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Existence Results for Nonlinear Boundary Value Problems

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Abstract. In the present paper, we are concerned to prove under some hypothesis the existence of fixed points of the operator L defined on $C(I)$ by

$$Lu(t) = \int_0^\omega G(t,s)h(s)f(u(s))ds, \ t \in I, \ \omega \in \{1, \infty\},$$

where the functions $f \in C([0,\infty);[0,\infty))$, $h \in C(I;[0,\infty))$, $G \in C(I \times I)$ and

$$\begin{cases} I = [0,1], & \text{if } \omega = 1, \\ I = [0,\infty), & \text{if } \omega = \infty. \end{cases}$$

By using Guo Krasnoselskii fixed point theorem, we establish the existence of at least one fixed point of the operator L.

1. Introduction

The existence of positive solutions for a second order differential equation of the form

$$u''(t) + h(t)f(u(t)) = 0 \tag{1}$$

or a third order differential equations of the form

$$u'''(t) + h(t)f(u(t)) = 0 \tag{2}$$

with suitable boundary conditions has proved to be important in theory and applications. The more general nonlinear multi-point boundary value problems have been studied by several authors by using the Guo Krasnoselskii fixed point theorem, we refer the readers to [2, 4–6, 10] for some recent results of nonlinear multi-point boundary value problems. Meanwhile, boundary value problems in an infinite interval arose in many applications and received much attention, see[1-9]. Due to the fact that an infinite interval is noncompact, the discussion about boundary value problems on the half-line is more complicated.
Our main idea in this paper is to change equations (1) and (2) into the Hammerstein equation of the form

$$u(t) = \int_0^\omega G(t,s)h(s)f(u(s))ds \equiv Lu(t), \ t \in I, \tag{3}$$

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where the functions $f \in C([0,\infty);[0,\infty))$, $h \in C(I;[0,\infty))$, $G \in C(I \times I)$ and

$$\begin{cases} I = [0,1]. \text{ if } \omega = 1, \\ I = [0,\infty), \text{ if } \omega = \infty. \end{cases}$$

We define

$$f^{\alpha} = \limsup_{x\to\alpha} \frac{f(x)}{x} \quad \text{and} \quad f_{\alpha} = \liminf_{x\to\alpha} \frac{f(x)}{x}, \tag{4}$$

where α denotes either 0 or ∞ and we always assume the following conditions:
(H_1) There exists tow positive functions p and q on I such that

$$\begin{cases} p = q \equiv 1, \text{ if } \omega = 1 \\ \lim_{t\to\infty} \frac{p(t)}{q(t)} = 0, \text{ if } \omega = \infty. \end{cases}$$

Moreover, there exists a nonnegative continuous function g on I positive in $(0,\omega)$ such that

$$\forall\ (t,s) \in I \times I,\ p(t)G(t,s) \le q(s)g(s).$$

(H_2) There exist $\gamma \in (0,1)$, $0 < a < b < \omega$ such that

$$\forall\ (t,s) \in [a,b] \times I,\ G(t,s) \ge \gamma q(s)g(s).$$

(H_3) $f \in C([0,\infty),[0,\infty))$.
(H_4) $h \in C(I,[0,\infty))$ such that

$$0 < \int_a^b q(s)g(s)h(s)ds \le \int_0^{\omega} q(s)g(s)h(s)ds < \infty.$$

Put

$$M := (\int_0^{\omega} \frac{q(s)}{p(s)} g(s)h(s)ds)^{-1} \text{ and } m := (\gamma^2 \int_a^b q(s)g(s)h(s)ds)^{-1}. \tag{5}$$

Then, the aim of this paper is to prove the following useful theorem:

Theorem 1.1. *Assume that $(H_1) - (H_4)$ are satisfied, then the operator L has at least one fixed point in the case*
(i) $0 \le f^0 \le M$ *and* $m \le f_\infty \le \infty$, *or*
(ii) $0 \le f^\infty \le M$ *and* $m \le f_0 \le \infty$.

This result can be consedered as a generalization of others, see for examples those contained in [6, 7].
This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on property of operator L. The proof of Theorem 1.1 is given in Section 3. In Section 4, we present two rigorous application of our main result.

2. Preliminaries

Assume that $p \in C(I,[0,\infty))$ and

$$E = \{x : I \to \mathbb{R} : x \text{ is continuous on } I \text{ and } \sup_{t\in I} |x(t)|p(t) < \infty\}. \tag{6}$$

For $x \in E$, we define

$$\|x\|_p := \sup_{t\in I} |x(t)|p(t).$$

Then E is a Banach space, for more details see [9].
The following theorem is needed in Section 3.

Theorem 2.1. *(See [9]) Let $\Omega \subset E$. If the function $x \in \Omega$ is locally equicontinuous on I and uniformly bounded in the sense of the norm*

$$\|x\|_q := \sup_{t\in I} |x(t)|q(t),$$

where the function q is positive and continuous on I such that

$$\begin{cases} p = q \equiv 1, & \text{if } \omega = 1, \\ \lim\limits_{t\to\infty} \dfrac{p(t)}{q(t)} = 0, & \text{if } \omega = \infty. \end{cases}$$

Then, Ω is relatively compact in E.

Now, we will give a result of completely continuous operator, to this aim, let γ be the constant given by hypotheses (H_2), we define a cone K as follows

$$K := \{u \in E : u(t) \geq 0, t \in I \text{ and } \min_{a\leq t\leq b} u(t) \geq \gamma \|u\|_p\}.$$

Then, we have the following theorem.

Theorem 2.2. *Assume that* $(\mathbf{H}_1) - (\mathbf{H}_4)$ *hold. Then, for any bounded set $\Omega \subset E$, we know that $L : \overline{\Omega} \cap K \longrightarrow K$ is completely continuous.*

Proof. If $\omega = 1$, it is easy to see that L is completely continuous.
If $\omega = \infty$, let us choose any bounded set $\Omega \subset E$.
Firstly, we prove that $L : \overline{\Omega} \cap K \longrightarrow K$. It is clear that

$$Lu(t) \geq 0, \ \forall u \in \overline{\Omega} \cap K, \ t \in I.$$

On the other hand, using $(\mathbf{H}_1)$, we have for all $t \in I$:

$$\begin{aligned} |Lu(t)|p(t) &= \int_0^\infty p(t)G(t,s)h(s)f(u(s))ds \\ &\leq \int_0^\infty q(s)g(s)h(s)f(u(s))ds \\ &\leq \|f\|_\infty \int_0^\infty q(s)g(s)h(s)ds. \end{aligned}$$

So, using $(\mathbf{H}_4)$, we obtain:

$$\sup_{t\in I} |Lu(t)|p(t) \leq \|f\|_\infty \int_0^\infty q(s)g(s)h(s)ds < \infty.$$

Thus

$$Lu \in E, \forall u \in \overline{\Omega} \cap K.$$

Moreover, from $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ we have for any $u \in \overline{\Omega} \cap K$ and $t_0 \in I$

$$\begin{aligned} \min_{a\leq t\leq b} Lu(t) &= \min_{a\leq t\leq b} \int_0^\infty G(t,s)h(s)f(u(s))ds \\ &\geq \gamma \int_0^\infty q(s)g(s)h(s)f(u(s))ds \\ &\geq \gamma \int_0^\infty p(t_0)G(t_0,s)h(s)f(u(s))ds \\ &\geq \gamma p(t_0)Lu(t_0). \end{aligned}$$

Therefore,

$$\min_{a\leq t\leq b} Lu(t) \geq \gamma\|Lu\|_p,\ u\in\overline{\Omega}\cap K.$$

Moreover, for any $T\in(0,\infty)$, the fact that

$$G\in C([0,\infty)\times[0,\infty)),\ \text{and } f,\ h\in C([0,\infty)),$$

and standard argument tells that $\{Lu:\ u\in\overline{\Omega}\cap K\}$ are equicontinuous in interval $[0,T]$. So $\{Lu:\ u\in\overline{\Omega}\cap K\}$ are equicontinuous on $[0,\infty)$, then, Theorem 2.1 implies that $L(\overline{\Omega}\cap K)$ is a precompact set in E. Hence L is completely continuous. □

The proof of our main results is based upon an application of the following fixed point theorems (See [4], [8]).

Theorem 2.3. *(Guo-Krasnoselskii [4, 8]) Let $(E,\|.\|)$ be a Banach space, and $P\subset E$ be a cone. Assume Ω_1, Ω_2 are bounded open subsets of E with $0\in\Omega_1$, $\overline{\Omega_1}\subset\Omega_2$, and let*

$$T:\ P\cap(\overline{\Omega_2}\backslash\Omega_1)\longrightarrow P$$

be a completely continuous operator such that either
(i) $\|Tu\|\leq\|u\|$ for $u\in P\cap\partial\Omega_1$ and $\|Tu\|\geq\|u\|$ for $u\in P\cap\partial\Omega_2$; or
(ii) $\|Tu\|\geq\|u\|$ for $u\in P\cap\partial\Omega_1$ and $\|Tu\|\leq\|u\|$ for $u\in P\cap\partial\Omega_2$.
Then T has a fixed point in $P\cap(\overline{\Omega_2}\backslash\Omega_1)$.

3. Proof of our main result

First, we will prove that L has a fixed point in K in the case:

$$0\leq f^0\leq M \ \text{ and }\ m\leq f_\infty\leq\infty,$$

where m and M are the constants given by hypothesis (H_4).
Since $0\leq f^0\leq M$, we may choose $R_1>0$ such that for each $0\leq x\leq R_1$ we have:

$$f(x)\leq Mx. \tag{7}$$

Put

$$\Omega_1=\{u\in E:\|u\|<R_1\},$$

then, it follows from (7) and $(H_1)-(H_4)$ that for all $(t,u)\in I\times(K\cap\partial\Omega_1)$

$$\begin{aligned}
p(t)Lu(t)=\int_0^\omega p(t)G(t,s)h(s)f(u(s))ds &\leq \int_0^\omega q(s)g(s)h(s)f(u(s))ds\\
&\leq M\int_0^\omega q(s)g(s)h(s)u(s)ds\\
&\leq M\|u\|\int_0^\omega \frac{q(s)}{p(s)}g(s)h(s)ds=\|u\|.
\end{aligned}$$

Hence, for all $u\in K\cap\partial\Omega_1$ we have

$$\|Lu\|\leq\|u\|.$$

On the other hand, since $m\leq f_\infty\leq\infty$, we may choose $R>0$ such that

$$f(x)\geq mx,\ \forall x\geq R. \tag{8}$$

Let $R_2 = max(2R, \frac{R}{\gamma})$ and

$$\Omega_2 = \{u \in E : \|u\| < R_2\}.$$

It follows that for all u in $K \cap \partial\Omega_2$ and t in $[a, b]$, we have

$$u(t) \geq \gamma\|u\| = \gamma R_2 \geq R.$$

So, we deduce by (8) and $(H_2) - (H_4)$ that

$$\begin{aligned} p(t)Lu(t) &= \int_0^\omega p(t)G(t,s)h(s)f(u(s))ds \\ &\geq \int_a^b p(t)G(t,s)h(s)f(u(s))ds \\ &\geq m\gamma \int_a^b q(s)g(s)h(s)u(s)ds \\ &\geq m\gamma^2\|u\| \int_a^b q(s)g(s)h(s)ds = \|u\|. \end{aligned}$$

Consequently,

$$\|Lu\| \geq \|u\| \quad \forall\ u \in K \cap \partial\Omega_2.$$

Therefore, it follows from the first part of Theorem 2.3 that L has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.
Now, we consider the case: $0 \leq f^\infty \leq M$ and $m \leq f_0 \leq \infty$.
Since $m \leq f_0 \leq \infty$, we may choose $R_3 > 0$ such that

$$f(x) \geq mx \quad \text{for all} \quad 0 \leq x \leq R_3. \tag{9}$$

Let

$$\Omega_3 = \{u \in E : \|u\| < R_3\}.$$

Then, using (9) and (H_2), we obtain for $u \in K \cap \partial\Omega_3$ and $t \in [a, b]$

$$\begin{aligned} p(t)Lu(t) &= \int_0^\omega p(t)G(t,s)h(s)f(u(s))ds \\ &\geq \int_a^b p(t)G(t,s)h(s)f(u(s))ds \\ &\geq m\gamma \int_a^b q(s)g(s)h(s)u(s)ds \\ &\geq m\gamma^2\|u\| \int_a^b q(s)g(s)h(s)ds = \|u\|. \end{aligned}$$

So,

$$\|Lu\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_3.$$

Now, by (H_1), there exists $R > 0$ such that $f(x) \leq R$ for all $x \in [0, \infty)$.
Let

$$R_4 = max\{2R_3, R \int_0^\omega q(s)g(s)h(s)ds\},$$

and put

$$\Omega_4 = \{u \in E : \|u\| < R_4\}.$$

Then, we obtain for any $u \in K \cap \partial\Omega_4$ and $t \in I$:

$$\begin{aligned} p(t)Lu(t) &= \int_0^{\omega} p(t)G(t,s)h(s)f(u(s))ds \\ &\leq R\int_0^{\omega} q(s)g(s)h(s)ds \\ &\leq R_4 = \|u\|. \end{aligned}$$

So,

$$\|Lu\| \leq \|u\|, \quad \forall u \in K \cap \partial\Omega_4.$$

Thus, from the second part of Theorem 2.3, we know that the operator L has a fixed point in $K \cap (\overline{\Omega_4} \backslash \Omega_3)$. This completes the proof.

4. Applications

As applications of the last theorem, we give the following theorems. In the first one, we generalize Theorem 3.1 proved in [6], where the others stated for sublinear or superlinear cases (i.e. $f^\infty = 0$ and $f_0 = \infty$ or $f^0 = 0$ and $f_\infty = \infty$). After, in the second application, we prove the existence of positive continuous solution of the following problem

$$(\mathbf{S_2}) \begin{cases} (Lu)'(t) + a(t)f(u(t)) = 0, \ t \in (0,1), \\ u(0) = Au'(0) = 0, \ Au'(1) = \alpha Au'(\eta), \end{cases}$$

where $Lu := \frac{1}{A}(Au')'$, $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\int_0^\eta A(s)ds}$.

4.1. *Third-order three-point boundary value problem:*

We will consider the existence of a positive solution to the third-order three-point boundary value problem

$$(\mathbf{S_1}) \begin{cases} u'''(t) + h(t)f(u(t)) = 0, \ t \in (0,1), \\ u(0) = u'(0) = 0, \ u'(1) = \alpha u'(\eta), \end{cases}$$

where $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$.

Theorem 4.1. *Assume* $(H_3) - (H_4)$*. Then, the boundary value problem* $(\mathbf{S_1})$ *has at least one positive solution in the case*
(i) $0 \leq f^0 \leq M$ *and* $m \leq f_\infty \leq \infty$*, or*
(ii) $0 \leq f^\infty \leq M$ *and* $m \leq f_0 \leq \infty$*.*

Proof. It well known (See [4, 5]) that a positive continuous function u in $[0,1]$ is a solution of the problem $(\mathbf{S_1})$ if and only if it is a fixed point of the operator L defined on E by:

$$Lu(t) = \int_0^1 G(t,s)h(s)f(u(s))ds, \ t \in [0,1], \tag{10}$$

where G is the Green function associated to the problem $(\mathbf{S_1})$.
Let

$$g(s) = \frac{1+\alpha}{1-\alpha\eta}s(1-s), \ s \in [0,1].$$

Then, it follows from Lemma 2.2 and Lemma 2.3 in [6], that

$$0 \leq G(t,s) \leq g(s), \ \forall (t,s) \in [0,1] \times [0,1]. \tag{11}$$

and

$$G(t,s) \geq \gamma g(s),\ \forall (t,s) \in [\frac{\eta}{\alpha}, \eta] \times [0,1], \tag{12}$$

where

$$0 < \gamma = \frac{\eta^2}{2\alpha^2(1+\alpha)} min\{\alpha - 1, 1\} < 1.$$

So, all the hypotheses of Theorem 1.1 are satisfied. Hence, the operator L has a fixed points which are the desired positive continuous solution of the problem ($\mathbf{S_1}$). □

4.2. A class of third-order three-point boundary value problem:

In the second corollary, we fixe a nonnegative continuous function A on $[0,1]$, positive and differentiable on $(0,1)$ such that

$$D(t) := \int_0^t \frac{1}{A(s)} ds < \infty,\ \forall t \in [0,1].$$

Without loss of generality we can assume that

$$\int_0^1 A(s)ds = 1.$$

We denoted by

$$Lu := \frac{1}{A}(Au')',$$

and we deal with the existence of positive continuous solution of the following problem

$$(\mathbf{S_2}) \begin{cases} (Lu)'(t) + a(t)f(u(t)) = 0,\ t \in (0,1), \\ u(0) = Au'(0) = 0,\ Au'(1) = \alpha Au'(\eta), \end{cases}$$

where $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\int_0^\eta A(s)ds}$. Firtly, we will give several important lemma.

Lemma 4.2. *The problem*

$$\begin{cases} (Lu)'(t) = h(t),\ t \in (0,1), \\ u(0) = Au'(0) = 0,\ Au'(1) = \alpha Au'(\eta), \end{cases}$$

has a unique solution $u(t) = \int_0^1 G(t,s)h(s)ds$, *where*

$$G(t,s) = \frac{1}{1-\alpha B(\eta)} \begin{cases} (\alpha - 1)C(t)B(s) + (1-\alpha B(\eta))[C(s) + B(s)(D(t) - D(s))],\ s \leq min(t,\eta), \\ C(t)(\alpha B(\eta) - B(s)) + (1 - \alpha B(\eta))[C(s) + B(s)(D(t) - D(s))],\ \eta \leq s \leq t, \\ C(t)(1 - \alpha B(\eta)) + (\alpha - 1)C(t)B(s),\ t \leq s \leq \eta, \\ C(t)(1 - B(s)),\ s \geq max(t,\eta), \end{cases}$$

is called the Green's function.

Proof. Let u be a solution of the following problem

$$\begin{cases} (Lu)'(t) = h(t),\ t \in (0,1), \\ u(0) = Au'(0) = 0,\ Au'(1) = \alpha Au'(\eta), \end{cases}$$

Then, by integration, we obtain

$$L(u)(t) = c + \int_0^t h(s)ds.$$

Multiplying by A and by integration, we obtain

$$\begin{aligned}\int_0^t A(s)L(u)(s)ds &= \int_0^t (A(s)u'(s))'ds\\ &= Au'(t)\\ &= cB(t) + \int_0^t A(s)\int_0^s h(\xi)d\xi ds\\ &= cB(t) + \int_0^t h(s)(B(t)-B(s))ds,\end{aligned}$$

where

$$B(t) = \int_0^t A(s)ds,\ \ t \in [0,1].$$

Since $Au'(1) = \alpha Au'(\eta)$, then we have

$$c = \frac{1}{1-\alpha B(\eta)}\left[\alpha\int_0^\eta h(s)(B(\eta)-B(s))ds - \int_0^1 h(s)(1-B(s))ds\right].$$

So,

$$\begin{aligned}(1-\alpha B(\eta))Au'(t) &= \alpha\int_0^\eta h(s)B(t)(B(\eta)-B(s))ds - \int_0^1 h(s)B(t)(1-B(s))ds\\ &\quad +(1-\alpha B(\eta))\int_0^t h(s)(B(t)-B(s))ds.\end{aligned}$$

Dividing by A and integrating, we obtain

$$\begin{aligned}(1-\alpha B(\eta))u(t) &= \alpha\int_0^\eta h(s)C(t)(B(\eta)-B(s))ds - \int_0^1 h(s)C(t)(1-B(s))ds\\ &\quad +(1-\alpha B(\eta))\int_0^t [C(t)-C(s)-B(s)(D(t)-D(s))]\,h(s)ds\\ &= (1-\alpha B(\eta))\int_0^1 G(t,s)h(s)ds,\end{aligned}$$

where

$$C(t) = \int_0^t \frac{B(s)}{A(s)}ds,\ \ t \in [0,1].$$

Finally G is defined on $[0,1]\times[0,1]$ by

$$G(t,s) = \frac{1}{1-\alpha B(\eta)}\begin{cases}(\alpha-1)C(t)B(s) + (1-\alpha B(\eta))[C(s)+B(s)(D(t)-D(s))],\ s \le min(t,\eta),\\ C(t)(\alpha B(\eta)-B(s)) + (1-\alpha B(\eta))[C(s)+B(s)(D(t)-D(s))],\ \eta \le s \le t,\\ C(t)(1-\alpha B(\eta)) + (\alpha-1)C(t)B(s),\ t \le s \le \eta,\\ C(t)(1-B(s)),\ s \ge max(t,\eta).\end{cases}$$

□

Theorem 4.3. *Assume* $(H_3)-(H_4)$, *then the boundary value problem* $(\mathbf{S_2})$ *has at least one positive continuous solution in* $[0,1]$ *in the case*
(i) $0 \le f^0 \le M$ *and* $m \le f_\infty \le \infty$, *or*
(ii) $0 \le f^\infty \le M$ *and* $m \le f_0 \le \infty$.

Proof. Let g be a function defined on $[0,1]$ by:

$$g(s) := \frac{(1+\alpha)D(1)}{(1-\alpha B(\eta))}B(s)(1-B(s)),\ s\in[0,1]$$

and

$$\gamma = \frac{C(\frac{\eta}{\alpha})}{(1+\alpha)D(1)}min(\alpha-1,1).$$

Then, we may prove that

$$0 \le G(t,s) \le g(s),\ \forall (t,s)\in[0,1]\times[0,1], \tag{13}$$

and

$$G(t,s) \ge \gamma g(s),\ \forall (t,s)\in[\frac{\eta}{\alpha},\eta]\times[0,1]. \tag{14}$$

Finally, let L be the operator defined on $C([0,1])$ by:

$$Lu(t) = \int_0^1 G(t,s)h(s)f(u(s))ds,\ t\in[0,1]. \tag{15}$$

Using Theorem 2.2, we prove that L has at least one positive continuous fixed point which is a desired solution of the problem **(S2)**. □

Remark 4.4. 1. *If $A = 1$, the problem **(S2)** becomes to the third order differential equation studied in [6].*

2. *In [7], the authors take $p(t) = e^{-kt}$ and $q(t) = e^{-\lambda t}$ $(k > \lambda > 0)$ and they prove that the Green function is*

$$G(t,s) = \frac{1}{2k}\begin{cases} e^{-ks}(e^{kt}-e^{-kt}) & for\ \ 0\le t\le s,\\ e^{-kt}(e^{ks}-e^{-ks}) & for\ \ 0\le s\le t.\end{cases} \tag{16}$$

Then, the hypothesis $(H_1)-(H_4)$ are satisfied the operator L defined in (1.3) with $\omega=\infty$ has at least one fixed point in the case
(i) $0\le f^0\le M$ and $m\le f_\infty\le\infty$, or
(ii) $0\le f^\infty\le M$ and $m\le f_0\le\infty$.

Which generalize the result given in [7].

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