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Almost Kenmotsu 3-Manifolds Satisfying Some Generalized Nullity Conditions

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Abstract. In this paper, a three-dimensional almost Kenmotsu manifold M^3 satisfying the generalized $(\kappa,\mu)'$-nullity condition is investigated. We mainly prove that on M^3 the following statements are equivalent: (1) M^3 is ϕ-symmetric; (2) the Ricci tensor of M^3 is cyclic-parallel; (3) the Ricci tensor of M^3 is of Codazzi type; (4) M^3 is conformally flat with scalar curvature invariant along the Reeb vector field; (5) M^3 is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$.

1. Introduction

As an extension of the well-known Kenmotsu manifolds (see [10]) and an analogy of almost Hermitian manifolds for manifolds of odd dimension, almost Kenmotsu manifolds defined in [9] are becoming an important research object in differential geometry of almost contact metric manifolds. For some recent results regarding such manifolds we refer the reader to [6, 7], [11–13] and also [16, 17]. Almost Kenmotsu manifolds satisfying the (κ,μ) and $(\kappa,\mu)'$-nullity conditions were firstly introduced and studied by Dileo and Pastore [7], where κ and μ both are constants. As a special case of the (κ,μ) and $(\kappa,\mu)'$-nullity conditions, κ-nullity condition defined on almost Kenmotsu manifolds was studied by Pastore and Saltarelli [12]. Later, Pastore and Saltarelli in [11] extended the above three nullity conditions to the corresponding generalized nullity conditions for which both κ and μ are assumed to be smooth functions. In particular, Saltarelli [13] studied three-dimensional almost Kenmotsu manifolds satisfying the generalized (κ,μ) and $(\kappa,\mu)'$-nullity conditions and described locally such manifolds under the condition $d\kappa\wedge\eta=0$ and $h\neq0$. We remark that a Kenmotsu manifold always satisfies the above all kinds of nullity conditions.

Three-dimensional Kenmotsu manifolds have been studied by De et al. in [3–5] from various points of view. In this paper, we aim to classify three-dimensional almost Kenmotsu manifolds satisfying the generalized $(\kappa,\mu)'$-nullity condition under some symmetry conditions. Our main results give some local classifications of such manifolds with some symmetry conditions restriction and this generalizes some corresponding results obtained by De and others. Precisely, we state that on any three-dimensional generalized $(\kappa,\mu)'$-almost Kenmotsu manifold M^3, the following assertions are equivalent to each other:

- M^3 is ϕ-symmetric.

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- The Ricci tensor of M^3 is cyclic-parallel.
- The Ricci tensor of M^3 is of Codazzi type.
- M^3 is conformally flat with its scalar curvature invariant along the Reeb vector field.
- M^3 is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$.

2. Almost Kenmotsu Manifolds

By an *almost contact metric structure* defined on a smooth differentiable manifold M^{2n+1} of dimension $(2n+1)$ we mean a (ϕ,ξ,η,g)-structure satisfying

$$\phi^2 = -\mathrm{id} + \eta\otimes\xi,\ \eta(\xi) = 1, \tag{1}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2}$$

for any vector fields X, Y, where ϕ is a $(1,1)$-type tensor field, ξ is a vector field called the Reeb vector field and η is a 1-form called the almost contact 1-form and g is a Riemannian metric called *compatible metric* with respect to the almost contact structure.

From Janssens and Vanhecke [9], in this paper by an *almost Kenmotsu manifold* we mean an almost contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ satisfying $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\eta\wedge\Phi$, where the *fundamental 2-form* Φ of the almost contact metric manifold M^{2n+1} is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any vector fields X and Y on M^{2n+1}. We consider the product $M^{2n+1}\times\mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$ and define on it an almost complex structure J by

$$J\Big(X, f\frac{\mathrm{d}}{\mathrm{d}t}\Big) = \Big(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\Big),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a C^∞-function on $M^{2n+1}\times\mathbb{R}$. We denote by $[\phi,\phi]$ the the Nijenhuis tensor of ϕ. If

$$[\phi,\phi] = -2\mathrm{d}\eta\otimes\xi$$

holds, or equivalently, J is integrable, then the almost contact metric structure is said to be *normal*. A normal almost Kenmotsu manifold is said to be a *Kenmotsu manifold* (cf. [9, 10]). It is well-known that an almost Kenmotsu manifold is a Kenmotsu manifold if and only if

$$(\nabla_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X$$

for any vector fields X, Y.

Let M^{2n+1} be an almost Kenmotsu manifold. We consider three tensor fields $l = R(\cdot,\xi)\xi$, $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $h' = h\circ\phi$ on M^{2n+1}, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. From Dileo and Pastore [6, 7], we know that the three $(1,1)$-type tensor fields l, h' and h are symmetric and satisfy $h\xi = 0$, $l\xi = 0$, $\mathrm{tr}h = 0$, $\mathrm{tr}(h') = 0$ and $h\phi + \phi h = 0$ and

$$\nabla_X\xi = X - \eta(X)\xi + h'X, \tag{3}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{4}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l, \tag{5}$$

$$\mathrm{tr}(l) = S(\xi,\xi) = g(Q\xi,\xi) = -2n - \mathrm{tr}h^2, \tag{6}$$

$$R(X,Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X \tag{7}$$

for any vector fields X, Y, where S, Q and ∇ denote the Ricci curvature tensor, the Ricci operator with respect to g and the Levi-Civita connection of g, respectively.

3. Generalized $(\kappa, \mu)'$-Nullity and Symmetry Conditions on Almost Kenmotsu 3-Manifolds

We first give the definition of the generalized $(\kappa, \mu)'$-nullity condition (cf. [11], [13]).

Definition 3.1. *A three-dimensional almost Kenmotsu manifold is said to be a generalized $(\kappa, \mu)'$-almost Kenmotsu 3-manifold if the Reeb vector field satisfies*

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y) \tag{8}$$

for any vector fields X, Y on M^3, where both κ and μ are smooth functions.

When both κ and μ in (8) are constants, then the generalized $(\kappa, \mu)'$-nullity condition is just the $(\kappa, \mu)'$-one (cf. [7]). We also need the following proposition in proofs of main results.

Proposition 3.2 ([6]). *An almost Kenmotsu 3-manifold is Kenmotsu if and only if h vanishes.*

On a Kenmotsu manifold using $h = 0$ in (7) we have $R(X,Y)\xi = -\eta(Y)X + \eta(X)Y$ for any vector fields X, Y. Then we say that a Kenmotsu manifold always satisfies the $(\kappa, \mu)'$-nullity condition with $\kappa = -1$ and μ an arbitrary function.

On a generalized $(\kappa, \mu)'$-almost Kenmotsu 3-manifold with $h \neq 0$, putting $Y = \xi$ in (8) we obtain $l = -\kappa\phi^2 + \mu h'$. From (1), it follows that $\phi l \phi = \kappa\phi^2 + \mu h'$. Using this in (4) we obtain $h^2 = (\kappa + 1)\phi^2$. We denote by λ the positive eigenvalue of h'. It follows that $\lambda = \sqrt{-1-\kappa}$ and also $-\lambda$ is another non-zero eigenvalue of h'. Moreover, from [13, pp. 441] we have

$$\xi(\kappa) = -2(\kappa + 1)(\mu + 2), \ \xi(\lambda) = -\lambda(\mu + 2). \tag{9}$$

Proposition 3.3 ([13, Proposition 5.1]). *Let M^3 be a generalized $(\kappa, \mu)'$-almost Kenmotsu 3-manifold with $h \neq 0$. Then we have*

$$\begin{aligned} &\nabla_e\xi = (1+\lambda)e, \ \nabla_{\phi e}\xi = (1-\lambda)\phi e, \ \nabla_e\phi e = -\frac{\phi e(\lambda)}{2\lambda}e, \\ &\nabla_{\phi e}e = -\frac{e(\lambda)}{2\lambda}\phi e, \ \nabla_\xi e = 0, \ \nabla_\xi \phi e = 0, \\ &\nabla_e e = \frac{\phi e(\lambda)}{2\lambda}\phi e - (1+\lambda)\xi, \ \nabla_{\phi e}\phi e = \frac{e(\lambda)}{2\lambda}e - (1-\lambda)\xi, \end{aligned} \tag{10}$$

where e and ϕe denote two eigenvector fields of h' corresponding the eigenvalues $\lambda > 0$ and $-\lambda$ respectively.

On a generalized $(\kappa, \mu)'$-almost Kenmotsu 3-manifold with $h \neq 0$ we also have (cf. [13, Lemma 3.3])

$$Q = \left(\frac{r}{2} - \kappa\right)\mathrm{id} + \left(3\kappa - \frac{r}{2}\right)\eta \otimes \xi + \mu h', \ h'(\mathrm{grad}\mu) = \mathrm{grad}\kappa - \xi(\kappa)\xi, \tag{11}$$

where r is the scalar curvature and grad denotes the usual gradient operator. From (11) and Proposition 3.3 we obtain

$$Q\xi = 2\kappa\xi, \ Qe = \left(\frac{r}{2} - \kappa + \lambda\mu\right)e, \ Q\phi e = \left(\frac{r}{2} - \kappa - \lambda\mu\right)\phi e. \tag{12}$$

Proposition 3.4. *A generalized $(\kappa, \mu)'$-almost Kenmotsu 3-manifold is Einstein if and only if it is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$.*

Proof. Let M^3 be a generalized $(\kappa, \mu)'$-almost Kenmotsu 3-manifold. If M^3 is Einstein and $h \neq 0$, from (12) we directly get

$$\mu = 0 \text{ and } r = 6\kappa,$$

where we have used $\lambda > 0$. Since the scalar curvature of an Einstein manifold of dimension greater than two is a constant, it follows from the second term of the above relation that κ is also a constant. Using this in (9) gives $\mu = -2$, a contradiction. If $h = 0$, from Proposition 3.2 it is seen that M^3 is a Kenmotsu 3-manifold. It is known that any three-dimensional Einstein manifold is of constant sectional curvature. Furthermore, K. Kenmotsu in [10, Corollary 6] proved that a locally symmetric Kenmotsu manifold is of constant sectional curvature −1. This completes the proof. □

Recently, Y. Wang [16] and J. T. Cho [2] gave a complete classification of almost Kenmotsu 3-manifolds under local symmetry condition.

Proposition 3.5 ([16, Theorem 3.4]). *An almost Kenmotsu 3-manifold is locally symmetric if and only if the manifold is locally isometric to either the hyperbolic space* $\mathbb{H}^3(-1)$ *or the product* $\mathbb{H}^2(-4) \times \mathbb{R}$.

Next we study almost Kenmotsu 3-manifolds under some symmetry conditions. Firstly, we give

Definition 3.6 ([14]). *An almost contact metric manifold is called ϕ-symmetric if*

$$\phi^2(\nabla_V R)(X, Y)Z = 0 \tag{13}$$

for any vector fields X, Y, Z, V.

Obviously, local symmetry condition (i.e., $\nabla R = 0$) implies ϕ-symmetry but the converse is not necessarily true. Applying the above preliminaries, we show that the above two kinds of symmetry on some almost Kenmotsu 3-manifolds are equivalent to each other.

Theorem 3.7. *A generalized $(\kappa, \mu)'$-almost Kenmotsu 3-manifold is ϕ-symmetric if and only if it is locally isometric to either $\mathbb{H}^3(-1)$ or $\mathbb{H}^2(-4) \times \mathbb{R}$.*

Proof. It is well-known that the curvature tensor R of a three-dimensional Riemannian manifold is given by

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y - \frac{r}{2}(g(Y, Z)X - g(X, Z)Y) \tag{14}$$

for any vector fields X, Y, Z, where r is the scalar curvature. Taking the covariant derivative of the above relation along V gives

$$\begin{aligned}&(\nabla_V R)(X, Y)Z\\ =&g(Y, Z)(\nabla_V Q)X - g(X, Z)(\nabla_V Q)Y + g((\nabla_V Q)Y, Z)X - g((\nabla_V Q)X, Z)Y - \frac{1}{2}V(r)(g(Y, Z)X - g(X, Z)Y)\end{aligned} \tag{15}$$

for any vector fields X, Y, Z, V. From (15) we see that local symmetry and Ricci symmetry (i.e., ∇Q=0) are equivalent. On the other hand, from (1) and (13) we see that an almost contact metric manifold is ϕ-symmetric if and only if

$$(\nabla_V R)(X, Y)Z = g((\nabla_V R)(X, Y)Z, \xi)\xi \tag{16}$$

for any vector fields X, Y, Z, V.

Let M^3 be a generalized $(\kappa, \mu)'$-almost Kenmotsu 3-manifold. If $h = 0$ from Proposition 3.2 it is seen that M^3 is a Kenmotsu manifold. U. C. De [3, Theorem 3.1] proved that a ϕ-symmetric Kenmotsu manifold is Einstein. Then, the proof follows from Proposition 3.4.

Next we consider the other case, i.e., non-Kenmotsu case. Let us assume that $h \neq 0$, or equivalently, $\kappa < -1$. Applying (12) and Proposition 3.3 we get

$$(\nabla_\xi Q)\xi = 2\xi(\kappa)\xi. \tag{17}$$

$$(\nabla_\xi Q)e = \Big(\frac{1}{2}\xi(r) - \xi(\kappa) + \mu\xi(\lambda) + \lambda\xi(\mu)\Big)e. \tag{18}$$

$$(\nabla_\xi Q)\phi e = \Big(\frac{1}{2}\xi(r) - \xi(\kappa) - \mu\xi(\lambda) - \lambda\xi(\mu)\Big)\phi e. \tag{19}$$

$$(\nabla_e Q)\xi = 2e(\kappa)\xi + (1+\lambda)\Big(3\kappa - \frac{r}{2} - \lambda\mu\Big)e. \tag{20}$$

$$(\nabla_e Q)e = (1+\lambda)\Big(3\kappa - \frac{r}{2} - \lambda\mu\Big)\xi + \Big(\frac{1}{2}e(r) - e(\kappa) + \mu e(\lambda) + \lambda e(\mu)\Big)e + \mu\phi e(\lambda)\phi e. \tag{21}$$

$$(\nabla_e Q)\phi e = \mu\phi e(\lambda)e + \Big(\frac{1}{2}e(r) - e(\kappa) - \mu e(\lambda) - \lambda e(\mu)\Big)\phi e. \tag{22}$$

$$(\nabla_{\phi e} Q)\xi = 2\phi e(\kappa)\xi + (1-\lambda)\Big(3\kappa - \frac{r}{2} + \lambda\mu\Big)\phi e, \tag{23}$$

$$(\nabla_{\phi e} Q)e = \Big(\frac{1}{2}\phi e(r) - \phi e(\kappa) + \mu\phi e(\lambda) + \lambda\phi e(\mu)\Big)e - \mu e(\lambda)\phi e. \tag{24}$$

$$(\nabla_{\phi e} Q)\phi e = (1-\lambda)\Big(3\kappa - \frac{r}{2} + \lambda\mu\Big)\xi - \mu e(\lambda)e + \Big(\frac{1}{2}\phi e(r) - \phi e(\kappa) - \mu\phi e(\lambda) - \lambda\phi e(\mu)\Big)\phi e. \tag{25}$$

Putting $X = \xi$, $Y = e$ and $Z = \phi e$ in (15) and using (16) give

$$g((\nabla_V Q)\xi, \phi e) = 0 \tag{26}$$

for any vector field V. Putting $V = \phi e$ in (26) and using (23) we have

$$(1-\lambda)\Big(3\kappa - \frac{r}{2} + \lambda\mu\Big) = 0. \tag{27}$$

Similarly, putting $X = \xi$, $Y = \phi e$ and $Z = e$ in (15) and using (16) give

$$g((\nabla_V Q)\xi, e) = 0 \tag{28}$$

for any vector field V. Putting $V = e$ in (28) and using (20) we have

$$3\kappa - \frac{r}{2} - \lambda\mu = 0, \tag{29}$$

where we have used $\lambda > 0$. Putting (29) in (27) gives $(1-\lambda)(6\kappa - r) = 0$.

We assume that $r = 6\kappa$ holds and hence using this in (29) we get $\mu = 0$. Now putting $X = \xi$ and $Y = Z = e$ in (15) and using (16) give

$$(\nabla_V Q)\xi = 0$$

for any vector field V, where we have used (28). Putting $V = \xi$ in the above relation and using (17) give $\xi(\kappa) = 0$. In view of (9) and $\mu = 0$ we have $\kappa = -1$. This implies $h = 0$ and M^3 is a Kenmotsu manifold, a contradiction. Thus we conclude that $\lambda = 1$ and hence $\kappa = -1 - \lambda^2 = -2$. Applying this in (9) gives $\mu = -2$. Also, it follows from (29) that $r = -8$. Finally, by a simple calculation we get from (17)-(25) that the Ricci tensor is parallel and hence the manifold is locally symmetric. Then the proof follows from Proposition 3.5. This completes the proof. □

Remark 3.8. *Theorem 3.7 is an extension of De [4, Theorem 3.1] for Kenmotsu* 3*-manifolds.*

Definition 3.9. *The Ricci tensor of an almost contact metric manifold is said to be cyclic-parallel if*

$$g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)Z, X) + g((\nabla_Z Q)X, Y) = 0 \tag{30}$$

for any vector fields X, Y, Z.

If the Ricci tensor is parallel (i.e., $\nabla Q = 0$), then (30) holds trivially. But the converse is not always true.

Theorem 3.10. *The Ricci tensor of any generalized* $(\kappa,\mu)'$*-almost Kenmotsu* 3*-manifold is cyclic-parallel if and only if the manifold is locally isometric to either* $\mathbb{H}^3(-1)$ *or* $\mathbb{H}^2(-4)\times\mathbb{R}$*.*

Proof. It is well-known that on a Riemannian manifold there holds

$$\frac{1}{2}X(r) = \mathrm{trace}\{e_i \to (\nabla_{e_i}Q)X\}, \tag{31}$$

where $\{e_i\}$ is a local orthogonal basics of the tangent space at certain point.

Applying (31) in (30) we obtain that the scalar curvature r is a constant. J. Inoguchi in [8, Proposition 3.1] proved that a Kenmotsu 3-manifold is of constant scalar curvature if and only if it is of constant sectional curvature -1. Then, the proof for Kenmotsu case follows.

Next, let M^3 be a generalized $(\kappa,\mu)'$-almost Kenmotsu 3-manifold with $h \neq 0$. Putting $X = Y = Z = \xi$ in (30) gives

$$g((\nabla_\xi Q)\xi, \xi) = 0.$$

Using (17) and (9) we have $\mu = -2$, where we have used the assumption $\kappa < -1$. Similarly, putting $X = \xi$ and $Y = Z = e$ in (30) gives

$$g((\nabla_\xi Q)e, e) + 2g((\nabla_e Q)e, \xi) = 0.$$

Using (18) and (21) in the above relation gives

$$3\kappa - \frac{r}{2} + 2\lambda = 0, \tag{32}$$

where we have used $\mu = -2$, $\lambda > 0$ and $r =$ constant. Similarly, putting $X = \xi$ and $Y = Z = \phi e$ in (30) gives

$$g((\nabla_\xi Q)\phi e, \phi e) + 2g((\nabla_{\phi e} Q)\phi e, \xi) = 0.$$

Using (19) and (25) in the above relation gives

$$(1-\lambda)\left(3\kappa - \frac{r}{2} - 2\lambda\right) = 0,$$

where we have used $\mu = -2$ and $r =$ constant. Putting (32) into the previous relation we obtain $(1-\lambda)(6\kappa - r) = 0$. Let us assume that $r = 6\kappa$. Using this in (32) we may obtain $\lambda = 0$, a contradiction. Thus, it follows that $\lambda = 1$ and hence $\kappa = -2$. The remaining proof follows from Theorem 3.7 and Proposition 3.5. This completes the proof. □

Remark 3.11. *Theorem 3.10 is an extension of De and Pathak [4, Theorem 5]. An almost Kenmotsu* 3*-h-manifold having a cyclic-parallel Ricci tensor was studied by W. Wang [15].*

Definition 3.12. *The Ricci tensor of an almost contact metric manifold is said to be of Codazzi type if*

$$(\nabla_X Q)Y = (\nabla_Y Q)X \tag{33}$$

for any vector fields X, Y*.*

Notice that (33) holds if and only if the curvature tensor R is harmonic, i.e., $\mathrm{div}R = 0$. Obviously, (33) can be viewed as an extension of a parallel Ricci tensor tensor.

Theorem 3.13. *The Ricci tensor of any generalized* $(\kappa,\mu)'$*-almost Kenmotsu* 3*-manifold is of Codazzi type if and only if the manifold is locally isometric to either* $\mathbb{H}^3(-1)$ *or* $\mathbb{H}^2(-4)\times\mathbb{R}$*.*

Proof. Applying (31) in (33) we see that the scalar curvature is a constant. Therefore, as seen in proof of Theorem 3.10, the proof for Kenmotsu case follows from J. Inoguchi [8, Proposition 3.1].

Let M^3 be a generalized $(\kappa,\mu)'$-almost Kenmotsu 3-manifold with $h \neq 0$. Putting $X = e$ and $Y = \xi$ in (33) gives

$$(\nabla_\xi Q)e = (\nabla_e Q)\xi.$$

Using (18) and (20) in the above relation gives

$$e(\kappa) = 0,\ (1+\lambda)\left(3\kappa - \frac{r}{2} - \lambda\mu\right) = -\xi(\kappa) + \mu\xi(\lambda) + \lambda\xi(\mu). \tag{34}$$

Similarly, putting $X = \phi e$ and $Y = \xi$ in (33) gives

$$(\nabla_\xi Q)\phi e = (\nabla_{\phi e} Q)\xi.$$

Using (19) and (23) in the above relation gives

$$\phi e(\kappa) = 0,\ (1-\lambda)\left(3\kappa - \frac{r}{2} + \lambda\mu\right) = -\xi(\kappa) - \mu\xi(\lambda) - \lambda\xi(\mu). \tag{35}$$

Similarly, putting $X = e$ and $Y = \phi e$ in (33) gives

$$(\nabla_e Q)\phi e = (\nabla_{\phi e} Q)e.$$

Using (22) and (24) in the above relation gives

$$e(\mu) = 0,\ \phi e(\mu) = 0,$$

this means $d\mu \wedge \eta = 0$ and where we have used the first terms of (34) and (35) and r = constant. Adding the second term of (34) to that of (35) gives

$$6\kappa - r = 2(\kappa+1)(\mu+4), \tag{36}$$

where we have used (9). Taking the action of ξ of (36), using again (9) and r = constant give

$$\xi(\mu) = 2(\mu+1)(\mu+2). \tag{37}$$

Subtracting the second term of (34) from that of (35) gives

$$-2\lambda\mu + \lambda(6\kappa - r) = 2\mu\xi(\lambda) + 2\lambda\xi(\mu).$$

Making use of (9), (36) and (37) in the above relation give

$$\lambda(\kappa-\mu)(\mu+4) = 0.$$

Now we first consider the possible subcase $\mu = -4$. Using this in (37) gives either $\mu = -1$ or $\mu = -2$, a contradiction. In view of $\lambda > 0$, we conclude that $\kappa = \mu$ and hence (37) becomes $\xi(\kappa) = 2(\kappa+1)(\kappa+2)$. On the other hand, using $\kappa = \mu$ in (9) gives $\xi(\kappa) = -2(\kappa+1)(\kappa+2)$. In view of $k < -1$, it follows that $\kappa = \mu = -2$. Then, the remaining proof follows from Theorem 3.7 and Proposition 3.5. This complete the proof. □

It is well-known that a three-dimensional Riemannian manifold M is said to be conformally flat if its Weyl-Schouten tensor is of Codazzi-type, or equivalently, its Ricci operator satisfies

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4}(X(r)Y - Y(r)X) \tag{38}$$

for any vector fields X, Y on M.

Theorem 3.14. *A generalized* $(\kappa,\mu)'$*-almost Kenmotsu* 3*-manifold is conformally flat with scalar curvature invariant along* ξ *if and only if the manifold is locally isometric to either* $\mathbb{H}^3(-1)$ *or* $\mathbb{H}^2(-4)\times\mathbb{R}$.

Proof. Let M^3 be a generalized $(\kappa,\mu)'$-almost Kenmotsu 3-manifold. Let us first consider the case $h=0$, i.e., M^3 is a Kenmotsu manifold. Using $h=0$ in (7) we have $R(X,Y)\xi=-\eta(Y)X+\eta(X)Y$ for any vector fields X,Y and hence $Q\xi=-2\xi$. Applying this and putting $Y=Z=\xi$ in (14) we obtain the Ricci operator

$$Q=\Big(\frac{r}{2}+1\Big)\mathrm{id}-\Big(\frac{r}{2}+3\Big)\eta\otimes\xi.$$

Using this in (31) we have $\xi(r)=-2(6+r)$. In view of the assumption of r invariant along the Reeb vector field we get $r=-6$ and hence we see that M^3 is Einstein. Then, the proof follows from Proposition 3.4.

Now we consider the other case $h\neq 0$. Putting $X=e$ and $Y=\xi$ in (38) we have

$$(\nabla_e Q)\xi-(\nabla_\xi Q)e=\frac{1}{4}e(r)\xi,$$

where we have used the assumption $\xi(r)=0$. Using (18) and (20) in the above relation gives

$$e(\kappa)=\frac{1}{8}e(r),\ (1+\lambda)\Big(3\kappa-\frac{r}{2}-\lambda\mu\Big)+\xi(\kappa)-\mu\xi(\lambda)-\lambda\xi(\mu)=0. \tag{39}$$

Similarly, putting $X=\phi e$ and $Y=\xi$ in (38) we have

$$(\nabla_{\phi e} Q)\xi-(\nabla_\xi Q)\phi e=\frac{1}{4}\phi e(r)\xi,$$

where we have used the assumption $\xi(r)=0$. Using (19) and (23) in the above relation gives

$$\phi e(\kappa)=\frac{1}{8}\phi e(r),\ (1-\lambda)\Big(3\kappa-\frac{r}{2}+\lambda\mu\Big)+\xi(\kappa)+\mu\xi(\lambda)+\lambda\xi(\mu)=0. \tag{40}$$

Similarly, putting $X=\phi e$ and $Y=e$ in (38) we have

$$(\nabla_{\phi e} Q)e-(\nabla_e Q)\phi e=\frac{1}{4}(\phi e(r)e-e(r)\phi e).$$

Using (22) and (24) in the above relation gives

$$\frac{1}{4}e(r)-e(\kappa)-\lambda e(\mu)=0,\ \frac{1}{4}\phi e(r)-\phi e(\kappa)+\lambda\phi e(\mu)=0. \tag{41}$$

Taking the inner product of the second term of (11) with e and ϕe, respectively, we have $e(\kappa)=\lambda e(\mu)$ and $\phi e(\kappa)=-\lambda\phi e(\mu)$. Applying this in (41) we get

$$e(\mu)=\frac{1}{8\lambda}e(r),\ \phi e(\mu)=-\frac{1}{8\lambda}\phi e(r), \tag{42}$$

where $\lambda=\sqrt{-1-\kappa}>0$. Adding the second term of (39) to that of (40) and using (9) we have

$$6\kappa-r-2(\kappa+1)(\mu+4)=0, \tag{43}$$

where we have used (9).

Taking the action of e of (43) and using the first terms of (39) and (42) we get

$$\frac{1}{4}(\mu+5-\lambda)e(r)=0.$$

From this we may assume that $e(r) \neq 0$ holds on certain open subset of M^3 and hence we get $\lambda = \mu + 5$. In view of $\lambda = \sqrt{-1-\kappa}$, it follows that $e(\lambda) = -\frac{1}{2\lambda}e(\kappa) = e(\mu)$. Making use of $e(\kappa) = \lambda e(\mu)$ we obtain from the first term of (42) that $e(r) = 8\lambda e(\mu) = 0$, a contradiction. Thus, we conclude that $e(r) = 0$ is true.

Taking the action of ϕe of (43) and using the first term of (40) and the second term of (42) we get

$$\frac{1}{4}(\mu + 5 + \lambda)\phi e(r) = 0.$$

From this we assume that $\phi e(r) \neq 0$ holds on some open subset of M^3 and hence we get $\lambda = -\mu + 5$. In view of $\lambda = \sqrt{-1-\kappa}$, it follows that $\phi e(\lambda) = -\frac{1}{2\lambda}\phi e(\kappa) = -\phi e(\mu)$. Making use of $\phi e(\kappa) = -\lambda\phi e(\mu)$ we obtain from the second term of (42) that $\phi e(r) = -8\lambda\phi e(\mu) = 0$, a contradiction. Thus, here we conclude that $\phi e(r) = 0$ is true.

Taking into account the assumption $\xi(r) = 0$ we see that the scalar curvature is a constant. From (38) it is seen that the Ricci tensor of M^3 is of Codazzi type. Then the proof follows from Theorem 3.13. □

The classification of conformally flat almost Kenmotsu manifolds was rarely studied. Very recently, Y. Wang in [17] proved that any CR-integrable almost Kenmotsu manifold of dimension > 3 with scalar curvature invariant along the Reeb vector field is conformally flat if and only if it is of constant sectional curvature −1.

Example 3.15. *We denote by (x, y, z) the usual canonical coordinates of $\mathbb{R}^3$. Let us consider*

$$M^3 := \{(x, y, z) \in \mathbb{R}^3 | z > 0\}.$$

On M^3 we define an almost contact metric structure (ϕ, ξ, η, g) as the following:

$$\xi := \frac{\partial}{\partial z},\ \eta := dz,$$
$$g = ze^{2z}dx^2 + \frac{e^{2z}}{z}dy^2 + dz^2,$$
$$\phi(\xi) = 0,\ \phi\left(\frac{\partial}{\partial x}\right) = z\frac{\partial}{\partial y},\ \phi\left(\frac{\partial}{\partial y}\right) = -\frac{1}{z}\frac{\partial}{\partial x}.$$

In [11, Section 6], it was shown that M^3 is a generalized $(k, \mu)'$-almost Kenmotsu 3-manifold with $k = -1 - \frac{1}{4z^2}$ and $\mu = -2 + \frac{1}{z}$. One can check that the scalar curvature of M^3 is not a constant and hence the Ricci tensor of M^3 is not cyclic-parallel and not of Codazzi type. Moreover, M^3 is neither ϕ-symmetric nor conformally flat.

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