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Complete Convergence and Complete Moment Convergence for Arrays of Rowwise Asymptotically Almost Negatively Associated Random Variables

Haiwu Huang[a,b], Qingxia Zhang[c], Hang Zou[a], Xiongtao Wu[a]

[a]*College of Mathematics and Statistics, Hengyang Normal University, Hengyang 421002, P.R.China*
[b]*Hunan Provincial Key Laboratory of Intelligent Information Processing and Application, Hengyang 421002, P.R. China*
[c]*School of Sciences, Southwest Petroleum University, Chengdu 610500, P.R.China*

Abstract. In this article, the authors investigate the complete convergence and complete moment convergence of the maximum partial sums for arrays of rowwise asymptotically almost negatively associated random variables without assumptions of identical distribution and stochastic domination, and obtain some new results, which not only generalize the corresponding theorems of Hu and Taylor (1997), Gan and Chen (2007), Wu (2012), but also improve them, respectively.

1. Introduction

Firstly, we shall restate the definitions of negatively associated random variables and asymptotically almost negatively associated random variables.

Definition 1.1. *A finite family of random variables $\{X_i; 1 \le i \le n\}$ is said to be negatively associated (NA, in short) if for any disjoint subsets A and B of $\{1, 2, \cdots, n\}$,*

$$Cov\left(f_1\left(X_i, i \in A\right), f_2\left(X_j, j \in B\right)\right) \le 0, \tag{1.1}$$

whenever f_1 and f_2 are any real coordinatewise nondecreasing functions such that this covariance exists. An infinite family of random variables $\{X_i; i \ge 1\}$ is NA if every finite sub-family is NA.

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Email addresses: haiwuhuang@126.com (Haiwu Huang), zqx121981@126.com (Qingxia Zhang), 13571036820qq.com (Hang Zou), 357891510@qq.com (Xiongtao Wu)

The concept of NA random variables was introduced by Block et al. [1] and carefully studied by Joag-Dev and Proschan [2]. Obviously, (1.1) holds if f_1 and f_2 are both real coordinatewise nonincreasing functions. By inspecting the proof of maximal inequality for the NA random variables in Matula [3], one also can allow negative correlations provided they are small. Primarily motivated by this, Chandra and Ghosal [4, 5] introduced the following dependence.

Definition 1.2. *A sequence of random variables $\{X_n; n \geq 1\}$ is said to be asymptotically almost negatively associated (AANA, in short) if there exists a nonnegative sequence $\mu(n) \to 0$ as $n \to \infty$ such that*

$$Cov\left(f_1\left(X_n\right), f_2\left(X_{n+1}, X_{n+2}, \cdots, X_{n+k}\right)\right) \leq \mu(n)\left(\mathrm{Var}\left(f_1(X_n)\right)\mathrm{Var}\left(f_2\left(X_{n+1}, X_{n+2}, \cdots, X_{n+k}\right)\right)\right)^{1/2} \tag{1.2}$$

for all $n, k \geq 1$ and for all coordinatewise nondecreasing continuous functions f_1 and f_2 whenever the variances exist.

An array of random variables $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ is called rowwise AANA random variables if for every $n \geq 1$, $\{X_{ni}; i \geq 1\}$ is a sequence of AANA random variables.

Obviously, AANA random variables contain independent random variables (with $\mu(n) = 0$ for $n \geq 1$) and NA random variables. Chandra and Ghosal [4] once pointed out that NA implies AANA, but AANA does not imply NA. Namely, AANA is much weaker than NA. NA has been applied to the reliability theory, multivariate statistical analysis and percolation theory, and attracted extensive attentions. Hence, extending the limit properties of NA random variables to the wider case of AANA random variables is very meaningful in the theory and applications.

Since the concept of AANA was introduced by Chandra and Ghosal [4], many applications have been established in various aspects. For more details, we can refer to Chandra and Ghosal [4, 5], Ko et al. [6], Yuan and An [7, 8] , Yuan and Wu [9], Wang et al [10–12], Yang et al. [13], Hu et al.[14], Tang [15], Shen and Wu [16], Huang et al. [17], Shen et al.[18], and so forth.

For a triangular array of rowwise random variables $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$, let $\{a_n; n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$, and let $\psi(t)$ be a positive, even function such that for some nonnegative integer p,

$$\frac{\psi(|t|)}{|t|^p} \uparrow \quad \text{and} \quad \frac{\psi(|t|)}{|t|^{p+1}} \downarrow \quad \text{as } |t| \uparrow . \tag{1.3}$$

Conditions are given as follows

$$EX_{ni} = 0, 1 \leq i \leq n, n \geq 1. \tag{1.4}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi(|X_{ni}|)}{\psi(a_n)} < \infty, \tag{1.5}$$

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E\left(\frac{X_{ni}}{a_n}\right)^2 \right)^{2k} < \infty, \tag{1.6}$$

where k is a positive integer.

In the case of independence, Hu and Taylor [19] obtained the following theorems.

Theorem 1.3. *Let $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise independent random variables and $\{a_n; n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$, let $\psi(t)$ be a positive, even function satisfying (1.3) for some integer $p > 2$. Then conditions (1.4), (1.5) and (1.6) imply*

$$\frac{1}{a_n} \sum_{i=1}^{n} X_{ni} \to 0 \quad \text{a.s.} \tag{1.7}$$

Theorem 1.4. *Let $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise independent random variables and $\{a_n; n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$, let $\psi(t)$ be a positive, even function satisfying (1.3) for $p = 1$. Then conditions (1.4) and (1.5) imply (1.7).*

Gan and Chen [20] extended and improved **Theorem 1.3** and **Theorem 1.4** to the case of NA random variables. Wu [21] investigated the complete moment convergence and the L^p convergence for arrays of rowwise NA random variables by using the different methods from Gan and Chen [20]. The results obtained by Wu [21] generalized the corresponding theorems by Gan and Chen [20]. However, according to our knowledge, the above subject for the complete convergence and the complete moment convergence for arrays of rowwise AANA random variables has not been studied. The main goal of this paper is to study the complete convergence and the complete moment convergence for arrays of rowwise AANA random variables. The main idea is inspired by Gan and Chen [20], Wu [21]. It is worth pointing out that the methods used in this article are different from those of Wu [21].

Definition 1.5. *A sequence of random variables $\{X_n; n \geq 1\}$ is said to converge completely to a constant λ if for $\forall \varepsilon > 0$,*

$$\sum_{n=1}^{\infty} P\left(|X_n - \lambda| > \varepsilon\right) < \infty. \tag{1.8}$$

for all $x \geq 0$ and $n \geq 1$. This notion was firstly given by Hsu and Robbins [22].

Definition 1.6. *Let $\{X_n; n \geq 1\}$ be a sequence of random variables, and let $a_n > 0$, $b_n > 0$, $q > 0$. If for some or all $\varepsilon \geq 0$,*

$$\sum_{n=1}^{\infty} a_n E\left(b_n^{-1}|X_n| - \varepsilon\right)_+^q < \infty. \tag{1.9}$$

Then (1.9) is called the complete moment convergence by Chow [23].

To prove the main results of this paper, the following two lemmas are needed.

Lemma 1.7. *(cf. Yuan and An [7]) Let $\{X_n; n \geq 1\}$ be a sequence of AANA random variables with the mixing coefficients $\{\mu(n); n \geq 1\}$, let $\{f_n; n \geq 1\}$ be a sequence of all nondecreasing (or all nonincreasing) continuous functions, then $\{f_n(X_n); n \geq 1\}$ is still a sequence of AANA random variables with the mixing coefficients $\{\mu(n); n \geq 1\}$.*

Lemma 1.8. *(cf. Yuan and An [7])) Let $p > 1$ and $\{X_n; n \geq 1\}$ be a sequence of AANA random variables with the mixing coefficients $\{\mu(n); n \geq 1\}$, $EX_n = 0$.*

If $\sum_{n=1}^{\infty} \mu^2(n) < \infty$, then there exists a positive constant $C = C(p)$ depending only on p such that for all $n \geq 1$ and $1 < p \leq 2$,

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C\left(\sum_{i=1}^{n} E|X_i|^p\right). \tag{1.10}$$

If $\sum_{n=1}^{\infty} \mu^{1/(p-1)}(n) < \infty$ for some $p \in \left(3\cdot 2^{k-1}, 4\cdot 2^{k-1}\right]$, where integer number $k \geq 1$, then there exists a positive constant $C = C(p)$ depending only on p such that for all $n \geq 1$,

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C\left(\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right). \tag{1.11}$$

Throughout the paper, let $I(A)$ be the indicator function of the set A. The symbol C denotes a positive constant, which may be different in various places, and $a_n = O(b_n)$ stands for $a_n \leq Cb_n$.

2. Main results

Let $\{X_{ni}; 1 \le i \le n, n \ge 1\}$ be an array of rowwise AANA random variables with the mixing coefficients $\{\mu(i); i \ge 1\}$ in each row and $\{a_n; n \ge 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\{\psi_n(t); n \ge 1\}$ be a sequence of positive, even functions such that for $1 \le q < p$

$$\frac{\psi_n(|t|)}{|t|^q} \uparrow \quad \text{and} \quad \frac{\psi_n(|t|)}{|t|^p} \downarrow \text{ as } |t| \uparrow . \tag{2.1}$$

Introduce the following conditions

$$EX_{ni} = 0, 1 \le i \le n, n \ge 1, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi_i(|X_{ni}|)}{\psi_i(a_n)} < \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E\left(\frac{X_{ni}}{a_n} \right)^r \right)^s < \infty, \tag{2.4}$$

where $0 < r \le 2$, $s > 0$.

Theorem 2.1. *Let $\{X_{ni}; 1 \le i \le n, n \ge 1\}$ be an array of rowwise AANA random variables with the mixing coefficients $\{\mu(i); i \ge 1\}$ in each row satisfying $\sum_{i=1}^{\infty} \mu^2(i) < \infty$, and let $\{a_n; n \ge 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\{\psi_n(t); n \ge 1\}$ be a sequence of positive, even functions satisfying (2.1) for $1 \le q < p \le 2$. Then conditions (2.2) and (2.3) imply*

$$\sum_{n=1}^{\infty} P\left(\frac{1}{a_n} \max_{1 \le j \le n} \left| \sum_{i=1}^{j} X_{ni} \right| > \varepsilon \right) < \infty \quad \text{for} \quad \forall \varepsilon > 0. \tag{2.5}$$

Theorem 2.2. *Let $\{X_{ni}; 1 \le i \le n, n \ge 1\}$ be an array of rowwise AANA random variables with the mixing coefficients $\{\mu(i); i \ge 1\}$ in each row satisfying $\sum_{i=1}^{\infty} \mu^{1/(p-1)}(i) < \infty$ for some $p \in \left(3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}\right]$, where integer number $k \ge 1$, and let $\{a_n; n \ge 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\{\psi_n(t); n \ge 1\}$ be a sequence of positive, even functions satisfying (2.1) for $1 \le q < p$ and $p > 2$. Then conditions (2.2), (2.3) and (2.4) imply (2.5).*

Remark 2.3. *Take $q = 1$ in (2.1), then the conditions of the above theorems are the same with those of* **Theorems 1** *and* **2** *in Gan and Chen [20]. The family of AANA sequence contains sequences of independent and NA random variables. So,* **Theorems 2.1** *and* **2.2** *are extensions and improvements of the corresponding results of Hu and Taylor [19], Gan and Chen [20].*

Theorem 2.4. *Let $\{X_{ni}; 1 \le i \le n, n \ge 1\}$ be an array of rowwise AANA random variables with the mixing coefficients $\{\mu(i); i \ge 1\}$ in each row satisfying $\sum_{i=1}^{\infty} \mu^2(i) < \infty$, and let $\{a_n; n \ge 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\{\psi_n(t); n \ge 1\}$ be a sequence of positive, even functions satisfying (2.1) for $1 \le q < p \le 2$. Then conditions (2.2) and (2.3) imply*

$$\sum_{n=1}^{\infty} a_n^{-q} E\left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} X_{ni} \right| - \varepsilon a_n \right)_+^q < \infty \quad \text{for} \quad \forall \varepsilon > 0. \tag{2.6}$$

Theorem 2.5. *Let $\{X_{ni}; 1 \le i \le n, n \ge 1\}$ be an array of rowwise AANA random variables with the mixing coefficients $\{\mu(i); i \ge 1\}$ in each row satisfying $\sum_{i=1}^{\infty} \mu^{1/(p-1)}(i) < \infty$ for some $p \in \left(3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}\right]$, where integer number $k \ge 1$, and let $\{a_n; n \ge 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\{\psi_n(t); n \ge 1\}$ be a sequence of positive, even functions satisfying (2.1) for $1 \le q < p$ and $p > 2$. Then conditions (2.2), (2.3) and (2.4) imply (2.6).*

Remark 2.6. *Compared with Wu [21], we study the complete moment convergence for arrays of rowwise AANA random variables under the same conditions. It is worth pointing out that the methods applied in this paper are different from those of Wu [21].*

Proof of Theorem 2.1 For fixed $n \geq 1$, define

$$Y_{ni} = -a_n I(X_{ni} < -a_n) + X_{ni} I(|X_{ni}| \leq a_n) + a_n I(X_{ni} > a_n),$$

$$Z_{ni} = X_{ni} - Y_{ni} = (X_{ni} + a_n) I(X_{ni} < -a_n) + (X_{ni} - a_n) I(X_{ni} > a_n).$$

To prove (2.5), we need only to show that

$$\sum_{n=1}^{\infty} P\left(\frac{1}{a_n} \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} Z_{ni}\right| > \varepsilon\right) < \infty; \tag{2.7}$$

$$\sum_{n=1}^{\infty} P\left(\frac{1}{a_n} \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} (Y_{ni} - EY_{ni})\right| > \varepsilon\right) < \infty; \tag{2.8}$$

$$\frac{1}{a_n} \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} EY_{ni}\right| \to 0 \text{ as } n \to \infty. \tag{2.9}$$

Firstly, we prove (2.7). When $X_{ni} > a_n$, $0 < Z_{ni} = X_{ni} - a_n < X_{ni}$. When $X_{ni} < -a_n$, $X_{ni} < Z_{ni} = X_{ni} + a_n < 0$. Hence, $|Z_{ni}| \leq |X_{ni}| I(|X_{ni}| > a_n)$. It follows from (2.1) and (2.3) that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\frac{1}{a_n} \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} Z_{ni}\right| > \varepsilon\right) &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(|Z_{ni}| > \varepsilon a_n) \\
&\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E|Z_{ni}|}{\varepsilon a_n} \\
&\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E|X_{ni}| I(|X_{ni}| > a_n)}{\varepsilon a_n} \\
&\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi_i(|X_{ni}|)}{\psi_i(a_n)} < \infty.
\end{aligned}$$

Secondly, we prove (2.8). By Lemma 1.7, $\{Y_{ni} - EY_{ni}; 1 \leq i \leq n, n \geq 1\}$ is an array of rowwise AANA random variables with mean zero. Note that $|Y_{ni}| \leq |X_{ni}|$ a.s. It follows from Markov inequality, (2.1), (2.3) and (1.10) of Lemma 1.8 for $1 < p \leq 2$ that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\frac{1}{a_n} \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} (Y_{ni} - EY_{ni})\right| > \varepsilon\right) &\leq C \frac{1}{\varepsilon^p} \sum_{n=1}^{\infty} E\left(\frac{1}{a_n} \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} (Y_{ni} - EY_{ni})\right|\right)^p \\
&\leq C \frac{1}{(a_n \varepsilon)^p} \sum_{n=1}^{\infty} \sum_{i=1}^{n} E|Y_{ni} - EY_{ni}|^p \\
&\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E|Y_{ni}|^p}{a_n^p} \\
&\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi_i(|Y_{ni}|)}{\psi_i(a_n)} \\
&\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi_i(|X_{ni}|)}{\psi_i(a_n)} < \infty.
\end{aligned}$$

Finally, we prove (2.9). For $1 \le i \le n$, $n \ge 1$, since $EX_{ni} = 0$, we can see that $EY_{ni} = -EZ_{ni}$. By a similar argument as the proof of (2.7), we can obtain that

$$\begin{aligned}
\frac{1}{a_n} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EY_{ni} \right| &= \frac{1}{a_n} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EZ_{ni} \right| \\
&\le C \sum_{i=1}^{n} \frac{E\left|X_{ni}\right| I\left(\left|X_{ni}\right| > a_n\right)}{a_n} \\
&\le C \sum_{i=1}^{n} \frac{E\psi_i\left(\left|X_{ni}\right| I\left(\left|X_{ni}\right| > a_n\right)\right)}{\psi_i\left(a_n\right)} \\
&\le C \sum_{i=1}^{n} \frac{E\psi_i\left(\left|X_{ni}\right|\right)}{\psi_i\left(a_n\right)} \to 0.
\end{aligned}$$

The proof of **Theorem 2.1** is completed.

Proof of Theorem 2.2 By using the same notations and the methods of proof of **Theorem 2.1**, we can see that (2.7) and (2.9) hold. It suffices to show that (2.8) holds.

Take $v = \max(p, 2s)$, it follows from Markov inequality and (1.11) of Lemma 1.8 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\frac{1}{a_n} \max_{1\le j\le n} \left| \sum_{i=1}^{j} \left(Y_{ni} - EY_{ni}\right) \right| > \varepsilon \right) \\
\le\ & C\frac{1}{\varepsilon^v} \sum_{n=1}^{\infty} E\left(\frac{1}{a_n} \max_{1\le j\le n} \left| \sum_{i=1}^{j} \left(Y_{ni} - EY_{ni}\right) \right| \right)^v \\
\le\ & C\frac{1}{a_n^v} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E\left|Y_{ni} - EY_{ni}\right|^v + \left(\sum_{i=1}^{n} E\left(Y_{ni} - EY_{ni}\right)^2 \right)^{v/2} \right) \\
\le\ & C\frac{1}{a_n^v} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} E\left|Y_{ni}\right|^v + \left(\sum_{i=1}^{n} EY_{ni}^2 \right)^{v/2} \right).
\end{aligned}$$

For $v \ge p$, it follows from (2.1) and (2.3) that

$$C\sum_{n=1}^{\infty} \frac{1}{a_n^v} \sum_{i=1}^{n} E\left|Y_{ni}\right|^v \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E\psi_i\left(\left|Y_{ni}\right|\right)}{\psi_i\left(a_n\right)} \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E\psi_i\left(\left|X_{ni}\right|\right)}{\psi_i\left(a_n\right)} < \infty.$$

For $0 < r \le 2$, $s > 0$, $v \ge 2s$, it follows from (2.4) that

$$\begin{aligned}
\frac{1}{a_n^v} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} EY_{ni}^2 \right)^{v/2} &= \sum_{n=1}^{\infty} \left(\left(\sum_{i=1}^{n} \frac{EY_{ni}^2}{a_n^2} \right)^s \right)^{v/2s} \\
&\le \left(\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{EY_{ni}^2}{a_n^2} \right)^s \right)^{v/2s} \\
&\le \left(\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{E|Y_{ni}|^r}{a_n^r} \right)^s \right)^{v/2s} \\
&\le \left(\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \frac{E|X_{ni}|^r}{a_n^r} \right)^s \right)^{v/2s} < \infty.
\end{aligned}$$

The proof of **Theorem 2.2** is completed.

Proof of Theorem 2.4 Without loss of generality, assume that $t > 0$. For fixed $n \geq 1$, define

$$Y_{ni} = -t^{1/q} I\left(X_{ni} < -t^{1/q}\right) + X_{ni} I\left(|X_{ni}| \leq t^{1/q}\right) + t^{1/q} I\left(X_{ni} > t^{1/q}\right),$$

$$Z_{ni} = X_{ni} - Y_{ni} = \left(X_{ni} + t^{1/q}\right) I\left(X_{ni} < -t^{1/q}\right) + \left(X_{ni} - t^{1/q}\right) I\left(X_{ni} > t^{1/q}\right).$$

Obviously, $X_{ni} = Z_{ni} + Y_{ni}$. When $|X_{ni}| \leq t^{1/q}$, $X_{ni} = Y_{ni}$. By Lemma **??**, $\{Y_{ni}; 1 \leq i \leq n, n \geq 1\}$ is an array of rowwise AANA random variables. It is easy to check that for $\forall \varepsilon > 0$,

$$\begin{aligned} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > t^{1/q}\right) &\leq P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > t^{1/q}, \bigcup_{i=1}^{n}\left(|X_{ni}| > t^{1/q}\right)\right) \\ &\quad + P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > t^{1/q}, \bigcap_{i=1}^{n}\left(|X_{ni}| \leq t^{1/q}\right)\right) \\ &\leq \sum_{i=1}^{n} P\left(|X_{ni}| > t^{1/q}\right) + P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_{ni}\right| > t^{1/q}\right). \end{aligned} \tag{2.10}$$

Since,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n^{-q} E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| - \varepsilon a_n\right)_+^q &= \sum_{n=1}^{\infty} a_n^{-q} \int_0^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| - \varepsilon a_n > t^{1/q}\right) dt \\ &= \sum_{n=1}^{\infty} a_n^{-q} \int_0^{a_n^q} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > \varepsilon a_n + t^{1/q}\right) dt \\ &\quad + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > \varepsilon a_n + t^{1/q}\right) dt \\ &\leq \sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > \varepsilon a_n\right) \\ &\quad + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}\right| > t^{1/q}\right) dt \\ &\doteq K_1 + K_2. \end{aligned} \tag{2.11}$$

By **Theorem 2.1**, we can easily obtain $K_1 < \infty$.
For K_2, it follows from (2.10) that

$$\begin{aligned} K_2 &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} P\left(|X_{ni}| > t^{1/q}\right) dt + C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_{ni}\right| > t^{1/q}\right) dt \\ &\doteq K_3 + K_4. \end{aligned}$$

For $t \geq a_n^q$,

$$\begin{aligned} K_3 &= C \sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} P\left(|X_{ni}| I\left(|X_{ni}| > a_n\right) > t^{1/q}\right) dt \\ &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-q} \int_0^{\infty} P\left(|X_{ni}| I\left(|X_{ni}| > a_n\right) > t^{1/q}\right) dt \\ &= C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E|X_{ni}|^q I\left(|X_{ni}| > a_n\right)}{a_n^q} \\ &\leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi_i\left(|X_{ni}|\right)}{\psi_i\left(a_n\right)} < \infty. \end{aligned} \tag{2.12}$$

It follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned}\max_{t\geq a_n^q}\max_{1\leq j\leq n}\left|t^{-1/q}\sum_{i=1}^{j}EY_{ni}\right| &= \max_{t\geq a_n^q}\max_{1\leq j\leq n}\left|t^{-1/q}\sum_{i=1}^{j}EZ_{ni}\right|\\ &\leq C\max_{t\geq a_n^q}t^{-1/q}\sum_{i=1}^{n}E\left|X_{ni}\right|I\left(\left|X_{ni}\right|>t^{1/q}\right)\\ &\leq C\sum_{i=1}^{n}\frac{E\left|X_{ni}\right|I\left(\left|X_{ni}\right|>a_n\right)}{a_n}\\ &\leq C\sum_{i=1}^{n}\frac{E\psi_i\left(\left|X_{ni}\right|\right)}{\psi_i\left(a_n\right)}\to 0.\end{aligned} \tag{2.13}$$

Hence, for n large enough and $t\geq a_n^q$, we can obtain that

$$\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}EY_{ni}\right|\leq\frac{t^{1/q}}{2}. \tag{2.14}$$

Take $d_n=[a_n]+1$. It follows from Markov inequality, (2.14) and (1.10) of Lemma 1.8 that

$$\begin{aligned}K_4 &\leq C\sum_{n=1}^{\infty}a_n^{-q}\int_{a_n^q}^{\infty}E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Y_{ni}\right|^2\right)t^{-2/q}dt\\ &\leq C\sum_{n=1}^{\infty}a_n^{-q}\sum_{i=1}^{n}\int_{a_n^q}^{\infty}EX_{ni}^2I\left(\left|X_{ni}\right|\leq d_n\right)t^{-2/q}dt\\ &\quad+C\sum_{n=1}^{\infty}a_n^{-q}\sum_{i=1}^{n}\int_{a_n^q}^{\infty}t^{-2/q}EX_{ni}^2I\left(d_n<\left|X_{ni}\right|\leq t^{1/q}\right)dt\\ &\quad+C\sum_{n=1}^{\infty}a_n^{-q}\sum_{i=1}^{n}\int_{a_n^q}^{\infty}P\left(\left|X_{ni}\right|>t^{1/q}\right)dt\\ &\doteq K_5+K_6+K_7.\end{aligned} \tag{2.15}$$

By the same argument to the proof of K_3, we can see that $K_7<\infty$. For $1\leq q<p\leq 2$ and $\frac{a_n+1}{a_n}\to 1$ as $n\to\infty$, we can obtain that

$$\begin{aligned}K_5 &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{EX_{ni}^2I\left(\left|X_{ni}\right|\leq d_n\right)}{a_n^2}\\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left(\frac{a_n+1}{a_n}\right)^2\frac{EX_{ni}^2I\left(\left|X_{ni}\right|\leq d_n\right)}{d_n^2}\\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\left|X_{ni}\right|^pI\left(\left|X_{ni}\right|\leq d_n\right)}{d_n^p}\\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi_i(\left|X_{ni}\right|)}{\psi_i(d_n)}\\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi_i(\left|X_{ni}\right|)}{\psi_i(a_n)}<\infty.\end{aligned} \tag{2.16}$$

Take $t = u^q$, it follows from $1 \le q < p \le 2$, (2.1) and (2.3) that

$$\begin{aligned}
K_6 &\doteq C\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} \left(\int_{a_n^q}^{d_n^q} + \int_{d_n^q}^{\infty} \right) t^{-2/q} E X_{ni}^2 I\left(d_n < |X_{ni}| \le t^{1/q}\right) dt \\
&\le C\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} \int_{d_n}^{\infty} u^{q-3} E X_{ni}^2 I\left(d_n < |X_{ni}| \le u\right) du \\
&= C\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} \sum_{s=d_n}^{\infty} \int_{s}^{s+1} u^{q-3} E X_{ni}^2 I\left(d_n < |X_{ni}| \le u\right) du \\
&\le C\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} \sum_{s=d_n}^{\infty} s^{q-3} E X_{ni}^2 I\left(d_n < |X_{ni}| \le s+1\right) \\
&= C\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} \sum_{s=d_n}^{\infty} s^{q-3} \sum_{m=d_n}^{s} E X_{ni}^2 I\left(m < |X_{ni}| \le m+1\right) \\
&\le C\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} \sum_{m=d_n}^{\infty} m^{q-2} E X_{ni}^2 I\left(m < |X_{ni}| \le m+1\right) \\
&\le C\sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} E|X_{ni}|^q I\left(|X_{ni}| > d_n\right) \\
&\le C\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E|X_{ni}|^q I\left(|X_{ni}| > a_n\right)}{a_n^q} \\
&\le C\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi_i(|X_{ni}|)}{\psi_i(a_n)} < \infty.
\end{aligned} \tag{2.17}$$

The proof of **Theorem 2.4** is completed.

Proof of Theorem 2.5 Following the same notations and the argument proofs of **Theorem 2.4**, we can easily obtain that $K_1 < \infty$ and $K_3 < \infty$. So, we need only to prove that $K_4 < \infty$. It follows from (2.14), Markov inequality and (1.11) of Lemma 1.8 that

$$\begin{aligned}
K_4 &\le C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} E\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} (Y_{ni} - EY_{ni}) \right|^p \right) t^{-p/q} dt \\
&\le C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\sum_{i=1}^{n} E|Y_{ni}|^p + \left(\sum_{i=1}^{n} E\left(Y_{ni}^2\right) \right)^{p/2} \right) t^{-p/q} dt \\
&\le C\sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} E|Y_{ni}|^p t^{-p/q} dt + C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(\sum_{i=1}^{n} EY_{ni}^2 \right)^{p/2} t^{-p/q} dt \\
&\doteq K_8 + K_9.
\end{aligned} \tag{2.18}$$

It follows that

$$\begin{aligned}
K_8 &= C\sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} E|X_{ni}|^p I\left(|X_{ni}| \le d_n\right) t^{-p/q} dt \\
&\quad + C\sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} E|X_{ni}|^p I\left(d_n < |X_{ni}| \le t^{1/q}\right) t^{-p/q} dt \\
&\quad + C\sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} P\left(|X_{ni}| > t^{1/q}\right) dt \\
&\doteq K_{81} + K_{82} + K_{83}.
\end{aligned}$$

By the similar argument as in the proofs of $K_5 < \infty$ and $K_6 < \infty$ (replacing 2 with p), and $K_7 < \infty$, we can obtain that $K_{81} < \infty$, $K_{82} < \infty$ and $K_{83} < \infty$.

It follows from the c_r inequality that

$$\begin{aligned} K_9 &\leq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\left(\sum_{i=1}^{n} EX_{ni}^2 I(|X_{ni}|\leq a_n)\right)^{p/2} t^{-p/q}dt \\ &+C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\left(\sum_{i=1}^{n} EX_{ni}^2 I\left(a_n<|X_{ni}|\leq t^{1/q}\right)\right)^{p/2} t^{-p/q}dt \\ &+C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\left(\sum_{i=1}^{n} P\left(|X_{ni}|>t^{1/q}\right)\right)^{p/2} dt \\ &\doteq K_{91}+K_{92}+K_{93}. \end{aligned}$$

For $p>q$, $p>2$ and (2.4),

$$K_{91}\leq C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{EX_{ni}^2 I(|X_{ni}|\leq a_n)}{a_n^2}\right)^{p/2}\leq C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{EX_{ni}^2}{a_n^2}\right)^{p/2}<\infty.$$

When $1\leq q\leq 2$ and $p>2$, it follows from (2.1) and (2.4) that

$$\begin{aligned} K_{92} &\doteq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\left(\sum_{i=1}^{n} t^{-2/q}E\left|X_{ni}^2\right| I\left(a_n<|X_{ni}|\leq t^{1/q}\right)\right)^{p/2} dt \\ &\leq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\left(\sum_{i=1}^{n} t^{-1}E|X_{ni}|^q I\left(a_n<|X_{ni}|\leq t^{1/q}\right)\right)^{p/2} dt \\ &\leq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\left(\sum_{i=1}^{n} t^{-1}E|X_{ni}|^q I(|X_{ni}|>a_n)\right)^{p/2} dt \\ &\leq C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E|X_{ni}|^q I(|X_{ni}|>a_n)}{a_n^q}\right)^{p/2}<\infty. \end{aligned}$$

When $2<q<p$, it follows from (2.1), (2.3) and c_r inequality again that

$$\begin{aligned} K_{92} &\doteq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\left(\sum_{i=1}^{n} t^{-2/q}E|X_{ni}|^2 I\left(a_n<|X_{ni}|\leq t^{1/q}\right)\right)^{p/2} dt \\ &\leq C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E|X_{ni}|^2 I(|X_{ni}|>a_n)}{a_n^2}\right)^{p/2} \\ &\leq C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{E|X_{ni}|^q I(|X_{ni}|>a_n)}{a_n^q}\right)^{p/2} \\ &\leq C\left(\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi_i(|X_{ni}|)}{\psi_i(a_n)}\right)^{p/2}<\infty. \end{aligned}$$

It follows from (2.1) that

$$\begin{aligned} \max_{t\geq a_n^q}\sum_{i=1}^{n} P\left(|X_{ni}|>t^{1/q}\right) &\leq \sum_{i=1}^{n} P(|X_{ni}|>a_n) \\ &\leq \sum_{i=1}^{n}\frac{E|X_{ni}|I(|X_{ni}|>a_n)}{a_n} \\ &\leq \sum_{i=1}^{n}\frac{E\psi_i(|X_{ni}|)}{\psi_i(a_n)}\to 0. \end{aligned}$$

Hence, for n large enough and $t \geq a_n^q$, we can obtain that

$$\sum_{i=1}^{n} P\left(|X_{ni}| > t^{1/q}\right) < \frac{1}{2},$$

which implies

$$K_{93} \leq C \sum_{n=1}^{\infty} a_n^{-q} \sum_{i=1}^{n} \int_{a_n^q}^{\infty} P\left(|X_{ni}| > t^{1/q}\right) dt.$$

By a similar argument proof of $K_3 < \infty$, we can obtain that $K_{93} < \infty$. The proof of **Theorem 2.5** is completed.

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