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On the Uniform Boundedness and Convergence of Generalized, Moore-Penrose and Group Inverses

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Abstract. This paper concerns the relationship between uniform boundedness and convergence of various generalized inverses. Using the stable perturbation for generalized inverse and the gap between closed linear subspaces, we prove the equivalence of the uniform boundedness and convergence for generalized inverses. Based on this, we consider the cases for the Moore-Penrose inverses and group inverses. Some new and concise expressions and convergence theorems are provided. The obtained results extend and improve known ones in operator theory and matrix theory.

1. Introduction and Preliminaries

Let X and Y be Banach spaces and $B(X, Y)$ the Banach space of all bounded linear operators from X into Y. We write $B(X)$ as $B(X, X)$. For any $T \in B(X, Y)$, we denote by $N(T)$ and $R(T)$ the null space and the range of T, respectively. The identity operator will be denoted by I.

Definition 1.1. *Let X and Y be Hilbert spaces. An operator $S \in B(Y, X)$ is called the Moore-Penrose inverse of $T \in B(X, Y)$ if S satisfies the four Penrose equations:*

$$(1)\ TST = T; \quad (2)\ STS = S; \quad (3)\ (TS)^* = TS; \quad (4)\ (ST)^* = ST,$$

where T^* denotes the adjoint operator of T. The Moore-Penrose inverse of T is always written by $T^\dagger$, which is uniquely determined if it exists.

Definition 1.2. *Let X and Y be Banach spaces. An operator $S \in B(Y, X)$ is called a generalized inverse of $T \in B(X, Y)$ if S satisfies:*

$$(1)\ TST = T \quad \text{and} \quad (2)\ STS = S,$$

The generalized inverse of T is always denoted by T^+. Furthermore, if $X = Y$ and S also satisfies

$$(5)\ TS = ST,$$

the corresponding generalized inverse is called the group inverse, denoted by $T^\sharp$, which is unique if it exists.

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Generalized inverses, Moore-Penrose inverses and group inverses have lots of applications in many fields, such as optimization, statistics and singular linear equations (see[3, 16, 17, 24]). For instance, let $T \in B(X, Y)$ and $b \in Y$. To consider the linear equation

$$Tx = b \tag{1.1}$$

with the unknown $x \in X$, we can investigate the approximating equation

$$T_n x = b_n \tag{1.2}$$

where $T_n \in B(X, Y)$ with $T_n \to T$ in the usual operator norm of $B(X, Y)$ and $b_n \in Y$ with $b_n \to b$ in Y as $n \to +\infty$. It is natural to ask whether the approximating solution (1.2) converges to the real solution of (1.1). For example, if T_n and T are invertible, does $T_n^{-1} b_n \to T^{-1} b$ or $T_n^{-1} \to T^{-1}$ hold? The following theorem is well-known.

Theorem 1.3. *Let $T \in B(X, Y)$ be invertible and T^{-1} its inverse. If $T_n \in B(X, Y)$ satisfies $T_n \to T$, then there exists $N \in \mathbf{N}$, such that for all $n \geq N$, T_n is invertible and*

$$T_n^{-1} \to T^{-1}.$$

In this case, $\sup\limits_{n \geq N} \|T_n^{-1}\| < +\infty$. Conversely, if T_n is invertible and $\sup\limits_{n \in \mathbf{N}} \|T_n^{-1}\| < +\infty$, we have

Theorem 1.4. *Let T_n and $T \in B(X, Y)$ satisfy $T_n \to T$. If T_n is invertible and $\sup\limits_{n \in \mathbf{N}} \|T_n^{-1}\| < +\infty$, then T is invertible and*

$$T_n^{-1} \to T^{-1}.$$

It can be claimed that in the case of invertible operators, the uniform boundedness of $\|T_n^{-1}\|$ can imply the invertibility of T and the convergence $T_n^{-1} \to T^{-1}$, i.e., under the condition of uniform boundedness, the approximating solution does converge to the real solution. If T is not invertible, what happens? Particularly, does Theorem 1.4 hold for the case of Moore-Penrose inverses, group inverses or generalized inverses? Much attention has been paid to the perturbation and convergence problem for Moore-Penrose, group and Drazin inverses [1, 2, 4–8, 10, 13–15, 19–23, 25]. For instance, J. Koliha [13], J. Benítez, D. Cvetković-Ilić and X. Liu [1] gave the following theorems (in C^*-algebra).

Theorem 1.5. [13] *Let X, Y be Hilbert spaces and $T_n, T \in B(X, Y)$ with $T_n \to T$. If T_n is Moore-Penrose invertible and $\sup\limits_{n \in \mathbf{N}} \|T_n^{\dagger}\| < +\infty$, then T is Moore-Penrose invertible and*

$$T_n^{\dagger} \to T^{\dagger}.$$

Theorem 1.6. [1] *Let X be Banach space and $T_n, T \in B(X)$ with $T_n \to T$. If the group inverse $T_n^{\sharp}$ exists and $\sup\limits_{n \in \mathbf{N}} \|T_n^{\sharp}\| < +\infty$, then $T^{\sharp}$ exists and*

$$T_n^{\sharp} \to T^{\sharp}.$$

That is, Theorem 1.4 holds for the Moore-Penrose inverses and group inverses. How about the generalized inverses?

Example 1.7. *Let*

$$T_n = \begin{pmatrix} 1 - \frac{1}{n} & 0 \\ \frac{1}{n} & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then $T_n \to T$, the Moore-Penrose inverse $T_n^{\dagger}$ and the group inverse $T_n^{\sharp}$ are

$$T_n^{\dagger} = \begin{pmatrix} \frac{n^2-n}{n^2-2n+2} & \frac{n}{n^2-2n+2} \\ 0 & 0 \end{pmatrix} \text{ and } T_n^{\sharp} = \begin{pmatrix} \frac{n}{n-1} & 0 \\ \frac{n}{(n-1)^2} & 0 \end{pmatrix},$$

respectively, which converge to the Moore-Penrose inverse and group inverse of T,

$$T^{\dagger} = T^{\sharp} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to verify that

$$T_n^{+} = \begin{pmatrix} 1 & 1 \\ \alpha_n & \alpha_n \end{pmatrix} (\forall\ \alpha_n \in R)$$

is a generalized inverse of T_n*. Let* $\alpha_n = (-1)^n$*, then* $\{T_n^+\}$ *is uniformly bounded but not convergent, i.e.,*

$$\sup_{n\in\mathbf{N}} \|T_n^+\| < +\infty \quad \textit{does not imply} \quad T_n^+ \to T^+,$$

although Rank T_n= Rank T.

It can be said that the case of generalized inverses is totally different from that of Moore-Penrose inverses and group inverses. In this paper, we shall use the stable perturbation to investigate the link between the uniform boundedness and convergence of generalized inverses. For the stable perturbation of generalized inverses, we have

Theorem 1.8. [Finite Rank Theorem][16] *Let* $T \in B(X, Y)$ *be of finite rank and* T^+ *a generalized inverse of* T*. Let* $\overline{T} = T + \delta T \in B(X, Y)$ *with* $\|T^+\delta T\| < 1$*. Then* $B = (I + T^+\delta T)^{-1}T^+$ *is a generalized inverse of* $\overline{T}$ *if and only if*

$$\text{Rank } \overline{T} = \text{Rank } T < \infty.$$

Theorem 1.9. [16] *Let* $T \in B(X, Y)$ *with a generalized inverse* $T^+ \in B(Y, X)$ *and* $\delta T \in B(X, Y)$ *with* $\|T^+\delta T\| < 1$*. Then the following statements are equivalent:*
(1) $B = (I + T^+\delta T)^{-1}T^+$ *is a generalized inverse of* $\overline{T} = T + \delta T$;
(2) $R(\overline{T}) \cap N(T^+) = \{0\}$;
(3) $Y = R(\overline{T}) \oplus N(T^+)$;
(4) $X = N(\overline{T}) \oplus R(T^+)$;
(5) $(I + \delta TT^+)^{-1}\overline{T}N(T) \subset R(T)$.

In the next section, we first give an equivalent condition for the uniform boundedness and convergence of generalized inverse. Applications to the Moore-Penrose inverse and group inverse are also considered. It is worth mentioning that our proof is brief and some concrete expressions are provided. Our results extend and improve many known ones in operator theory and matrix theory.

2. Main Results

Following T. Kato[12], for any closed linear subspaces M and N of X, we define the gap between M and N by

$$\text{gap}(M, N) = \max\{\delta(M, N), \delta(N, M)\}$$

where $\delta(\{0\}, N) = 0$ and

$$\delta(M, N) = \sup\{d(u, N) : u \in M, \|u\| = 1\}, \quad M \neq \{0\},$$

and $d(u, N) = \inf\{\|u - x\| : x \in N\}$.

Lemma 2.1. *Let* $T_n, T \in B(X, Y)$ *have generalized inverses* $T_n^+, T^+ \in B(Y, X)$*, respectively. Then*

$$\delta(R(T_n), R(T)) \leq \|I - TT^+\|\|T_n - T\|\|T_n^+\|.$$

Proof. If $R(T_n) = \{0\}$, then $T_n = 0$. Hence $T_n^+ = 0$ and the inequality holds. If $R(T_n) \neq \{0\}$, let $u \in R(T_n)$ and $\|u\| = 1$, then

$$\begin{aligned} d(u, R(T)) &= \inf\{\|u - TT^+y\| : y \in Y\} \\ &\leq \|u - TT^+u\| \\ &= \|(I - TT^+)u\| \\ &= \|(I - TT^+)T_nT_n^+u\| \\ &= \|(I - TT^+)[T + (T_n - T)]T_n^+u\| \\ &= \|(I - TT^+)(T_n - T)T_n^+u\| \\ &\leq \|I - TT^+\|\|T_n - T\|\|T_n^+\|\|u\| \\ &= \|I - TT^+\|\|T_n - T\|\|T_n^+\|. \end{aligned}$$

Thus we get what we desired. □

Theorem 2.2. *Let $T_n, T \in B(X, Y)$ be generalized invertible and $T_n \to T$. If the generalized inverse T_n^+ satisfies $\sup\limits_{n \in \mathbf{N}} \|T_n^+\| < +\infty$, then for any generalized inverse T^+, there exists a generalized inverse $T_n^\oplus$ of T_n, such that*

$$T_n^\oplus \to T^+.$$

Proof. From Theorem 1.9, it is enough to prove that for all sufficiently large n,

$$R(T_n) \cap N(T^+) = \{0\}. \tag{2.1}$$

In fact, if so,

$$T_n^\oplus = T^+[I + (T_n - T)T^+]^{-1} = [I + T^+(T_n - T)]^{-1}T^+$$

is a generalized inverse of T_n and obviously, $T_n^\oplus \to T^+$. Assume that (2.1) does not hold, then for any $k \in \mathbf{N}$, there always is an $n_k > k$, such that

$$R(T_{n_k}) \cap N(T^+) \neq \{0\}.$$

We can take some $y_{n_k} \in R(T_{n_k}) \cap N(T^+)$ satisfying $\|y_{n_k}\| = 1$. Hence $(I - TT^+)y_{n_k} = y_{n_k} \neq 0$, and so $I - TT^+ \neq 0$. Thus for all $x \in X$,

$$\begin{aligned} \|y_{n_k} - Tx\| &\geq \|I - TT^+\|^{-1}\|(I - TT^+)(y_{n_k} - Tx)\| \\ &= \|I - TT^+\|^{-1}\|y_{n_k}\| \\ &= \|I - TT^+\|^{-1}. \end{aligned}$$

This means

$$d(y_{n_k}, R(T)) \geq \|I - TT^+\|^{-1}$$

and therefore

$$\delta(R(T_{n_k}), R(T)) \geq \|I - TT^+\|^{-1}.$$

Combining it with Lemma 2.1, we can obtain

$$\|I - TT^+\|\|T_{n_k} - T\|\|T_{n_k}^+\| \geq \delta(R(T_{n_k}), R(T)) \geq \|I - TT^+\|^{-1},$$

which implies

$$\|T_{n_k} - T\|\|T_{n_k}^+\| \geq \|I - TT^+\|^{-2}.$$

Noting $\sup\limits_{n \in \mathbf{N}} \|T_n^+\| < +\infty$, we get a contradiction. □

Remark 2.3. *Even if $\sup\limits_{n \in \mathbf{N}} \|T_n^+\| < +\infty$, $\{T_n^+\}$ may not be convergent. But we can find another convergent generalized inverses $\{T_n^\oplus\}$.*

Example 2.4. *In Example 1.7,* $\{T_n^+\}$ $(\alpha_n = (-1)^n)$ *is uniformly bounded and not convergent, but*

$$T_n^{\oplus} = T^{\dagger}[I + (T_n - T)T^{\dagger}]^{-1} = \begin{pmatrix} \frac{n}{n-1} & 0 \\ 0 & 0 \end{pmatrix}$$

is a generalized inverse of T_n *and converges to* $T^{\dagger}$.

Corollary 2.5. *Let* $T \in B(X, Y)$ *be of finite rank. If* $T_n \in B(X, Y)$ *and* $T_n \to T$*, then the following statements are equivalent:*
(1) Rank T_n = Rank T *for all sufficiently large n;*
(2) *there exists* $N \in \mathbf{N}$*, such that for all* $n \geq N$, T_n *has a generalized inverse* T_n^+ *satisfying*

$$\sup_{n\geq N} \|T_n^+\| < +\infty;$$

(3) *for any generalized inverse* T^+ *of* T*, there exists* $N \in \mathbf{N}$*, such that for all* $n \geq N$, T_n *has a generalized inverse* T_n^+ *satisfying*

$$T_n^+ \to T^+.$$

Proof. It is enough to prove (1) ⇔ (3). Noting that (1) ⇒ (3) comes from Theorem 1.8, we only need to show (3) ⇒ (1). If T_n has a generalized inverse T_n^+ satisfying $T_n^+ \to T^+$, then projectors $T_nT_n^+ \to TT^+$. Hence by Lemma 4.10 in [12], there exists $N \in \mathbf{N}$, such that for all $n \geq N$, $\dim R(T_nT_n^+) = \dim R(TT^+)$, i.e., Rank T_n = Rank T. □

Remark 2.6. *It should be noted that in (2) of Corollary 2.5, not every generalized inverse* T_n^+ *satisfies* $\sup_{n\in\mathbf{N}} \|T_n^+\| < +\infty$.

Example 2.7. *In Example 1.7, if we take* $\alpha_n = n$*, then*

$$T_n^+ = \begin{pmatrix} 1 & 1 \\ n & n \end{pmatrix}$$

is a generalized inverse of T_n *and* $\sup_{n\in\mathbf{N}} \|T_n^+\| = +\infty$*, although* Rank T_n = Rank T.

It should be pointed out that the generalized invertibility of T in Theorem 2.2 can not be deleted. But in the case of Hilbert space, it can be done.

Lemma 2.8. [11] *Let* X, Y *be Hilbert spaces and* $T \in B(X, Y)$ *with a generalized inverse* $T^+ \in B(Y, X)$*. Then* T *has the Moore–Penrose inverse* $T^{\dagger}$ *and*

$$T^{\dagger} = [I - T^+T - (T^+T)^*]^{-1}T^+[I - TT^+ - (TT^+)^*]^{-1}.$$

Theorem 2.9. *Let* X, Y *be Hilbert spaces and* $T_n, T \in B(X, Y)$ *with* $T_n \to T$*. If* T_n *has a generalized inverse* T_n^+ *satisfying* $\sup_{n\in\mathbf{N}} \|T_n^+\| < +\infty$*, then* T *has a generalized inverse. Moreover, for any generalized inverse* T^+*, there exists a generalized inverse* $T_n^{\oplus}$ *of* T_n *such that*

$$T_n^{\oplus} \to T^+.$$

Proof. By Lemma 2.8, the Moore-Penrose inverse $T_n^{\dagger}$ exists. It follows from

$$T_n^{\dagger} = T_n^{\dagger}T_nT_n^{\dagger} = T_n^{\dagger}T_nT_n^{+}T_nT_n^{\dagger}$$

that $\|T_n^{\dagger}\| \leq \|T_n^{\dagger}T_n\|\|T_n^+\|\|T_nT_n^{\dagger}\| = \|T_n^+\|$ and so $\sup_{n\in\mathbf{N}} \|T_n^{\dagger}\| < +\infty$. Utilizing the equality (2.1) in [11], i.e.,

$$T_m^{\dagger} - T_n^{\dagger} = -T_m^{\dagger}(T_m - T_n)T_n^{\dagger} + (I - T_m^{\dagger}T_m)(T_m^* - T_n^*)(T_n^{\dagger})^*T_n^{\dagger} + T_m^{\dagger}(T_m^{\dagger})^*(T_m^* - T_n^*)(I - T_nT_n^{\dagger}),$$

we know that $\{T_n^{\dagger}\}$ is a Cauchy sequence in $B(Y, X)$. Assuming $T_n^{\dagger} \to S \in B(Y, X)$, we take the limit in the four Penrose equations and hence S is the Moore-Penrose inverse of T. By Theorem 2.2, we can get the conclusion. □

In fact, we have proved Theorem 1.5 and obtained the expression of the Moore-Penrose inverse $T_n^{\dagger}$.

Theorem 2.10. *Let X, Y be Hilbert spaces and $T_n, T \in B(X, Y)$ with $T_n \to T$. If T_n is Moore-Penrose invertible and $\sup\limits_{n\in\mathbf{N}} \|T_n^{\dagger}\| < +\infty$, then T is Moore-Penrose invertible,*

$$T_n^{\dagger} \to T^{\dagger}$$

and for all sufficiently large n,

$$T_n^{\dagger} = [I - B_nT_n - (B_nT_n)^*]^{-1}B_n[I - T_nB_n - (T_nB_n)^*]^{-1},$$

where $B_n = [I + T^{\dagger}(T_n - T)]^{-1}T^{\dagger}$.

Proof. From Theorem 2.9, T is generalized invertible and then it is Moore-Penrose invertible. Then by the proof of Theorem 2.2, $B_n = [I + T^{\dagger}(T_n - T)]^{-1}T^{\dagger}$ is a generalized inverse of T_n. It follows from Lemma 2.8 that

$$T_n^{\dagger} = [I - B_nT_n - (B_nT_n)^*]^{-1}B_n[I - T_nB_n - (T_nB_n)^*]^{-1}$$

and obviously, $T_n^{\dagger} \to T^{\dagger}$. □

For finite-rank operators between Hilbert spaces, we have

Corollary 2.11. *Let X, Y be Hilbert spaces and $T \in B(X, Y)$ be of finite rank. If $T_n \in B(X, Y)$ and $T_n \to T$, then the following statements are equivalent:*

(1) Rank T_n = Rank T *for all sufficiently large n;*

(2) *there exists $N \in \mathbf{N}$, such that for all $n \geq N$, T_n is Moore-Penrose invertible and the Moore-Penrose inverse $T_n^{\dagger}$ satisfies $T_n^{\dagger} \to T^{\dagger}$;*

(3) *there exists $N \in \mathbf{N}$, such that for all $n \geq N$, T_n is Moore-Penrose invertible and the Moore-Penrose inverse $T_n^{\dagger}$ satisfies*

$$\sup_{n\in\mathbf{N}} \|T_n^{\dagger}\| < +\infty;$$

(4) *there exists $N \in \mathbf{N}$, such that for all $n \geq N$, T_n has a generalized inverse T_n^{+} satisfying*

$$\sup_{n\in\mathbf{N}} \|T_n^{+}\| < +\infty;$$

(5) *for any generalized inverse T^{+} of T, there exists $N \in \mathbf{N}$, such that for all $n \geq N$, T_n has a generalized inverse T_n^{+} satisfying*

$$T_n^{+} \to T^{+}.$$

Proof. Obviously, (1) ⇔ (5) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ((5) ⇒ (2) comes from Lemma 2.8). □

Next we shall discuss the uniform boundedness and convergence of group inverse. We first prove the following convergence theorem which is parallel to Theorem 2.3 in [9].

Theorem 2.12. *Let X be a Banach space and $T \in B(X)$ be group invertible. Let $T_n \in B(X)$ satisfy $T_n \to T$. Then the following statements are equivalent:*

(1) *for all sufficiently large n,*

$$R(T_n) \cap N(T^{\sharp}) = \{0\};$$

(2) *there exists $N \in \mathbf{N}$, such that for all $n \geq N$, T_n is group invertible with*

$$T_n^{\sharp} \to T^{\sharp}.$$

In this case, for all sufficiently large n,

$$T_n^{\sharp} = B_nW_n^{-1} + (I - B_nT_n)W_n^{-1}B_nW_n^{-1}, \tag{2.2}$$

where $B_n = [I + T^{\sharp}(T_n - T)]^{-1}T^{\sharp}$ and $W_n = B_nT_n + T_nB_n - I$.

Proof. (1) ⇒ (2). It follows from Theorem 1.9 that for all sufficiently large n,

$$B_n \doteq T^\sharp[I + (T_n - T)T^\sharp]^{-1} = [I + T^\sharp(T_n - T)]^{-1}T^\sharp$$

is a generalized inverse of T_n. Noticing $(2TT^\sharp - I)^2 = I$, we get that $2TT^\sharp - I$ is invertible and $(2TT^\sharp - I)^{-1} = 2TT^\sharp - I$. Since

$$\begin{aligned} B_nT_n + T_nB_n - I &= [I + T^\sharp(T_n - T)]^{-1}T^\sharp T_n + T_nT^\sharp[I + (T_n - T)T^\sharp]^{-1} - I \\ &= [I + T^\sharp(T_n - T)]^{-1}\{T^\sharp T_n[I + (T_n - T)T^\sharp] + [I + T^\sharp(T_n - T)]T_nT^\sharp \\ &\quad -[I + T^\sharp(T_n - T)][I + (T_n - T)T^\sharp]\}[I + (T_n - T)T^\sharp]^{-1} \\ &= [I + T^\sharp(T_n - T)]^{-1}[T^\sharp T + TT^\sharp - I + T^\sharp(T_n^2 - T^2)T^\sharp][I + (T_n - T)T^\sharp]^{-1} \\ &= [I + T^\sharp(T_n - T)]^{-1}[2TT^\sharp - I + T^\sharp(T_n^2 - T^2)T^\sharp][I + (T_n - T)T^\sharp]^{-1}, \end{aligned}$$

we know that for all sufficiently large n, $W_n \doteq B_nT_n + T_nB_n - I$ is invertible. In the following, we shall show that

$$S_n \doteq B_nW_n^{-1} + (I - B_nT_n)W_n^{-1}B_nW_n^{-1} = T_nB_nW_n^{-1}B_nW_n^{-1}$$

is the group inverse of T_n. In fact, if we set $P_n = B_nT_n$ and $Q_n = T_nB_n$, then P_n and Q_n are idempotent operators, and

$$\begin{aligned} P_nW_n &= P_nQ_n = W_nQ_n, \\ Q_nW_n &= Q_nP_n = W_nP_n, \\ W_nT_n &= B_nT_n^2 = P_nT_n. \end{aligned}$$

Hence

$$W_n^{-1}P_n = Q_nW_n^{-1},\ W_n^{-1}Q_n = P_nW_n^{-1},\ P_nQ_nW_n^{-1} = P_n,\ W_n^{-1}P_nT_n = T_n.$$

Therefore, by the definition of S_n, we obtain $T_nS_n = T_nB_nW_n^{-1} = Q_nW_n^{-1}$,

$$T_nS_nT_n = Q_nW_n^{-1}T_n = W_n^{-1}P_nT_n = T_n,$$

$$\begin{aligned} S_nT_nS_n &= S_nT_nB_nW_n^{-1} \\ &= S_n(T_nB_nW_n^{-1} - I) + S_n \\ &= S_n(T_nB_n - W_n)W_n^{-1} + S_n \\ &= S_n(I - B_nT_n)W_n^{-1} + S_n \\ &= T_nB_nW_n^{-1}B_nW_n^{-1}(I - B_nT_n)W_n^{-1} + S_n \\ &= T_nB_nW_n^{-1}B_nQ_nW_n^{-1}(I - B_nT_n)W_n^{-1} + S_n \\ &= T_nB_nW_n^{-1}B_nW_n^{-1}P_n(I - B_nT_n)W_n^{-1} + S_n \\ &= S_n \end{aligned}$$

and $B_nW_n^{-1}T_n = B_nQ_nW_n^{-1}T_n = B_nW_n^{-1}P_nT_n = B_nT_n = P_n$,

$$\begin{aligned} S_nT_n &= B_nW_n^{-1}T_n + (I - B_nT_n)W_n^{-1}B_nW_n^{-1}T_n \\ &= P_n + W_n^{-1}B_nW_n^{-1}T_n - P_nW_n^{-1}B_nW_n^{-1}T_n \\ &= P_n + W_n^{-1}P_n - P_nW_n^{-1}P_n \\ &= P_n + W_n^{-1}P_n - P_nQ_nW_n^{-1} \\ &= P_n + W_n^{-1}P_n - P_n \\ &= W_n^{-1}P_n \\ &= Q_nW_n^{-1} \\ &= T_nS_n. \end{aligned}$$

Thus we have concluded that S_n is the group inverse of T_n and obviously,

$$\begin{aligned} S_n &\to T^\sharp(2TT^\sharp - I)^{-1} + (I - T^\sharp T)(2TT^\sharp - I)^{-1}T^\sharp(2TT^\sharp - I)^{-1} \\ &= T^\sharp(2TT^\sharp - I) + (I - T^\sharp T)(2TT^\sharp - I)T^\sharp(2TT^\sharp - I) \\ &= T^\sharp. \end{aligned}$$

(2) ⇒ (1). Without loss of generality, we can suppose that

$$\|T_n^\sharp T_n - T^\sharp T\| < 1 \quad \text{and} \quad \|T_n - T\|\|T^\sharp\| < 1.$$

Then $I - (T_n^\sharp T_n - T^\sharp T)$ and $I + (T_n - T)T^\sharp$ are invertible. For any $x \in N(T)$, set

$$y_n = (I - T_n^\sharp T_n)[I - (T_n^\sharp T_n - T^\sharp T)]^{-1}x,$$

then $y_n \in N(T_n)$,

$$\begin{aligned} x &= (I - T^\sharp T)\{T^\sharp T[I - (T_n^\sharp T_n - T^\sharp T)]^{-1}x + x\} \\ &= (I - T^\sharp T)(I - T_n^\sharp T_n + 2T^\sharp T)[I - (T_n^\sharp T_n - T^\sharp T)]^{-1}x \\ &= (I - T^\sharp T)(I - T_n^\sharp T_n)[I - (T_n^\sharp T_n - T^\sharp T)]^{-1}x \\ &= (I - T^\sharp T)y_n, \end{aligned}$$

and

$$T_n x = T_n(I - T^\sharp T)y_n = -T_n T^\sharp T y_n = -[I + (T_n - T)T^\sharp]Ty_n.$$

Hence we have proved $[I + (T_n - T)T^\sharp]^{-1}T_n x = -Ty_n$, and so

$$[I + (T_n - T)T^\sharp]^{-1}T_n N(T) \subset R(T).$$

By Theorem 1.9, we get $R(T_n) \cap N(T^\sharp) = \{0\}$. □

Remark 2.13. *The invertibility of W_n used in Theorem 2.12 is inspired from [18]. It is worth to point out that the statement $R(T_n) \cap N(T^\sharp) = \{0\}$ is called to be a stable perturbation of T which is an extension of rank-preserving perturbation and used widely in perturbation theory of generalized inverses [4].*

Now we give a concise proof of Theorem 1.6 and furthermore, we obtain a concrete expression of $T_n^\sharp$.

Theorem 2.14. *Let X be a Banach space and $T_n, T \in B(X)$ with $T_n \to T$. If the group inverses $T_n^\sharp$ exist and $\sup\limits_{n\in\mathbf{N}} \|T_n^\sharp\| < +\infty$, then T has the group inverse $T^\sharp$ satisfying*

$$T_n^\sharp \to T^\sharp,$$

and for all sufficiently large n, the expression (2.2) holds.

Proof. It follows from $\sup\limits_{n\in\mathbf{N}} \|T_n^\sharp\| < +\infty$ and

$$\begin{aligned} T_m^\sharp - T_n^\sharp &= T_m^\sharp - T_m^\sharp T_n^\sharp T_n - T_n^\sharp + T_m^\sharp T_m T_n^\sharp + T_m^\sharp T_n^\sharp T_n - T_m^\sharp T_m T_n^\sharp \\ &= T_m^\sharp(I - T_n^\sharp T_n) - (I - T_m^\sharp T_m)T_n^\sharp - T_m^\sharp(T_m - T_n)T_n^\sharp \\ &= (T_m^\sharp)^2 T_m(I - T_n^\sharp T_n) - (I - T_m^\sharp T_m)T_n(T_n^\sharp)^2 - T_m^\sharp(T_m - T_n)T_n^\sharp \\ &= (T_m^\sharp)^2(T_m - T_n)(I - T_n^\sharp T_n) + (I - T_m^\sharp T_m)(T_m - T_n)(T_n^\sharp)^2 - T_m^\sharp(T_m - T_n)T_n^\sharp \end{aligned}$$

that $\{T_n^\sharp\}$ is a Cauchy sequence in $B(X)$. Assuming $T_n^\sharp \to S \in B(X)$, we take the limit in three equalities in the definition of $T_n^\sharp$. Then S is the group inverse of T and $T_n^\sharp \to S = T^\sharp$. By Theorem 2.12, we get the conclusion. □

Next, we can give succinct expressions of group inverses in some special cases.

Theorem 2.15. *Let X be a Banach space and $T \in B(X)$ be group invertible. If $T_n \in B(X)$ satisfies $T_n \to T$ and $T_n = TT^\sharp T_n$, then for all sufficiently large n, T_n is group invertible and*

$$T_n^\sharp = B_n^2 T_n = \{[I + T^\sharp(T_n - T)]^{-1} T^\sharp\}^2 T_n = \{T^\sharp [I + (T_n - T)T^\sharp]^{-1}\}^2 T_n.$$

Proof. Since $T_n = TT^\sharp T_n$, we have $R(T_n) \subset R(T)$ and $R(T_n) \cap N(T^\sharp) \subset R(T) \cap N(T^\sharp) = \{0\}$. By Theorem 1.9 and Theorem 2.12, B_n is a generalized inverse of T_n, T_n is group invertible, and

$$T_n^\sharp = B_n W_n^{-1} + (I - B_n T_n) W_n^{-1} B_n W_n^{-1} = T_n B_n W_n^{-1} B_n W_n^{-1}.$$

Noting $T_n T^\sharp = TT^\sharp T_n T^\sharp = TT^\sharp [I + (T_n - T)T^\sharp]$ and $T^\sharp T_n TT^\sharp = [I + T^\sharp(T_n - T)]TT^\sharp$, we get

$$T_n B_n = T_n T^\sharp [I + (T_n - T)T^\sharp]^{-1} = TT^\sharp \quad \text{and} \quad B_n T_n TT^\sharp = TT^\sharp.$$

Hence $W_n = B_n T_n + T_n B_n - I = B_n T_n + TT^\sharp - I$ and by $TT^\sharp B_n T_n = B_n T_n$,

$$W_n^2 = (B_n T_n + TT^\sharp - I)(B_n T_n + TT^\sharp - I) = I$$

which implies $W_n^{-1} = W_n$. Thus $T_n B_n W_n = TT^\sharp W_n = TT^\sharp B_n T_n$ and

$$T_n^\sharp = T_n B_n W_n B_n W_n = TT^\sharp B_n T_n B_n W_n = B_n T_n B_n W_n = B_n W_n = B_n^2 T_n.$$

□

Remark 2.16. *We can verify directly that $T_n(B_n^2 T_n)T_n = T_n$, $(B_n^2 T_n)T_n(B_n^2 T_n) = (B_n^2 T_n)$ and $(B_n^2 T_n)T_n = T_n(B_n^2 T_n)$.*

Theorem 2.17. *Let X be a Banach space and $T \in B(X)$ be group invertible. If $T_n \in B(X)$ satisfies $T_n \to T$ and $T_n = T_n T^\sharp T$, then for all sufficiently large n, T_n is group invertible and*

$$T_n^\sharp = T_n B_n^2 = T_n \{[I + T^\sharp(T_n - T)]^{-1} T^\sharp\}^2 = T_n \{T^\sharp [I + (T_n - T)T^\sharp]^{-1}\}^2.$$

Proof. Since $T_n = T_n T^\sharp T$, we have $N(T) \subset N(T_n)$ and by (5) in Theorem 1.9 and Theorem 2.12, T_n is group invertible and $T_n^\sharp = T_n B_n W_n^{-1} B_n W_n^{-1}$. Noting $T^\sharp T_n = T^\sharp T_n T^\sharp T = [I + T^\sharp(T_n - T)]T^\sharp T$ and $TT^\sharp T_n T^\sharp = TT^\sharp [I + (T_n - T)T^\sharp]$, we get

$$B_n T_n = [I + T^\sharp(T_n - T)]^{-1} T^\sharp T_n = T^\sharp T \quad \text{and} \quad TT^\sharp T_n B_n = TT^\sharp.$$

Hence $W_n = T_n B_n + T^\sharp T - I$ and by $B_n TT^\sharp = B_n$,

$$W_n^2 = (T_n B_n + T^\sharp T - I)(T_n B_n + T^\sharp T - I) = I$$

which implies $W_n^{-1} = W_n$. Thus $B_n W_n = B_n T^\sharp T = B_n$ and

$$T_n^\sharp = T_n B_n W_n B_n W_n = T_n B_n T^\sharp T B_n W_n = T_n B_n B_n W_n = T_n B_n^2.$$

□

Remark 2.18. *The simpler expressions obtained in Theorems 2.15 and 2.17 are consistent with those given in [14] for matrices.*

From the above theorems, we can obtain a characterization for $T_n^\sharp$ to have the simplest possible expression.

Theorem 2.19. *Let X be a Banach space and $T \in B(X)$ be group invertible. Let $T_n \in B(X)$ satisfy $T_n \to T$, then for all sufficiently large n, $T_n = T_nT^\sharp T = TT^\sharp T_n$ if and only if T_n is group invertible and*

$$T_n^\sharp = B_n = [I + T^\sharp(T_n - T)]^{-1}T^\sharp = T^\sharp[I + (T_n - T)T^\sharp]^{-1}.$$

Proof. If $T_n = T_nT^\sharp T = TT^\sharp T_n$, then $T_n^\sharp = T_nB_n^2 = TT^\sharp B_n = B_n$. Conversely, if $T_n^\sharp = [I + T^\sharp(T_n - T)]^{-1}T^\sharp$, then

$$R(I - T^\sharp T) = N(T^\sharp T) = N(TT^\sharp) = N(T^\sharp) = N(T_n^\sharp) = N(T_n)$$

and

$$N(I - TT^\sharp) = R(TT^\sharp) = R(T^\sharp T) = R(T^\sharp) = R(T_n^\sharp) = R(T_n).$$

Hence $T_n(I - T^\sharp T) = 0$ and $(I - TT^\sharp)T_n = 0$, i.e., $T_n = T_nT^\sharp T = TT^\sharp T_n$. □

Remark 2.20. *The condition $T_n = T_nT^\sharp T = TT^\sharp T_n$ is called condition (W) in [23], which appears as a sufficient condition for the case of Drazin inverses.*

Corollary 2.21. *Let $T \in B(X)$ be of finite rank. If $T_n \to T$ and T is group invertible, then the following statements are equivalent:*
(1) Rank T_n = Rank T *for all sufficiently large n;*
(2) *there exists $N \in \mathbf{N}$, such that for all $n \geq N$, T_n is group invertible and*

$$T_n^\sharp \to T^\sharp;$$

(3) *there exists $N \in \mathbf{N}$, such that for all $n \geq N$, T_n is group invertible and*

$$\sup_{n \geq N} \|T_n^\sharp\| < +\infty.$$

In this case, for all sufficiently large n, the expression (2.2) holds.

Proof. If Rank T_n = Rank T, then by Theorems 1.8 and 1.9, we get $R(T_n) \cap N(T^\sharp) = \{0\}$. Hence by Theorem 2.12, we obtain (2). Thus (1) ⇒ (2) holds. It is easy to see (2) ⇒ (3) and that (3) ⇒ (1) follows from Corollary 2.11. □

Remark 2.22. *It is worth pointing out that in Corollary 2.21, we can conclude that if T is group invertible, $T_n \to T$ and* Rank T_n = Rank T, *then for all sufficiently large n, T_n is group invertible and $T_n^\sharp \to T^\sharp$. In the case of matrices, a well-known result is that if T_n and T are group invertible, $T_n \to T$ and* Rank T_n = Rank T, *then $T_n^\sharp \to T^\sharp$ [3, 22].*

The following example shows that the condition that T is group invertible in Corollary 2.21 can not be deleted.

Example 2.23. *Let*

$$T_n = \begin{pmatrix} \frac{1}{n} & 0 \\ 1 - \frac{1}{n} & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $T_n \to T$ and

$$T_n^\sharp = \begin{pmatrix} n & 0 \\ n(n-1) & 0 \end{pmatrix} \text{ is the group inverse of } T_n \text{ which is}$$

unbounded, although Rank T_n = Rank T. *It should be noted that* index $T = 2$ *and T is not group invertible.*

Remark 2.24. *Example 2.23 shows that Corollary 2.21 does not hold for Drazin inverses.*

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