}\Big|\mathcal{Q}[F_u]-\mathcal{Q}[F_v]\Big|}{\Lambda\Gamma(\omega-1)\Gamma(\beta)}\sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}e^{\lambda z}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}} \\
&\quad +\frac{h^2e^{-\lambda t}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}\sum_{z=[\alpha+\beta-2+\frac{\omega}{h}]h}^{t-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}e^{\lambda z}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}\times \\
&\quad F\left(x+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h,u\left(x+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h\right),(\Psi_h^{\gamma}u)\left(x+\left(\alpha+\beta+\gamma-1+\frac{\omega}{h}\right)h\right)\right)\Bigg| \\
&\le \|u-v\|_C\Omega_1\frac{e^{-\lambda}(t-\alpha-\beta)^{\underline{\omega-1}}}{|\Lambda|\Gamma(\omega)}\left(L_1+L_2\frac{\varphi_0\big(t-\omega-(\alpha+\beta-2)h\big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right) \\
&\quad +\|u-v\|_C\,\Omega_2\frac{e^{-\lambda}(t-\alpha-\beta)^{\underline{\omega-1}}(t-\omega+(1-\alpha)h)_h^{\underline{\beta}}}{|\Lambda|\Gamma(\omega)\Gamma(\beta+1)}\left(L_1+L_2\frac{\varphi_0\big(t-\omega-(\alpha+\beta-2)h\big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right) \\
&\quad +\left(L_1+L_2\frac{\varphi_0\big(t-\omega-(\alpha+\beta-2)h\big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u-v\|_C\left[\frac{e^{-\lambda}(t-\alpha-\beta)^{\underline{\omega-1}}(t-\omega+(1-\alpha)h)_h^{\underline{\beta}}\left(\frac{t-\omega}{h}-\beta\right)_h^{\underline{\alpha}}}{\Gamma(\omega)\Gamma(\beta+1)\Gamma(\alpha+1)}\right] \\
&\le \|u-v\|_C\left(L_1+L_2\frac{\varphi_0\big((T+2)h\big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\left\{\frac{e^{-\lambda}\Big(Th+\omega+(h-1)(\alpha+\beta)\Big)^{\underline{\omega-1}}}{|\Lambda|\Gamma(\omega)}\left(\Omega_1+\Omega_2\frac{\big((T+\beta+1)h\big)_h^{\underline{\beta}}}{\Gamma(\beta+1)}\right)+\Omega_3\right\} \\
&= \|u-v\|_C\,\chi.
\end{aligned}$$

From (H_4), we have

$$\Big\|(\mathcal{A}u)(t)-(\mathcal{A}v)(t)\Big\|_C<\|u-v\|_C.$$

Consequently, $\mathcal{A}$ is a contraction. By the Banach contraction principle, we hence get that $\mathcal{A}$ has a fixed point which is a unique solution of the problem (4) on $t \in (h\mathbb{N})_{[\alpha+\beta-2+\frac{\omega-1}{h}]h,[T+\alpha+\beta+\frac{\omega}{h}]h}$. □

The second result, we deduce the existence of at least of solution to (4) by using the following the Schauder's fixed point theorem.

Lemma 3.2. *[44] (Arzelá-Ascoli theorem) A set of functions in $C[a,b]$ with the sup norm, is relatively compact if and only it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 3.3. *[44] If a set is closed and relatively compact then it is compact.*

Lemma 3.4. *[45] (Schauder fixed point theorem) Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T : D \to D$ be the map such that the set $Tu : u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

Theorem 3.5. *Suppose that $(H_1)-(H_3)$ hold. Hence, problem (4) has at least one solution on $\mathbb{N}_{\alpha-n,T+\alpha}$.*

Proof. We divide the proof into three steps.

Step I. Verify $\mathcal{A}$ map bounded sets into bounded sets in $B_R = \{u \in C : \|u\|_C \le R\}$. We consider

$$B_R = \left\{u \in C\Big((h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega-1}{h}\right]h, \left[T+\alpha+\beta+\frac{\omega}{h}\right]h} \Big) : \|u\|_C \le R\right\}.$$

Set $\max_{t\in(h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega-1}{h}\right]h, \left[T+\alpha+\beta+\frac{\omega}{h}\right]h}} |F(t,0,0)| = \mathcal{M}$ and choose a constant

$$R \ge \frac{\chi\mathcal{M}}{1-\chi\left[L_1 + L_2\frac{\varphi_0\left((T+2)h\right)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right]}, \tag{26}$$

where χ satisfies $(H3)$. Denote that

$$\begin{aligned}\left|\mathcal{S}(t,u,0)\right| &= \Bigg|F\left(t+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h, u\left(t+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h\right), (\Psi_h^\gamma u)\left(t+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h\right)\right)\\ &\quad - F\left(t+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h,0,0\right)\Bigg| + \left|F\left(t+\left(\alpha+\beta-1+\frac{\omega}{h}\right)h,0,0\right)\right|,\end{aligned}$$

for each $u \in B_R$. We obtain

$$\begin{aligned}\left|\mathcal{P}[F_u]\right| &= \Bigg|\Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{he^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}}{\Lambda\Gamma(\omega-1)\Gamma(\beta)}\Bigg]\times\\ &\quad\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta+\theta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^3e^{\lambda(z-r)}\left(\left[T+\alpha+\beta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}}{\Gamma(\theta)\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}\times\\ &\quad(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}\,g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right)\left|\mathcal{S}(x,u,0)\right|\Bigg]\\ &\quad-\Bigg[\sum_{r=\alpha+\beta-2+\frac{\omega}{h}}^{T+\alpha+\beta+\frac{\omega}{h}}\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{r-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\frac{\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h-\sigma(hr)\right)_h^{\underline{\theta-1}}e^{\lambda(z-r)}}{\Gamma(\theta)}\times\\ &\quad\frac{h^2(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}}{\Gamma(\omega-1)\Gamma(\beta)}\,g\left(\left[T+\alpha+\beta+\theta+\frac{\omega}{h}\right]h\right)\Bigg]\times\\ &\quad\Bigg[\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{\eta-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha}\frac{h^2e^{\lambda(z-\eta)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}\left|\mathcal{S}(x,u,0)\right|\Bigg]\Bigg|\\ &\le \left[\left(L_1+L_2\frac{\varphi_0\left((T+2)h\right)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u\|_C+\mathcal{M}\right]\Omega_1.\end{aligned}$$

Similarly, we have

$$\left|\mathcal{Q}[F_u]\right| \le \left[\left(L_1+L_2\frac{\varphi_0\left((T+2)h\right)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u\|_C+\mathcal{M}\right]\Omega_2,$$

and

$$\left\|\mathcal{A}u\right\|_C \leq \left[\left(L_1 + L_2\frac{\varphi_0\Big((T+2)h\Big)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\|u\|_C + \mathcal{M}\right]\chi.$$

Using (26), we get that $\|\mathcal{A}u\|_C \leq R$. This implies that $\mathcal{A}$ is uniformly bounded.

Step II. Show that $\mathcal{A}$ is continuous on B_R. From the continuity of F and g, imply that the operator $\mathcal{A}$ is continuous on B_R.

Step III. Examine $\mathcal{A}$ is equicontinuous with B_R. For any $\epsilon > 0$, there exist a positive constant $\delta^* = \max\{\delta_1, \delta_2, \delta_3\}$ such that for $t_1, t_2 \in (h\mathbb{N})_{\left[\alpha+\beta-2+\frac{\omega-1}{h}\right]h, \left[T+\alpha+\beta+\frac{\omega}{h}\right]h}$,

$$\left|(t_2-\alpha-\beta)^{\underline{\omega-1}} - (t_1-\alpha-\beta)^{\underline{\omega-1}}\right| < \frac{\epsilon|\Lambda|\Gamma(\omega)}{\Omega_1 e^{-\lambda}\|F\|}, \text{ whenever } |t_2-t_1| < \delta_1,$$

$$\left|(t_2-\alpha-\beta)^{\underline{\omega-1}}(t_2-\omega+(1-\alpha)h)_h^{\underline{\beta}} - (t_1-\alpha-\beta)^{\underline{\omega-1}}(t_1-\omega+(1-\alpha)h)_h^{\underline{\beta}}\right| < \frac{\epsilon|\Lambda|\Gamma(\omega)\Gamma(\beta+1)}{\Omega_2 e^{-\lambda}\|F\|},$$

whenever $|t_2-t_1| < \delta_2$, and

$$\left|(t_2-\alpha-\beta)^{\underline{\omega-1}}(t_2-\omega+(1-\alpha)h)_h^{\underline{\beta}}\left(\tfrac{t_2-\omega}{h}-\beta\right)_h^{\underline{\alpha}} - (t_1-\alpha-\beta)^{\underline{\omega-1}}(t_1-\omega+(1-\alpha)h)_h^{\underline{\beta}}\left(\tfrac{t_1-\omega}{h}-\beta\right)_h^{\underline{\alpha}}\right| < \frac{\epsilon\Gamma(\omega)\Gamma(\beta+1)\Gamma(\alpha+1)}{e^{-\lambda}\|F\|},$$

whenever $|t_2-t_1| < \delta_3$.

Then we obtain

$$\begin{aligned}
&\Big|(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)\Big| \\
&\leq \frac{\mathcal{P}[F_u]}{|\Lambda|\Gamma(\omega-1)}\left|\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{t_2-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1} e^{\lambda(z-t_2)}(z-\sigma(s))^{\underline{\omega-2}} - \sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{t_1-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1} e^{\lambda(z-t_1)}(z-\sigma(s))^{\underline{\omega-2}}\right| \\
&\quad + \frac{hQ[F_u]}{|\Lambda|\Gamma(\omega-1)\Gamma(\beta)}\left|\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{t_2-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta} e^{\lambda(z-t_2)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}\right. \\
&\quad \left. - \sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{t_1-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta} e^{\lambda(z-t_1)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}\right| \\
&\quad + \frac{h^2\|F\|}{\Gamma(\omega-1)\Gamma(\beta)\Gamma(\alpha)}\left|\sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{t_2-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha} e^{\lambda(z-t_2)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}\right. \\
&\quad \left. - \sum_{z=\left[\alpha+\beta-2+\frac{\omega}{h}\right]h}^{t_1-1}\sum_{s=\alpha+\beta-2}^{z-\omega+1}\sum_{v=\alpha-1}^{\frac{s}{h}-\beta}\sum_{x=0}^{\frac{v}{h}-\alpha} e^{\lambda(z-t_1)}(z-\sigma(s))^{\underline{\omega-2}}(s-\sigma(hv))_h^{\underline{\beta-1}}(v-\sigma(hx))_h^{\underline{\alpha-1}}\right| \\
&\leq \frac{\Omega_1 e^{-\lambda}\|F\|}{|\Lambda|\Gamma(\omega)}\left|(t_2-\alpha-\beta)^{\underline{\omega-1}} - (t_1-\alpha-\beta)^{\underline{\omega-1}}\right| + \frac{\Omega_2 e^{-\lambda}\|F\|}{|\Lambda|\Gamma(\omega)\Gamma(\beta+1)}\times \\
&\quad \left|(t_2-\alpha-\beta)^{\underline{\omega-1}}(t_2-\omega+(1-\alpha)h)_h^{\underline{\beta}} - (t_1-\alpha-\beta)^{\underline{\omega-1}}(t_1-\omega+(1-\alpha)h)_h^{\underline{\beta}}\right| + \frac{e^{-\lambda}\|F\|}{\Gamma(\omega)\Gamma(\beta+1)\Gamma(\alpha+1)}\times \\
&\quad \left|(t_2-\alpha-\beta)^{\underline{\omega-1}}(t_2-\omega+(1-\alpha)h)_h^{\underline{\beta}}\left(\frac{t_2-\omega}{h}-\beta\right)_h^{\underline{\alpha}} - (t_1-\alpha-\beta)^{\underline{\omega-1}}(t_1-\omega+(1-\alpha)h)_h^{\underline{\beta}}\left(\frac{t_1-\omega}{h}-\beta\right)_h^{\underline{\alpha}}\right| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Therefore, $\mathcal{A}(B_R)$ is an equicontinuous set. As a consequence of Steps I to III together with the Arzelá-Ascoli theorem, we get that $\mathcal{A}: C \to C$ is completely continuous. Using the Schauder's fixed point theorem, we can conclude that problem (4) has at least one solution. Our proof is ended. □

4. An example

In this section, an example to illustrate our result is provided as follow. Consider the following fractional h-difference boundary value problem

$$
\begin{aligned}
&{}_C D_{\frac{1}{2}}^{\frac{2}{3}}\left[{}_C D_{\frac{1}{2}}^{\frac{4}{5}}\left(D_C^{\frac{5}{6}} + (e^2-1) D_C^{-\frac{1}{6}} E\right)\right] u(t) = \frac{e^{-\cos^2\left(2\pi t + \frac{16}{15}\right)}\left|u\left(t+\frac{16}{15}\right)\right| + \left|\Psi_{\frac{1}{2}}^{\frac{1}{2}} u\left(t+\frac{16}{15}\right)\right|}{\left(t+\frac{61}{15}\right)^4\left[1+\left|u\left(t+\frac{16}{15}\right)\right|\right]},\\
&u\left(\frac{17}{30}\right) = u\left(\frac{77}{30}\right) = 0, \qquad D_{\frac{1}{2}}^{-\frac{3}{4}} e^{\sin\left(2\pi\frac{293}{120}\right)} u\left(\frac{293}{120}\right) = 0,
\end{aligned} \tag{27}
$$

where $t \in \left(\frac{1}{2}\mathbb{N}\right)_{0,6}$ and $\left(\Psi_{\frac{1}{2}}^{\frac{1}{2}} u\right)(t) = \displaystyle\sum_{s=\frac{17}{15}}^{2t-\frac{1}{2}} \frac{\left(t-\sigma\left(\frac{s}{2}\right)\right)^{\underline{-\frac{1}{2}}}}{2\Gamma\left(\frac{1}{2}\right)} \frac{e^{-\frac{s}{2}+\frac{1}{4})}}{(t+2)^2} u\left(\frac{s}{2}\right).$

Letting $h = \frac{1}{2}$, $\alpha = \frac{2}{3}$, $\beta = \frac{5}{6}$, $\gamma = \frac{1}{2}$, $\omega = \frac{5}{6}$, $\theta = \frac{3}{4}$, $T = 6$, $\lambda = 2$, $\eta = \frac{77}{30}$, $g(t) = e^{\sin(2\pi t)}$, $\varphi(t+\gamma h, s) = \frac{e^{-s}}{\left(t+\frac{9}{2}\right)^2}$ and $F = \frac{e^{-\cos^2(2\pi t)}|u(t)| + |\Psi_{\frac{1}{2}}^{\frac{1}{2}} u(t)|}{(t+3)^4[1+|u(t)|]}$,

we can show that

$$|\Lambda| = 0.0601, \quad \Omega_1 = 15.0543, \quad \Omega_2 = 1.9134, \quad \Omega_3 = 0.8616 \quad \text{and} \quad \chi = 130.312.$$

$(H_1) - (H_2)$ hold, for each $t \in \left(\frac{1}{2}\mathbb{N}\right)_{\frac{17}{30}, \frac{293}{120}}$, because we obtain

$$
\begin{gathered}
\left|F[t, u, \Psi_{\frac{1}{2}}^{\frac{1}{2}} u] - F[t, v, \Psi_{\frac{1}{2}}^{\frac{1}{2}} v]\right| \leq \tfrac{e}{570}|u-v| + \tfrac{1}{570}|\Psi_{\frac{1}{2}}^{\frac{1}{2}} u - \Psi_{\frac{1}{2}}^{\frac{1}{2}} v|,\\
\tfrac{1}{e} < g(t) < e \ \text{ and } \ \varphi_0 \leq \tfrac{e}{6}.
\end{gathered}
$$

Finally, we can show that

$$\left(L_1 + L_2 \frac{\varphi_0\left((T+2)h\right)_h^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right)\chi \;=\; 0.859 \;<\; 1.$$

Hence, by Theorem 3.1, the problem 27 has a unique solution on $\left(\frac{1}{2}\mathbb{N}\right)_{\frac{17}{30}, \frac{293}{120}}$. □

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