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Module Amenability of Restricted Semigroup Algebras Under Module Actions

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Abstract. In this article, we show that module amenability with the canonical action of restricted semigroup algebra $l^1_r(S)$ and semigroup algebra $l^1(S_r)$ are equivalent, where S_r is the restricted semigroup of associated to the inverse semigroup S. We use this to give a characterization of module amenability of restricted semigroup algebra $l^1_r(S)$ with the canonical action, where S is a Clifford semigroup.

1. Introduction

The notion of module amenability for a Banach algebra $\mathcal{A}$ which is a Banach module over another Banach algebra $\mathcal{U}$ is defined by Amini in [1]. He showed that for an inverse semigroup S, the semigroup algebra $l^1(S)$ is module amenable as a $l^1(E)$- module with the multiplication right action and the trivial left action, where E is the set of idempotents of S if and only if S is amenable.

In this paper we show that module amenability of $l^1(S)$ as an $l^1(E)$-module with the canonical action implies its module amenability as an $l^1(E)$-module with the trivial left action. The main difference is that the corresponding equivalence relation leads a Clifford homomorphic image. We characterize module amenability of the restricted semigroup algebra $l^1_r(S)$ as an $l^1_r(E)$-module with the canonical action, for each Clifford semigroup S. Also we show that in the canonical action, the module amenability of the semigroup algebra $l^1(S_r)$ and the restricted semigroup algebra $l^1_r(S)$ are equivalent. This could be considered as the module version of a result of [6], [9], which asserts that the amenability of the semigroup algebra $l^1(S_r)$ and the restricted semigroup algebra $l^1_r(S)$ are equivalent.

Throughout this paper, $\mathcal{A}$ and $\mathcal{U}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module with compatible actions

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \qquad (a,\ b \in \mathcal{A}, \alpha \in \mathcal{U}),$$

and

$$\alpha \cdot (\beta \cdot a) = (\alpha\beta) \cdot a, \quad (a \cdot \beta) \cdot \alpha = a \cdot (\beta\alpha) \quad (a \in \mathcal{A}, \alpha,\ \beta \in \mathcal{U}).$$

The Banach algebra $\mathcal{U}$ acts trivially on $\mathcal{A}$ from left (right) if for each $\alpha \in \mathcal{U}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$ ($a \cdot \alpha = f(\alpha)a$), where f is a continuous character on $\mathcal{U}$.

Let X be a Banach $\mathcal{A}$-module and a Banach $\mathcal{U}$-module with compatible actions

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x,\ a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,$$

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$$(\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \ (a \in \mathcal{A}, \alpha \in \mathfrak{U}, x \in X);$$

and similarly for the right and two sided actions. We call X a $\mathcal{A}$-$\mathfrak{U}$-module. If in addition,

$$\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathfrak{U}, x \in X)$$

then X is called a commutative $\mathcal{A}$-$\mathfrak{U}$-module. If X is a commutative $\mathcal{A}$-$\mathfrak{U}$-module, then so is X^*, under the actions

$$(\alpha \cdot f)(x) = f(x \cdot \alpha), \ (a \cdot f)(x) = f(x \cdot a) \ (a \in \mathcal{A}, \alpha \in \mathfrak{U}, x \in X, f \in X^*)$$

and similarly for the right actions.

Let J be the closed ideal of $\mathcal{A}$ generated by elements of the form $\alpha \cdot ab - ab \cdot \alpha$ for $\alpha \in \mathfrak{U}$, $a, b \in \mathcal{A}$.

Let $\mathcal{A}$, $\mathfrak{U}$ and X be as above. A bounded map $D : \mathcal{A} \to X$ is called a module derivation if

$$D(a \pm b) = D(a) \pm D(b) \ , \ D(ab) = Da \cdot b + a \cdot Db \quad (a,\ b \in \mathcal{A})$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a) \ , \ D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in \mathcal{A}, \alpha \in \mathfrak{U}).$$

Note that D is not necessarily linear, but still its boundedness implies its norm continuity (since D preserves subtraction). When X is commutative, each $x \in X$ defines a module derivation

$$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These are called inner module derivations.

Definition 1.1. *$\mathcal{A}$ is called module amenable (as an $\mathfrak{U}$-module) if for any commutative $\mathcal{A}$-$\mathfrak{U}$-module X, each module derivation $D : \mathcal{A} \to X^*$ is inner.*

Definition 1.2. *A discrete semigroup S is called an inverse semigroup if for each $x \in S$ there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $e \in S$ is called an idempotent if $e = e^2$.*

Throughout this paper S is an inverse semigroup with the set of idempotents E. An inverse semigroup whose idempotents are in the center is called a Clifford semigroup [3]. A Clifford semigroup S is called a semilattice if each element of S is idempotent [7]. It is easy to see that E is a commutative subsemigroup of S and $l^1(E)$ can be regarded as a subalgebra of $l^1(S)$.
Let $l^1(E)$ acts on $l^1(S)$ by the multiplication from right and trivially from left, that is

$$\delta_e * \delta_s = \delta_s, \ \ \delta_s \cdot \delta_e = \delta_{se} \ (s \in S, e \in E).$$

In this case, J is the closed ideal generated by

$$\{\delta_s - \delta_{se} : \ s \in S, e \in E\}.$$

Consider an equivalence relation on S as follows

$$h \approx k \Leftrightarrow \delta_h - \delta_k \in J \quad (h,\ k \in S).$$

It is shown in [8] that the quotient $S/\approx$ is a discrete group.

2. Module amenability of restricted semigroup algebras

Here we consider $l^1(E)$ acts on $l^1(S)$ with canonical actions, that is

$$\delta_e \cdot \delta_s = \delta_{es}, \ \delta_s \cdot \delta_e = \delta_{se} \quad (s \in S, e \in E).$$

The closed ideal J_c of $l^1(S)$ is generated by

$$\{\delta_{es} - \delta_{se} : s \in S, e \in E\}.$$

We consider an equivalence relation on S as follows

$$s \sim t \Leftrightarrow \delta_s - \delta_t \in J_c \quad (s, t \in S).$$

An equivalence relation R on a semigroup S is called a congruence if

$$(s, t) \in R \Rightarrow (as, at),\ (sa, ta) \in R \quad (s,\ t,\ a \in S).$$

Congruences on any semigroup provide some information about its homomorphic images[2].

Let ρ be a congruence on S and P a property of homomorphic image S/ρ, we call ρ a P congruence. A least congruence ρ such that S/ρ is a P congruence is called the least P congruence.

Lemma 2.1. *$\sim$ is the least Clifford congruence on S.*

Proof. Since J_c is an ideal of $l^1(S)$, $\sim$ is a congruence. From definition of $\sim$, it follows that $es \sim se$. Thus $S/\sim$ is a Clifford semigroup. Hence the least Clifford congruence $\xi \subseteq \sim$.

Let I_γ be the closed ideal of $l^1(S)$ generated by

$$\{\delta_s - \delta_t : (s, t) \in \gamma\},$$

for each Clifford congruence γ on S. Clearly

$$s\gamma\, t \Leftrightarrow \delta_s - \delta_t \in I_\gamma \quad (s,\ t \in S).$$

Since $(es, se) \in \gamma$, it follows that $\delta_{es} - \delta_{se} \in I_\gamma$. Thus $J_c \subseteq I_\gamma$ and so $\sim \subseteq \gamma$, for each Clifford congruence γ. Hence $\sim \subseteq \xi$. □

Let X be a commutative $l^1(S)$-$l^1(E)$-module. Throughout this paper we denote by $\bullet$ the left and right actions of $l^1(E)$ on X and by $\cdot$ the left and right actions of $l^1(S)$ on X.

Proposition 2.2. *If $l^1(S)$ is module amenable as an $l^1(E)$-module with the canonical action then $l^1(S)$ is module amenable as an $l^1(E)$-module with the trivial left action.*

Proof. Suppose that $l^1(E)$ acts on $l^1(S)$ with the trivial left action and the multiplication right action. Let X be a commutative $l^1(S)$-$l^1(E)$-module and $D : l^1(S) \to X^*$ be a module derivation. We have

$$\begin{aligned} \delta_{se} \cdot x &= \delta_s \cdot (\delta_e \bullet x) = \delta_s \cdot (x \bullet \delta_e) \\ &= (\delta_s \cdot x) \bullet \delta_e = \delta_e \bullet (\delta_s \cdot x) \\ &= (\delta_e \bullet \delta_s) \cdot x \\ &= \delta_s \cdot x. \end{aligned}$$

Thus $J \cdot X = 0$ and similarly $X \cdot J = 0$. Now since $S/\approx$ is a group, $es \approx se$ and so $\delta_{es} - \delta_{se} \in J$. It follows that $X \cdot J_c = J_c \cdot X = 0$ and even if $l^1(E)$ acts on $l^1(S)$ with the canonical action, X is a commutative $l^1(S)$-$l^1(E)$-module. In additions, we have

$$D(\delta_{se}) = D(\delta_s) \bullet \delta_e = \delta_e \bullet D(\delta_s) = D(\delta_s).$$

Therefore $D|_J = 0$ and so $D(\delta_{es} - \delta_{se}) = 0$. Now if $l^1(E)$ acts on $l^1(S)$ with the canonical action, then we have

$$D(\delta_f \cdot \delta_s) = D(\delta_s \cdot \delta_f) = D(\delta_s) \bullet \delta_f = \delta_f \bullet D(\delta_s) \quad (f \in E,\ s \in S).$$

Hence D is a module derivation. So by assumption it is inner. □

Similar to the Proposition 2.1.5 of [10] we have the following Lemma.

Lemma 2.3. *Let $l^1(S)$ has a bounded approximate identity. Then $l^1(S)$ is module amenable as an $l^1(E)$-module with the canonical action if and only if each module derivation $D : l^1(S) \to X^*$ is inner, for each pseudo-unital $l^1(S)$-$l^1(E)$-module X.*

Theorem 2.4. *Let S be a semilattice. Then $l^1(S)$ is module amenable as an $l^1(E)$-module with the canonical action if and only if $l^1(S)$ admits a bounded approximate identity.*

Proof. Suppose that $l^1(S)$ admits a bounded approximate identity (λ_i). Consider a module derivation $D : l^1(S) \to X^*$. For each $e \in S$ and $\lambda \in \mathbb{C}$ we have

$$D(\lambda\delta_e) = \lambda\delta_e \bullet D(\delta_e) = \lambda D(\delta_e).$$

Thus D is a derivation. By Lemma 2.3, we may suppose that X is pseudo-unital $l^1(S)$-module. That is, for each $x \in X$, there exist $f, g \in l^1(S)$ and there is $y \in X$ such that $x = f \cdot y \cdot g$. It follows that

$$\begin{aligned} D(\delta_e)(x) &= D(\delta_e)(f \cdot y \cdot g) \\ &= D(\delta_e)((\lim_i \lambda_i \cdot f) \cdot y \cdot g) \\ &= D(\delta_e)(\lim_i \lambda_i \cdot (f \cdot y \cdot g)) \\ &= \lim_i D(\delta_e) \cdot \lambda_i(f \cdot y \cdot g) \\ &= \lim_i D(\delta_e) \cdot \lambda_i(x). \end{aligned}$$

Similarly $D(\delta_e)(x) = \lim_i \lambda_i \cdot D(\delta_e)(x)$. From the equalities $D(\delta_{ef}) = D(\delta_e) \bullet \delta_f = D(\delta_f) \bullet \delta_e$, it follows that

$$D(\delta_e) \cdot \delta_{ef} = D(\delta_f) \cdot \delta_e, \tag{1}$$

for each $e, f \in S$. In addition, we have

$$\begin{aligned} D(\delta_{ef}) &= \delta_e \bullet D(\delta_{ef}) = \delta_e \bullet (\delta_e \cdot D(\delta_f) + D(\delta_e) \cdot \delta_f) \\ &= \delta_e \cdot D(\delta_f) + D(\delta_e) \cdot \delta_{ef}. \end{aligned}$$

Thus

$$D(\delta_e) \cdot \delta_{ef} = D(\delta_e) \cdot \delta_f. \tag{2}$$

From (1), (2), it follows that $D(\delta_e)\cdot\delta_f = D(\delta_f)\cdot\delta_e$, for each $e, f \in S$. Thus we have for each λ_i, $D(\lambda_i)\cdot\delta_e = D(\delta_e)\cdot\lambda_i$ and so $D(\delta_e) = \lim_i D(\lambda_i \cdot \delta_e) = \lim_i(D(\delta_e) \cdot \lambda_i + \lambda_i \cdot D(\delta_e))$. This implies that $D(\delta_e)(x) = \lim_i D(\delta_e) \cdot \lambda_i(x) + \lim_i \lambda_i \cdot D(\delta_e)(x) = 2D(\delta_e)(x)$, for each $x \in X$. Hence $D(\delta_e)(x) = 0$, for $x \in X$ and so $D(\delta_e) = 0$. Conversely, since $l^1(S)$ is a commutative $l^1(S)$-module, it has a bounded approximate identity by [1]. □

Note that by the above theorem, for semilattice $S = (\mathbb{N}, \vee)$, $l^1(S)$ is module amenable as an $l^1(E)$-module with the canonical action. This example shows that module amenability of a semilattice algebra does not imply finiteness of the semilattice.

Consider the multiplication $\circ$ on the Banach space $l^1(S)$ by

$$\sum_{s\in S} f(s)\delta_s \circ \sum_{t\in S} g(t)\delta_t = \sum_{r\in S} \sum_{st=r,\, s^*s=tt^*} f(s)g(t)\delta_r,$$

if there are no elements $t, s \in S$ with $st = r$ and $s^*s = tt^*$, the multiplication is taken as zero. Under the usual l^1-norm, $(l^1(S), \circ)$ is a Banach algebra. We denote this Banach algebra by $l^1_r(S)$ as in [6]. In the particular case,

$$\delta_s \circ \delta_t = \begin{cases} \delta_{st} & s^*s = tt^* \\ 0 & \textit{otherwise.} \end{cases}$$

Note that $l_r^1(E)$ could be regarded as a subalgebra of $l_r^1(S)$. Here we consider $l_r^1(E)$ acts on $l_r^1(S)$ with the canonical actions. The closed ideal J_B of $l_r^1(S)$ is generated by

$$\{\delta_s|\ s \in S,\ ss^* \neq s^*s\}.$$

Consider an equivalence relation $\sim_B$ on S as follows

$$s \sim_B t \Longleftrightarrow \delta_s - \delta_t \in J_B\ \ (s,\ t \in S).$$

Note that in general, $\sim_B$ is not a congruence.
Let X be a commutative $l_r^1(S)$-$l_r^1(E)$- module. Throughout the rest of this paper we denote left and right actions of $l_r^1(E)$ on X by $\bullet$ and left and right actions of $l_r^1(S)$ on X by $\cdot$.

Proposition 2.5. *$l_r^1(S)$ is module amenable as an $l_r^1(E)$-module with the canonical action if and only if $l_r^1(S)/J_B$ is module amenable as an $l_r^1(E)$-module.*

Proof. Let X be a commutative $l_r^1(S)$-$l_r^1(E)$-module. Consider a module derivation $D : l_r^1(S) \to X^*$. For each $s \in S$ such that $ss^* \neq s^*s$, we have

$$\begin{aligned} D(\delta_s) &= D(\delta_s \circ \delta_{s^*s}) = D(\delta_s) \bullet \delta_{s^*s} \\ &= \delta_{s^*s} \bullet D(\delta_s) = D(\delta_{s^*s} \circ \delta_s) \\ &= 0. \end{aligned}$$

Thus $D|_{J_B} = 0$ and so $\tilde{D} : l_r^1(S)/J_B \to X^*$ defined by $\tilde{D}(\delta_s + J_B) = D(\delta_s)$ is a module derivation. We conclude that if $l_r^1(S)/J_B$ is module amenable as $l_r^1(E)$-module with the canonical action, then $l_r^1(S)$ is module amenable as $l_r^1(E)$-module with the canonical action. The converse follows using the module homomorphism $\pi : l_r^1(S) \to l_r^1(S)/J_B$ and Proposition 2.5 of [1]. □

Proposition 2.6. *Let S be a Clifford semigroup. Then $l_r^1(S)$ is module amenable as an $l_r^1(E)$-module with the canonical action if and only if $l^1(S)$ is amenable.*

Proof. Suppose that $l_r^1(S)$ is module amenable as an $l_r^1(E)$-module with the canonical action. Since S is a Clifford semigroup, $l_r^1(S)$ is a commutative $l_r^1(E)$-module with the canonical action. It follows from Proposition 2.2 of [1] that $l_r^1(S)$ has a bounded approximate identity. From [6], it follows that E is finite. Let I be the closed principal ideal of S generated by $e \in E$. Thereby $l_r^1(I)$ is an $l_r^1(E)$-module with the following compatible actions

$$\delta_f \cdot \delta_i := \delta_f \circ \delta_i,\ \delta_i \cdot \delta_f := \delta_i \circ \delta_f\ \ (f \in E, i \in I).$$

Consider the module homomorphism $\varphi : l_r^1(S) \to l_r^1(I)$ defined by $\varphi(\delta_s) = \delta_s \circ \delta_e$. Thus $l_r^1(I)$ is module amenable as an $l_r^1(E)$-module with the canonical action. Now put $I_e = \{b \in I : SbS \subsetneq I\}$. Similarly I_e is an ideal of I and $\psi : l_r^1(I) \to l_r^1(I/I_e)$ is a module homomorphism and so $l_r^1(I/I_e)$ is module amenable as an $l_r^1(E)$-module with the canonical action, by Proposition 2.5 of [1]. Similarly $I/I_e \cong \{0\} \cup G_e$ and $l_r^1(\{0\} \cup G_e)$ is module amenable as an $l_r^1(E)$-module with the canonical action. We claim that $l^1(G_e)$ is amenable. Let X be a $l^1(G_e)$-module and $D : l^1(G_e) \to X^*$ be a derivation. Since $l_r^1(\{0\} \cup G_e) = l^1(\{0\} \cup G_e) = l^1(G_e) \oplus \mathbb{C}\delta_0$, with the following new definition, X is a commutative $l^1(\{0\} \cup G_e)$-$l^1(\{0, e\})$-module with the compatible actions

$$x \cdot \delta_0 = \delta_0 \cdot x = 0.$$

Consider $\tilde{D} : l^1(\{0\} \cup G_e) \to X^*$ defined by $\tilde{D}(\delta_g) = D(\delta_g)(g \in G_e)$ and $D(\delta_0) = 0$. Clearly if $l^1(S)$ is an $l^1(E)$-module with the canonical action, then $\tilde{D}$ is a module derivation and so it is inner. Therefore D is an inner derivation and this proves that $l^1(G_e)$ is amenable. It follows that G_e is amenable and by [5], $l^1(S)$ is amenable. The converse is clear. □

Corollary 2.7. *Let S be a semilattice. Then $l_r^1(S)$ is module amenable as an $l_r^1(E)$-module with the canonical action if and only if S is finite.*

Proof. It follows from the above proposition that $l^1(S)$ is amenable. Since S is semilattice, S is finite. The converse is clear. □

Proposition 2.6 means that if $l^1_r(S)$ is module amenable as an $l^1_r(E)$-module with the canonical action then $l^1(S)$ is module amenable with the canonical action for each Clifford semigroup. The converse fails in general. For example let $(\mathbb{N}, \vee)$ be the semigroup of positive integers with maximum operation, that is $m \vee n = \max(m, n)$. By Theorem 2.4, $l^1(S)$ is module amenable as an $l^1_r(E)$-module with the canonical action but $l^1_r(S)$ is not module amenable as an $l^1_r(E)$-module with the canonical action by Corollary 2.7.

For an arbitrary inverse semigroup S with the set of idempotents E, the restricted product of elements x and y of S is xy if $x^*x = yy^*$ and undefined, otherwise. The set S with this restricted product forms a discrete groupoid [4]. If we adjoin a zero element 0 to this groupoid and put $0^* = 0$, then we get an inverse semigroup S_r with the multiplication rule

$$x \diamond y = \begin{cases} xy & x^*x = yy^* \\ 0 & otherwise, \end{cases}$$

for each $x,\ y \in S \cup \{0\}$. The inverse semigroup S_r is called the restricted semigroup of S (see[6]).
Note that $(l^1(E \cup \{0\}), \diamond)$ could be regarded as a subalgebra of $l^1(S_r)$ and we denote this Banach algebra by $l^1(E_r)$. Thereby $l^1(S_r)$ is an $l^1(E_r)$- module with the canonical action. The closed ideal J_r of $l^1(S_r)$ is generated by

$$\{\delta_s - \delta_0 |\, s \in S,\ s^*s \neq ss^*\}.$$

We consider an equivalence relation $\sim_r$ on S_r as follows

$$s \sim_r t \Longleftrightarrow \delta_s - \delta_t \in J_r \ \ (s, t \in S_r).$$

Proposition 2.8. *$\sim_r$ is the least Clifford congruence on S_r.*

Proof. From definition of J_r, $s \sim_r t$ for each $s,\ t \in S$ such that $ss^* \neq s^*s$ and $tt^* \neq t^*t$. Since each element s such that $ss^* = s^*s$ is contained in a maximal subgroup of S, S is a semilattice of groups. Thus $S/\sim_r$ is a Clifford semigroup by Theorem 4.2.1 of [3]. Thus $\sim_r$ is a Clifford equivalence. Suppose that $s \sim_r\ t$ and $l \in S$. Since $\delta_s - \delta_t \in J_r$, it follows that $\delta_l \cdot (\delta_s - \delta_t) \in J_r$ and $(\delta_s - \delta_t) \cdot \delta_l \in J_r$. Thus $\delta_{l\diamond s} - \delta_{l\diamond t} \in J_r$ and $\delta_{s\diamond l} - \delta_{t\diamond l} \in J_r$ and so $\sim_r$ is a congruence on S_r. Finally suppose that ρ is a Clifford congruence on S_r. Let I_ρ be the closed ideal of $l^1(S_r)$ generated by $\{\delta_s - \delta_t :\ (s, t) \in \rho\}$. Clearly

$$s\rho\, t \Leftrightarrow \delta_s - \delta_t \in I_\rho.$$

Since for each $s \in S$ such that $ss^* \neq s^*s$ we have $(s, 0) \in \rho$, it follows that $\delta_s - \delta_0 \in I_\rho$, for each Clifford congruence ρ. Thus $J_r \subseteq I_\rho$ and so for each Clifford congruence ρ, $\sim_r \subseteq \rho$. Hence $\sim_r$ is the least Clifford congruence on S_r. □

Note that $\delta_s \circ \delta_t = 0$ in $l^1_r(S)$ but $\delta_s \cdot \delta_t = \delta_0$ in $l^1(S_r)$, for each $s, t \in S$ such that $s^*s \neq tt^*$. Thus $l^1_r(S)$ is not a subalgebra of $l^1(S_r)$.

Proposition 2.9. *Let S be an inverse semigroup. Then the following statements are equivalent:*
(i): $l^1_r(S)$ is module amenable as an $l^1_r(E)$-module with the canonical action.
(ii): $l^1(S_r)$ is module amenable as an $l^1(E_r)$-module with the canonical action.
(iii): $l^1(S_r/\sim_r)$ is amenable.

Proof. $(i) \Rightarrow (ii)$ Suppose that X is a commutative $l^1(S_r)$-$l^1(E_r)$-module and $D : l^1(S_r) \to X^*$ is a module derivation. Then the following module actions are well-defined

$$\delta_s *_B x = \begin{cases} 0 & \delta_s \cdot x = \delta_0 \cdot y (for\ some\ y \in X) \\ \delta_s \cdot x & otherwise, \end{cases}$$

for each $s \in S$ and similarly for the right action. Also $l^1_r(E)$ acts on X by the following action

$$\delta_e \bullet_B x = \begin{cases} 0 & \delta_e \bullet x = \delta_0 \bullet y (for\ some\ y \in X) \\ \delta_s \bullet x & otherwise. \end{cases}$$

Therefore X is a commutative $l^1_r(S)$-$l^1_r(E)$-module. Consider $\tilde{D} : l^1_r(S) \to X^*$ defined by

$$\tilde{D}(\delta_s) = \begin{cases} D(\delta_s) & D(\delta_s) \neq D(\delta_0) \\ 0 & otherwise. \end{cases}$$

$\tilde{D}$ extends to a module derivation and so it is inner. Therefore D is inner.
$(ii) \Rightarrow (i)$ Suppose that X is a commutative $l^1_r(S)$-$l^1_r(E)$- module. It is enough to define $\delta_0 \cdot x = \delta_0 \bullet x = 0$, then X is a commutative $l^1(S_r)$-$l^1(E_r)$- module. Let $D : l^1_r(S) \to X^*$ be a module derivation. Consider $\tilde{D} : l^1(S_r) \to X^*$ defined by

$$\tilde{D}(\delta_s) = \begin{cases} D(\delta_s) & s \in S \\ 0 & s = 0. \end{cases}$$

It is easy to sea that $\tilde{D}$ extends to a module derivation and so it is inner. Therefore D is inner.
$(ii) \Rightarrow (iii)$ Since $l^1(S_r)$ is module amenable as an $l^1(E_r)$-module with the canonical action, it follows from Proposition 2.5 of [1] that $l^1(S_r/\sim_r)$ is module amenable as an $l^1_r(E_r)$-module with the canonical action. Now by Propositions 2.6, 2.8, $l^1(S_r/\sim_r)$ is amenable.
$(iii) \Rightarrow (ii)$ Let X be a commutative $l^1(S_r)$-$l^1(E_r)$-module. Since $J_r \cdot X = X \cdot J_r = 0$, the following module actions are well-defined

$$(\delta_s + J_r) \cdot x := \delta_s + J_r, \quad x \cdot (\delta_s + J_r) := x \cdot \delta_s \ \ (x \in X, \delta_s \in l^1(S_r)),$$

therefore X is an $l^1(S)/J_r$-module. Suppose that $D : l^1(S_r) \to X^*$ is a module derivation, and consider $\tilde{D} : l^1(S_r)/J_r \to X^*$ defined by $\tilde{D}(\delta_s + J_r) = D(\delta_s)$ $(s \in S_r)$. We have

$$\begin{aligned} D(\delta_s) &= D(\delta_s \cdot \delta_{s^*s}) \\ &= D(\delta_s) \bullet \delta_{s^*s} \\ &= \delta_{s^*s} \bullet D(\delta_s) \\ &= D(\delta_{s^*s} \cdot \delta_s) \\ &= D(\delta_{s^*s \diamond s}) \\ &= 0. \end{aligned}$$

By the above observation, $\tilde{D}$ is also well-defined. Moreover,

$$D(\lambda\delta_s) = \lambda\delta_s s^* \bullet D(\delta_s) = \lambda D(\delta_s) \ \ (\lambda \in \mathbb{C}).$$

Thus D is linear and so $\tilde{D}$ is linear. Hence $\tilde{D}$ is inner. Therefore D is an inner module derivation. So $l^1(S_r)/J_r$ is module amenable as an $l^1(E_r)$-module with the canonical action and it follows from proposition 2.5 of [1] and $l^1(S_r/\sim_r) \cong l^1(S_r)/J_r$ that $l^1(S_r/\sim_r)$ is module amenable as an $l^1(E_r)$-module with the canonical action. □

References

[1] M. Amini, Module amenability for semigroup algebras, Semigroup Forum 69 (2004) 243-254.
[2] D. G. Green, The lattice of congruence on inverse semigroup, Pacific Journal of Mathematics 57 (1975) 141-152.
[3] J. M. Howie, An introduction to semigroup theory, Academic Press, London, 1976.
[4] M. V. Lawson, Inverse semigroups. The theory of partial symmetries, World Scientific, Singapore, 1998.
[5] M. Sadr, Pseudo-amenability of Brandet semigroup algebras, Comment. Math. Univ. Carolin. 50.3 (2009) 413-419.
[6] M. Mehran, M. Amini, Amenability of restricted semigroup algebras, Int Journal of Math Analysis 4.1 (2010) 17-28.
[7] N. R. Reilly, H. E. Scheiblich, Congruence on regular semigroups, Pacific J of Mathematics 23 (1967) 349-360.
[8] R. Rezavand, M. Amini, M. H. Sattari, D. Ebrahimi Bagha, Module Arense regularity for semigroup algebras, Semigroup Forum 77 (2008) 300-305.
[9] M. Rostami, A. Pourabbas, M. Essmaili, Approximate amenability of certain semigroup algebras, Acta Mathematica Scientia 33B(2) (2013) 565-577.
[10] V. Runde, Lectures on amenability, Lecture Notes in Mathematics, Springer-Verlag, New York, 2002.