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On Generalization of Trapezoid Type Inequalities for s-Convex Functions with Generalized Fractional Integral Operators

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Abstract. By using contemporary theory of inequalities, this study is devoted to propose a number of refinements inequalities for the Hermite−Hadamard's type inequality and conclude explicit bounds for the trapezoid inequalities in terms of s-convex mappings, at most second derivative through the instrument of generalized fractional integral operator and a considerable amount of results for special means. The results of this study which are the generalization of those given in earlier works are obtained for functions f where $|f'|$ and $|f''|$ (or $|f'|^q$ and $|f''|^q$ for $q \geq 1$) are s-convex hold by applying the Hölder inequality and the power mean inequality.

1. Introduction

The Hermite-Hadamard inequality is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. Numerous mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[6], [12, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [2], [3], [5], [7], [9], [11], [14]-[20]) and the references cited therein.

The overall structure of the study takes the form of six sections including introduction. The remaining part of the paper proceeds as follows: In Section 2, the generalised version of fractional integral operator

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is summarised, along with the needed definitions. In section 3, the Hermite-Hadamard type inequalities for s convex functions via generalized fractional integral operators are introduced while in section 4 and 5 trapezoid type inequalities for functions whose first and second derivatives in absolute value are s-convex with generalized fractional integral operators are presented and we also provide some corollaries for theorems. Some conclusions and further directions of research are discussed in Section 6.

2. Definitions and Basic Properties

In this section we will give a brief overview of the basic definitions which will be used in the proof of our main cumulative results.

Definition 2.1. *(s-Convex Functions in The Second Sense) [4] A function $f:[0,\infty)\to[0,\infty)$ is said to be s-convex (in the second sense), or that f belongs to the class K_s^2, if*

$$f(\lambda x+(1-\lambda)y)\leq\lambda^s f(x)+(1-\lambda)^s f(y)$$

for all $x,y\in[0,\infty)$ with $\lambda\in[0,1]$ and for some fixed $s\in(0,1]$.

An s-convex function was introduced in Breckner's paper [4] and a number of properties and connections with s-convexity in the first sense were discussed in paper [8]. Also, we note that, it can be easily seen that for $s=1$, s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

2.1. Generalized Fractional Integral Operators

In addition to this, in [13], Raina defined the following results connected with the general class of fractional integral operators.

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x)=\mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x)=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k \quad (\rho,\lambda>0;|x|<\mathcal{R}), \tag{2}$$

where the coefficients $\sigma(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers and $\mathcal{R}$ is the set of real numbers. With the help of (2), in [13] and [1], Raina and Agarwal et. al defined the following left-sided and right-sided fractional integral operators, respectively, as follows:

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(x)=\int_a^x(x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(x-t)^{\rho}\right]f(t)dt,\ \ x>a, \tag{3}$$

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(x)=\int_x^b(t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-x)^{\rho}\right]f(t)dt,\ \ x<b, \tag{4}$$

where $\lambda,\rho>0,\omega\in\mathbb{R}$, and $f(t)$ is such that the integrals on the right side exists.

It is easy to verify that $\mathcal{J}^{\alpha}_{\rho,\lambda,a+;\omega}f(x)$ and $\mathcal{J}^{\alpha}_{\rho,\lambda,b-;\omega}f(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M}:=\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]<\infty. \tag{5}$$

In fact, for $f\in L(a,b)$, we have

$$\left\|\mathcal{J}^{\alpha}_{\rho,\lambda,a+;\omega}f(x)\right\|_1\leq\mathfrak{M}(b-a)^{\lambda}\left\|f\right\|_1 \tag{6}$$

and

$$\left\|\mathcal{J}^{\alpha}_{\rho,\lambda,b^-;\omega}f(x)\right\|_1\leq\mathfrak{M}(b-a)^{\lambda}\left\|f\right\|_1, \tag{7}$$

where

$$\|f\|_p := \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann-Liouville fractional integrals $I_{a^+}^{\alpha}$ and $I_{b^-}^{\alpha}$ of order α defined by (see, [10])

$$\left(I_{a^+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \quad (x > a; \alpha > 0) \tag{8}$$

and

$$\left(I_{b^-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt \quad (x < b; \alpha > 0) \tag{9}$$

follow easily by setting

$$\lambda = \alpha, \ \sigma(0) = 1, \ \text{and } w = 0 \tag{10}$$

in (3) and (4), and the boundedness of (8) and (9) on $L(a,b)$ is also inherited from (6) and (7), (see, [1]).

In [21], Yaldiz and Sarikaya gave the following useful identity for the generalized fractional integral operators:

Lemma 2.2. *Let $f : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$. If $f' \in L[a,b]$, then we have the following identity for generalized fractional integral operators:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[w(b-a)^{\rho}]} \left[\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma} f(a) + \mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma} f(b) \right] \tag{11}$$

$$= \frac{b-a}{2\mathcal{F}_{\rho,\lambda+1}^{\sigma}[w(b-a)^{\rho}]} \left[\int_0^1 (1-t)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[w(b-a)^{\rho}(1-t)^{\rho}] f'(ta+(1-t)b)\,dt \right.$$

$$\left. - \int_0^1 t^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[w(b-a)^{\rho} t^{\rho}] f'(ta+(1-t)b)\,dt \right].$$

The main concern in this paper is to investigate Hermite-Hadamard's inequalities for functions whose first and second derivatives in absolute value are s-convex with the aid of generalized fractional integral operators and therefore obtains explicit bounds through the use of Hölder and power mean inequalities and the modern theory of inequalities.

3. Hermite-Hadamard Type Inequalities for s-convex functions via Generalized Fractional Integral Operators

In this section, we will present a theorem for Hermite-Hadamard type inequalities with generalized fractional integral operators which is the generalization of previous work. In other words the main result of this section is the following refinement of the classical Hermite-Hadamard inequality for fractional integral operators.

Theorem 3.1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is s-convex function in the second sense on $[a,b]$, then for all $\rho,\lambda,>0$ and $w\in\mathbb{R}$, we have the following inequalities for generalized fractional integral operators:*

$$2^s f\left(\frac{a+b}{2}\right)\leq\frac{1}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)\right] \tag{12}$$

$$\leq\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]}\left[A_1(\lambda,s)+\mathcal{F}^{\sigma_{0,s}}_{\rho,\lambda}\left[\omega(b-a)\right]\right]\left[f(a)+f(b)\right]$$

where $\sigma_{0,s}(k)=\frac{\sigma(k)}{\rho k+s+\lambda}$, $k=0,1,2,\ldots$ and

$$A_1(\lambda,s)=\int_0^1 t^{\lambda-1}(1-t)^s\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]dt.$$

Proof. Since f is s-convex function in the second sense on $[a,b]$, we have for $x,y\in[a,b]$

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2^s}.$$

For $x=ta+(1-t)b$ and $y=(1-t)a+tb$, we obtain

$$2^s f\left(\frac{a+b}{2}\right)\leq f(ta+(1-t)b)+f((1-t)a+tb). \tag{13}$$

Multiplying both sides of (13) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$2^s f\left(\frac{a+b}{2}\right)\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]dt$$

$$\leq\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]f(ta+(1-t)b)\,dt+\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]f((1-t)a+tb)\,dt.$$

For $u=ta+(1-t)b$ and $v=(1-t)a+tb$, we obtain

$$2^s f\left(\frac{a+b}{2}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]$$

$$\leq\frac{1}{b-a}\int_a^b\left(\frac{1}{b-a}(b-u)\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}\left(\frac{1}{b-a}(b-u)\right)^{\rho}\right]f(u)\,du$$

$$+\frac{1}{b-a}\int_a^b\left(\frac{1}{b-a}(v-a)\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}\left(\frac{1}{b-a}(v-a)\right)^{\rho}\right]f(v)\,dv$$

$$=\left(\frac{1}{b-a}\right)^{\lambda}\int_a^b(b-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-u)^{\rho}\right]f(u)\,du+\left(\frac{1}{b-a}\right)^{\lambda}\int_a^b(v-a)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-a)^{\rho}\right]f(v)\,dv$$

$$=\left(\frac{1}{b-a}\right)^{\lambda}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)\right]$$

and the first inequality is proved.

For the proof of the second inequality (12), we first note that if f is s-convex function in the second sense, it yields

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b)$$

and

$$f((1-t)a + tb) \leq (1-t)^s f(a) + t^s f(b).$$

By adding these inequalities together, one has the following inequality:

$$f(ta + (1-t)b) + f((1-t)a + tb) \leq [f(a) + f(b)]\left[t^s + (1-t)^s\right]. \tag{14}$$

Then multiplying both sides of (14) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho} t^{\rho}\right]$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho} t^{\rho}\right] f(ta + (1-t)b)\,dt + \int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho} t^{\rho}\right] f((1-t)a + tb)\,dt \\
&\leq [f(a) + f(b)]\int_0^1 \left[t^s + (1-t)^s\right] t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho} t^{\rho}\right] dt \\
&= \left[A_1(\lambda,s) + \mathcal{F}^{\sigma_{0,s}}_{\rho,\lambda}\left[\omega(b-a)\right]\right][f(a) + f(b)].
\end{aligned}$$

That is,

$$\left(\frac{1}{b-a}\right)^{\lambda}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} f(b) + \mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega} f(a)\right] \leq \left[A_1(\lambda,s) + \mathcal{F}^{\sigma_{0,s}}_{\rho,\lambda}\left[\omega(b-a)\right]\right][f(a) + f(b)].$$

Hence, the proof is completed. □

Remark 3.2. *If we choose $s = 1$ in Theorem 3.1, then for all $\rho, \lambda, > 0$ and $w \in \mathbb{R}$, we have the following inequality*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} f(b) + \mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega} f(a)\right] \\
&\leq \frac{f(a) + f(b)}{2}
\end{aligned}$$

which was given by Yaldız and Sarıkaya in [21].

Remark 3.3. *If we choose $\lambda = \alpha$, $\sigma(0) = 1$, $w = 0$ in Theorem 3.1, then Theorem 3.1 reduces the Theorem 3 proved by Set et al. in [18].*

Remark 3.4. *If we choose $\lambda = \alpha$, $\sigma(0) = 1$, $w = 0$ and $s = 1$ in Theorem 3.1, then Theorem 3.1 reduces the Theorem 2 proved by Sarıkaya et al. in [15].*

4. Trapezoid Type Inequalities for Differentiable Functions with Generalized Fractional Integral Operators

In this section we will present some refinements of the classical trapezoid type inequalities for function whose first derivative in absolute value is s-convex via generalized fractional integral operators.

Theorem 4.1. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $|f'|$ is s-convex function in the second sense, then for all $\rho,\lambda,>0$ and $w\geq 0$, we have the following inequality for generalized fractional integral operators:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right| \leq \frac{b-a}{2}\frac{\mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+1}[w(b-a)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[|f'(a)|+|f'(b)|\right]$$

where

$$\sigma_{1,s}(k)=\sigma(k)\left[\beta\left(\frac{1}{2};s+1,\rho k+\lambda+1\right)-\beta\left(\frac{1}{2};\rho k+\lambda+1,s+1\right)+\frac{2^{\rho k+\lambda+s}-1}{(\rho k+\lambda+s+1)2^{\rho k+\lambda+s}}\right]$$

for $k=0,1,2,....$

Proof. Using Lemma 2.2 and generalized triangle inequality we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right| \\ &= \frac{b-a}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left|\sum_{k=0}^{\infty}\frac{\sigma(k)w^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\int_0^1\left[(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right]f'(ta+(1-t)b)\,dt\right| \\ &\leq \frac{b-a}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\sum_{k=0}^{\infty}\frac{\sigma(k)w^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|\left|f'(ta+(1-t)b)\right|dt\end{aligned} \tag{15}$$

Then, using the s-convexity of $|f'|$ we find that

$$\begin{aligned}&\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|\left|f'(ta+(1-t)b)\right|dt \\ &= \int_0^{\frac{1}{2}}\left[(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right]\left|f'(ta+(1-t)b)\right| + \int_{\frac{1}{2}}^1\left[t^{\rho k+\lambda}-(1-t)^{\rho k+\lambda}\right]\left|f'(ta+(1-t)b)\right|dt \\ &\leq \int_0^{\frac{1}{2}}\left[(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right]\left[t^{s}|f'(a)|+(1-t)^{s}|f'(b)|\right]dt \\ &\quad+\int_{\frac{1}{2}}^1\left[t^{\rho k+\lambda}-(1-t)^{\rho k+\lambda}\right]\left[t^{s}|f'(a)|+(1-t)^{s}|f'(b)|\right]dt \\ &= \left[\beta\left(\frac{1}{2};s+1,\rho k+\lambda+1\right)-\beta\left(\frac{1}{2};\rho k+\lambda+1,s+1\right)+\frac{2^{\rho k+\lambda+s}-1}{(\rho k+\lambda+s+1)2^{\rho k+\lambda+s}}\right]\left[|f'(a)|+|f'(b)|\right].\end{aligned} \tag{16}$$

By subsituting the inequality (16) into (15), we find that

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right| \\
&\leq \frac{b-a}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \\
&\quad\times\sum_{k=0}^{\infty}\frac{\sigma(k)w^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left[\beta\left(\frac{1}{2};s+1,\rho k+\lambda+1\right)-\beta\left(\frac{1}{2};\rho k+\lambda+1,s+1\right)+\frac{2^{\rho k+\lambda+s}-1}{(\rho k+\lambda+s+1)2^{\rho k+\lambda+s}}\right] \\
&=\frac{b-a}{2}\frac{\mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]
\end{aligned}$$

which completes the proof. □

Remark 4.2. *If we choose $s=1$ in Theorem 4.1, then for all $\rho,\lambda,>0$ and $w\geq 0$, we have the following inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right| \\
&\leq\frac{b-a}{2}\frac{\mathcal{F}^{\sigma_{1,1}}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]
\end{aligned}$$

where

$$\sigma_{1,1}(k)=\frac{\sigma(k)}{\rho k+\lambda+1}\left(1-\frac{1}{2^{\rho k+\lambda}}\right),$$

which is the same result given by Yaldiz and Sarikaya in [21].

Corollary 4.3. *If we choose $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ in Theorem 4.1, then we have the following inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(I^{\alpha}_{b-}f\right)(a)+\left(I^{\alpha}_{a+}f\right)(b)\right]\right| \\
&\leq\frac{b-a}{2}\left[\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\beta\left(\frac{1}{2};\alpha+1,s+1\right)+\frac{2^{\alpha+s}-1}{(\alpha+s+1)2^{\alpha+s}}\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right].
\end{aligned}$$

Remark 4.4. *If we choose $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ and $s=1$ in Theorem 4.1, then Theorem 4.1 reduces the Theorem 3 proved by Sarikaya et al. in [15].*

Theorem 4.5. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $\left|f'\right|^{q}$, $q>1$, is s-convex function in the second sense, then for all $\rho,\lambda,>0$ and $w\geq 0$, we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right| \\
&\leq\frac{b-a}{2}\frac{\mathcal{F}^{\sigma_{2}}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{s+1}\right]^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\sigma_2(k)=\sigma(k)\left(\frac{2}{p\left(\rho k+\lambda\right)+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{p\left(\rho k+\lambda\right)}}\right)^{\frac{1}{p}}$$

for $k=0,1,2,....$

Proof. Using the well known Hölder inequality, we obtain

$$\begin{aligned}
&\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|\left|f'\left(ta+(1-t)b\right)\right|dt \qquad (17)\\
&\le\left(\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\le\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right]^p dt+\int_{\frac{1}{2}}^1\left[t^{\rho k+\lambda}-(1-t)^{\rho k+\lambda}\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Now using the fact that $(A-B)^p\le A^p-B^p$, for any $A>B\ge 0$ and $p\ge 1$, we find that

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left[(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right]^p dt+\int_{\frac{1}{2}}^1\left[t^{\rho k+\lambda}-(1-t)^{\rho k+\lambda}\right]^p dt \qquad (18)\\
&\le\int_0^{\frac{1}{2}}\left[(1-t)^{p\left(\rho k+\lambda\right)}-t^{p\left(\rho k+\lambda\right)}\right]dt+\int_{\frac{1}{2}}^1\left[t^{p\left(\rho k+\lambda\right)}-(1-t)^{p\left(\rho k+\lambda\right)}\right]dt\\
&=\frac{2}{p\left(\rho k+\lambda\right)+1}\left[1-\frac{1}{2^{p\left(\rho k+\lambda\right)}}\right].
\end{aligned}$$

Since $\left|f'\right|^q$, $q>1$, is s-convex function in the second sense, we have

$$\int_0^1\left|f'\left(ta+(1-t)b\right)\right|^q dt\le\int_0^1\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt=\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{s+1} \qquad (19)$$

If we put the inequality (18) and (19) in (17), we get

$$\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|\left|f'\left(ta+(1-t)b\right)\right|dt\le\left(\frac{2}{p\left(\rho k+\lambda\right)+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{p\left(\rho k+\lambda\right)}}\right)^{\frac{1}{p}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}. \qquad (20)$$

By substituting the inequality (20) into (15), we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|\\&\leq\frac{b-a}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\frac{|f'(a)|^{q}+|f'(b)|^{q}}{s+1}\right]^{\frac{1}{q}}\sum_{k=0}^{\infty}\frac{\sigma(k)w^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left(\frac{2}{p(\rho k+\lambda)+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{p(\rho k+\lambda)}}\right)^{\frac{1}{p}}\\&=\frac{b-a}{2}\frac{\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[w(b-a)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\frac{|f'(a)|^{q}+|f'(b)|^{q}}{s+1}\right]^{\frac{1}{q}}.\end{aligned}$$

Thus, the proof is completed. □

Corollary 4.6. *If we choose $s=1$ in Theorem 4.5, then for all $\rho,\lambda,>0$ and $w\geq 0$, we have the following inequality*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|\\&\leq\frac{b-a}{2}\frac{\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[w(b-a)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right]^{\frac{1}{q}}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 4.7. *If we choose $\lambda=\alpha$, $\sigma(0)=1$, $w=0$ in Theorem 4.5, then we have the following inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(I^{\alpha}_{b^-}f\right)(a)+\left(I^{\alpha}_{a^+}f\right)(b)\right]\right|\\&\leq\frac{b-a}{2}\left(\frac{2}{p\alpha+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^{q}+|f'(b)|^{q}}{s+1}\right]^{\frac{1}{q}}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 4.8. *Choosing $s=1$ in Corollary 4.7, we have the the following inequality*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(I^{\alpha}_{b^-}f\right)(a)+\left(I^{\alpha}_{a^+}f\right)(b)\right]\right|\\&\leq\frac{b-a}{2}\left(\frac{2}{p\alpha+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{p\alpha}}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right]^{\frac{1}{q}}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 4.9. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $|f'|^{q}$, $q\geq 1$, is s-convex function in the second sense, then for all $\rho,\lambda,>0$ and $w\geq 0$, we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|\\&\leq\frac{b-a}{2}\frac{\mathcal{F}^{\sigma_{3,s}}_{\rho,\lambda+1}[w(b-a)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left(\left[|f'(a)|^{q}+|f'(b)|^{q}\right]\right)^{\frac{1}{q}}\end{aligned}$$

where

$$\sigma_{3,s}(k) = \sigma(k)\left(\frac{2}{\rho k+\lambda+1}\left[1-\frac{1}{2^{\rho k+\lambda}}\right]\right)^{1-\frac{1}{q}}$$
$$\times\left(\left[\beta\left(\frac{1}{2};s+1,\rho k+\lambda+1\right)-\beta\left(\frac{1}{2};\rho k+\lambda+1,s+1\right)+\frac{2^{\rho k+\lambda+s}-1}{(\rho k+\lambda+s+1)2^{\rho k+\lambda+s}}\right]\right)^{\frac{1}{q}}$$

for $k=0,1,2,...$ and $\beta(z;x,y)$ denotes incomplete Beta function.

Proof. Using the well known power mean inequality, we obtain

$$\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|\left|f'(ta+(1-t)b)\right|dt \tag{21}$$
$$\le\left(\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$\le\left(\frac{2}{\rho k+\lambda+1}\left[1-\frac{1}{2^{\rho k+\lambda}}\right]\right)^{1-\frac{1}{q}}\left(\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

As $|f'|^q$, $q\ge1$, is s-convex function in the second sense, we have

$$\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|\left|f'(ta+(1-t)b)\right|^q dt \tag{22}$$
$$\le\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt$$
$$=\left[\beta\left(\frac{1}{2};s+1,\rho k+\lambda+1\right)-\beta\left(\frac{1}{2};\rho k+\lambda+1,s+1\right)+\frac{2^{\rho k+\lambda+s}-1}{(\rho k+\lambda+s+1)2^{\rho k+\lambda+s}}\right]\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right].$$

By substituting inequalities (21) and (22) into (15), we find that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|$$
$$\le\frac{b-a}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left(\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]\right)^{\frac{1}{q}}$$
$$\times\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left(\frac{2}{\rho k+\lambda+1}\left[1-\frac{1}{2^{\rho k+\lambda}}\right]\right)^{1-\frac{1}{q}}$$
$$\times\left(\left[\beta\left(\frac{1}{2};s+1,\rho k+\lambda+1\right)-\beta\left(\frac{1}{2};\rho k+\lambda+1,s+1\right)-\frac{2^{\rho k+\lambda+s}-1}{(\rho k+\lambda+s+1)2^{\rho k+\lambda+s}}\right]\right)^{\frac{1}{q}}$$
$$=\frac{b-a}{2}\frac{\mathcal{F}^{\sigma_{3,s}}_{\rho,\lambda+1}[w(b-a)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left(\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]\right)^{\frac{1}{q}}$$

which completes the proof. □

Corollary 4.10. *If we choose $s = 1$ in Theorem 4.9, then for all $\rho, \lambda, > 0$ and $w \geq 0$, we have the following inequality*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a) + \mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right| \\ &\leq \frac{b-a}{2}\frac{\mathcal{F}^{\sigma_{3,1}}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left(\left[\left|f'(a)\right|^{q} + \left|f'(b)\right|^{q}\right]\right)^{\frac{1}{q}} \end{aligned}$$

where

$$\sigma_{3,1}(k) = \sigma(k)2^{1-\frac{1}{q}}\left(\frac{1}{\rho k+\lambda+1}\left(1-\frac{1}{2^{\rho k+\lambda}}\right)\right)$$

for $k = 0, 1, 2, ...$

Remark 4.11. *If we choose $\lambda = \alpha$, $\sigma(0) = 1$, $w = 0$ in Theorem 4.9, then Theorem 4.9 reduce to Theorem 4 proved by Set et al. in [18].*

Corollary 4.12. *Choosing $\lambda = \alpha$, $\sigma(0) = 1$, $w = 0$ in Corollary 4.10, we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(I^{\alpha}_{b^-}f\right)(a) + \left(I^{\alpha}_{a^+}f\right)(b)\right]\right| \leq \frac{b-a}{2^{\frac{1}{q}}}\left(\frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right)\left(\left[\left|f'(a)\right|^{q} + \left|f'(b)\right|^{q}\right]\right)^{\frac{1}{q}}.$$

5. Trapezoid Type Inequalities for Twice Differentiable Functions with Generalized Fractional Integral Operators

Now, similarly, we will present some refinements of the classical trapezoid type inequalities for function whose second derivative in absolute value is s convex via generalized fractional integral operators.

Lemma 5.1. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable function on (a, b) with $a < b$. If $f'' \in L[a, b]$, then for all $\rho, \lambda, > 0$ and $w \geq 0$, we have the following identity for generalized fractional integral operators:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a) + \mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right] \qquad (23) \\ &= \frac{(b-a)^2}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\int_0^1 \mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right]f''(ta+(1-t)b)\,dt\right. \\ &-\int_0^1 (1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}(1-t)^{\rho}\right]f''(ta+(1-t)b)\,dt \\ &\left.-\int_0^1 t^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}t^{\rho}\right]f''(ta+(1-t)b)\,dt\right], \end{aligned}$$

Proof. We have

$$\begin{aligned} K &= \int_0^1 \mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right] f''(ta+(1-t)b)\,dt \\ &\quad - \int_0^1 (1-t)^{\lambda+1}\,\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}(1-t)^{\rho}\right] f''(ta+(1-t)b)\,dt \\ &\quad - \int_0^1 t^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}t^{\rho}\right] f''(ta+(1-t)b)\,dt \\ &= K_1 - K_2 - K_3. \end{aligned} \tag{24}$$

Integrating K_1 as:

$$K_1 = \mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right] \int_0^1 f''(ta+(1-t)b)\,dt = \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right]}{b-a}\left[f'(b)-f'(a)\right]. \tag{25}$$

Using integration by parts twice, we have

$$\begin{aligned} K_2 &= \int_0^1 (1-t)^{\lambda+1}\,\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}(1-t)^{\rho}\right] f''(ta+(1-t)b)\,dt \\ &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right]}{b-a} f'(b) - \int_0^1 (1-t)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}(1-t)^{\rho}\right] f'(ta+(1-t)b)\,dt \\ &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right]}{b-a} f'(b) - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{(b-a)^2} f(b) \\ &\quad + \frac{1}{(b-a)^2}\int_0^1 (1-t)^{\lambda-1}\,\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}(1-t)^{\rho}\right] f(ta+(1-t)b)\,dt \\ &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right]}{b-a} f'(b) - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{(b-a)^2} f(b) \\ &\quad + \frac{1}{(b-a)^{\lambda+2}}\int_a^b (x-a)^{\lambda-1}\,\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-a)^{\rho}\right] f(x)\,dt, \end{aligned} \tag{26}$$

and similarly

$$\begin{aligned} K_3 &= \int_0^1 t^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}t^{\rho}\right] f''(ta+(1-t)b)\,dt \\ &= -\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right]}{b-a} f'(a) - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{(b-a)^2} f(a) \\ &\quad + \frac{1}{(b-a)^{\lambda+2}}\int_a^b (b-x)^{\lambda-1}\,\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-x)^{\rho}\right] f(x)\,dt. \end{aligned} \tag{27}$$

If we put the equalities (25), (26) and (27) in (24), then we obtain

$$\begin{aligned} K &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{(b-a)^{2}}\left[f(a)+f(b)\right] \\ &\quad -\frac{1}{(b-a)^{\lambda+2}}\int_{a}^{b}(x-a)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-a)^{\rho}\right]f(x)\,dt \\ &\quad -\frac{1}{(b-a)^{\lambda+2}}\int_{a}^{b}(b-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-x)^{\rho}\right]f(x)\,dt \\ &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}{(b-a)^{2}}\left[f(a)+f(b)\right]-\frac{1}{(b-a)^{\lambda+2}}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]. \end{aligned} \tag{28}$$

Mutiplying both sides of (28) by $\frac{(b-a)^2}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}$, we can obtain desired identity (23). □

Theorem 5.2. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable function on (a,b) with $a<b$. If $\left|f''\right|$ is s-convex function in the second sense, then for all $\rho,\lambda,>0$ and $w\geq 0$, we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right| \\ &\leq \frac{(b-a)^{2}}{2}\frac{\mathcal{F}^{\sigma_{4,s}}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\left|f''(a)\right|+\left|f''(b)\right|\right] \end{aligned}$$

where

$$\sigma_{4,s}(k)=\sigma(k)\left[\frac{\rho k+\lambda+1}{(s+1)(\rho k+\lambda+s+2)}-\beta(s+1,\rho k+\lambda+2)\right]$$

for $k=0,1,2,...$ and $\beta(x,y)$ is the Beta function defined by

$$\beta(x,y)=\int_{0}^{1}t^{x-1}(1-t)^{y-1}dt.$$

Proof. Taking modulus both sides of (23) and using the generalized triangle inequality, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right| \\ &= \frac{(b-a)^{2}}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left|\sum_{k=0}^{\infty}\frac{\sigma(k)w^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+2)}\int_{0}^{1}\left[1-(1-t)^{\rho k+\lambda+1}-t^{\rho k+\lambda+1}\right]f''(ta+(1-t)b)\,dt\right| \\ &\leq \frac{(b-a)^{2}}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)w^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+2)}\int_{0}^{1}\left[1-(1-t)^{\rho k+\lambda+1}-t^{\rho k+\lambda+1}\right]\left|f''(ta+(1-t)b)\right|dt \end{aligned} \tag{29}$$

because $1-(1-t)^{\rho k+\lambda+1}-t^{\rho k+\lambda+1}\geq 0$ for all $t\in[0,1]$.

Since $\left|f''\right|$ is s-convex function in the second sense, we obtain

$$\int_0^1 \left[1-(1-t)^{\rho k+\lambda+1}-t^{\rho k+\lambda+1}\right]\left|f''(ta+(1-t)b)\right|dt \tag{30}$$
$$\leq \int_0^1 \left[1-(1-t)^{\rho k+\lambda+1}-t^{\rho k+\lambda+1}\right]\left[t^s\left|f''(a)\right|+(1-t)^s\left|f''(b)\right|\right]dt$$
$$=\left[\frac{\rho k+\lambda+1}{(s+1)(\rho k+\lambda+s+2)}-\beta(s+1,\rho k+\lambda+2)\right]\left[\left|f''(a)\right|+\left|f''(b)\right|\right]$$

where $\beta(x,y)$ is the Beta function. Using the inequality (30) in (29), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]$$
$$\times\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+2)}\left[\frac{\rho k+\lambda+1}{(s+1)(\rho k+\lambda+s+2)}-\beta(s+1,\rho k+\lambda+2)\right]$$
$$=\frac{(b-a)^2}{2}\frac{\mathcal{F}^{\sigma_{4,s}}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]$$

which completes the proof. □

Corollary 5.3. *If we choose $s=1$ in Theorem 5.2, then for all $\rho,\lambda,>0$ and $w\geq 0$, we have the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{4}\frac{\mathcal{F}^{\sigma 5}_{\rho,\lambda+3}\left[w(b-a)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]$$

where $\sigma_5(k)=(\rho k+\lambda)\sigma(k)$, $k=0,1,2,\ldots$.

Corollary 5.4. *If we take $\lambda=\alpha$, $\sigma(0)=1$, $w=0$ in Theorem 5.2, then we have the following inequality for Riemann-Liouville fractional integral*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(I^{\alpha}_{b^-}f\right)(a)+\left(I^{\alpha}_{a^+}f\right)(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{2}\left[\frac{1}{(s+1)(\alpha+s+2)}-\frac{\beta(s+1,\alpha+2)}{\alpha+1}\right]\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Corollary 5.5. *Choosing $s=1$ in Corollary 5.4, we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(I^{\alpha}_{b^-}f\right)(a)+\left(I^{\alpha}_{a^+}f\right)(b)\right]\right|\leq\frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right].$$

Theorem 5.6. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a < b$. If $\left|f''\right|^q$, $q > 1$, is s-convex function in the second sense, then for all $\rho, \lambda, > 0$ and $w \geq 0$, we have the following inequality for generalized fractional integral operators:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a) + \mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{2}\frac{\mathcal{F}^{\sigma_6}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left(\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\sigma_6(k) = \sigma(k)\left(1 - \frac{2}{p(\rho k + \lambda + 1) + 1}\right)^{\frac{1}{p}}.$$

for $k = 0, 1, 2, ...$

Proof. Using the well known Hölder inequality, we have

$$\int_0^1 \left[1-(1-t)^{\rho k+\lambda+1} - t^{\rho k+\lambda+1}\right]\left|f''(ta+(1-t)b)\right|dt$$
$$\leq \left(\int_0^1 \left[1-(1-t)^{\rho k+\lambda+1} - t^{\rho k+\lambda+1}\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Using the fact that

$$(A-B)^p \leq A^p - B^p$$

for any $A > B \geq 0$ and $p \geq 1$, we get

$$\left[1-(1-t)^{\rho k+\lambda+1} - t^{\rho k+\lambda+1}\right]^p \leq 1-(1-t)^{p(\rho k+\lambda+1)} - t^{p(\rho k+\lambda+1)} \tag{31}$$

for any $t \in [0,1]$. Using the inequality (31) and s-convexity of $\left|f''\right|^q$, we obtain

$$\int_0^1 \left[1-(1-t)^{\rho k+\lambda+1} - t^{\rho k+\lambda+1}\right]\left|f''(ta+(1-t)b)\right|dt \tag{32}$$
$$\leq \left(\int_0^1 \left[1-(1-t)^{p(\rho k+\lambda+1)} - t^{p(\rho k+\lambda+1)}\right]dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[t^s\left|f''(a)\right|^q + (1-t)^s\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$= \left(1 - \frac{2}{p(\rho k + \lambda + 1) + 1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{s+1}\right)^{\frac{1}{q}}.$$

By substituting inequality (32) into (29), we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|\\ &\leq\frac{(b-a)^{2}}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left(\frac{\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\sum_{k=0}^{\infty}\frac{\sigma(k)w^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+2)}\left(1-\frac{2}{p(\rho k+\lambda+1)+1}\right)^{\frac{1}{p}}\\ &=\frac{(b-a)^{2}}{2}\frac{\mathcal{F}^{\sigma_{6}}_{\rho,\lambda+2}[w(b-a)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left(\frac{\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\end{aligned}$$

where $\sigma_6(k)$, $k = 0, 1, 2, ...$ are defined as Theorem 5.6. The proof is completed. □

Corollary 5.7. *If we choose $s = 1$ in Theorem 5.6, then for all $\rho, \lambda, > 0$ and $w \geq 0$, we have the following inequality*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|\\ &\leq\frac{(b-a)^{2}}{2}\frac{\mathcal{F}^{\sigma_{6}}_{\rho,\lambda+2}[w(b-a)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left(\frac{\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}}{2}\right)^{\frac{1}{q}}\end{aligned}$$

where $\sigma_6(k)$, $k = 0, 1, 2, ...$ are defined as Theorem 5.6.

Corollary 5.8. *If we take $\lambda = \alpha$, $\sigma(0) = 1$, $w = 0$ in Theorem 5.6, then we have the following inequality for Riemann-Liouville fractional integral*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(I^{\alpha}_{b^-}f\right)(a)+\left(I^{\alpha}_{a^+}f\right)(b)\right]\right|\\ &\leq\frac{(b-a)^{2}}{2(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}}{s+1}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 5.9. *Choosing $s = 1$ in Corollary 5.8, we obtain*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(I^{\alpha}_{b^-}f\right)(a)+\left(I^{\alpha}_{a^+}f\right)(b)\right]\right|\\ &\leq\frac{(b-a)^{2}}{2(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 5.10. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable function on (a, b) with $a < b$. If $\left|f''\right|^q$, $q \geq 1$, is s-convex function in the second sense, then for all $\rho, \lambda, > 0$ and $w \geq 0$, we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|\\ &\leq\frac{(b-a)^{2}}{2}\frac{\mathcal{F}^{\sigma_{7,s}}_{\rho,\lambda+2}[w(b-a)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}\end{aligned}$$

where

$$\sigma_{7,s}(k)=\sigma(k)\left(1-\frac{2}{\rho k+\lambda+2}\right)^{1-\frac{1}{q}}\left[\frac{\rho k+\lambda+1}{(s+1)(\rho k+\lambda+s+2)}-\beta(s+1,\rho k+\lambda+2)\right]^{\frac{1}{q}}$$

for $k=0,1,2,....$

Proof. Using the well known power mean inequality and s-convexity of $\left|f''\right|^q$ we have

$$\int_0^1\left[1-(1-t)^{\rho k+\lambda+1}-t^{\rho k+\lambda+1}\right]\left|f''(ta+(1-t)b)\right|dt \tag{33}$$

$$\leq\left(\int_0^1\left[1-(1-t)^{\rho k+\lambda+1}-t^{\rho k+\lambda+1}\right]dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[1-(1-t)^{\rho k+\lambda+1}-t^{\rho k+\lambda+1}\right]\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\left(1-\frac{2}{\rho k+\lambda+2}\right)^{1-\frac{1}{q}}\left(\int_0^1\left[1-(1-t)^{\rho k+\lambda+1}-t^{\rho k+\lambda+1}\right]\left[t^s\left|f''(a)\right|^q+(1-t)^s\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\left(1-\frac{2}{\rho k+\lambda+2}\right)^{1-\frac{1}{q}}\left[\frac{\rho k+\lambda+1}{(s+1)(\rho k+\lambda+s+2)}-\beta(s+1,\rho k+\lambda+2)\right]^{\frac{1}{q}}\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}$$

where $\beta(x,y)$ is the Beta function. By substituting inequality (33) into (29), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^\lambda\mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho]}\left[\mathcal{J}^\sigma_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^\sigma_{\rho,\lambda,a+;\omega}f(b)\right]\right|$$

$$\leq\frac{(b-a)^2}{2\mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho]}\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}$$

$$\times\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+2)}\left(1-\frac{2}{\rho k+\lambda+2}\right)^{1-\frac{1}{q}}\left[\frac{\rho k+\lambda+1}{(s+1)(\rho k+\lambda+s+2)}-\beta(s+1,\rho k+\lambda+2)\right]^{\frac{1}{q}}$$

$$=\frac{(b-a)^2}{2}\frac{\mathcal{F}^{\sigma_{7,s}}_{\rho,\lambda+2}[w(b-a)^\rho]}{\mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho]}\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}$$

which completes the proof. □

Corollary 5.11. *If we choose* $s=1$ *in Theorem 5.10, then for all* $\rho,\lambda,>0$ *and* $w\geq 0$, *we have the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^\lambda\mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho]}\left[\mathcal{J}^\sigma_{\rho,\lambda,b-;\omega}f(a)+\mathcal{J}^\sigma_{\rho,\lambda,a+;\omega}f(b)\right]\right|$$

$$\leq\frac{(b-a)^2}{2}\frac{\mathcal{F}^{\sigma_{7,1}}_{\rho,\lambda+2}[w(b-a)^\rho]}{\mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho]}\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}$$

where

$$\sigma_{7,1}(k)=\sigma(k)\left(1-\frac{2}{\rho k+\lambda+2}\right)^{1-\frac{1}{q}}\left[\frac{1}{\rho k+\lambda+3}\left(\frac{\rho k+\lambda+1}{2}-\frac{1}{\rho k+\lambda+2}\right)\right]^{\frac{1}{q}}$$

for $k=0,1,2,....$

Corollary 5.12. *If we take $\lambda = \alpha$, $\sigma(0) = 1$, $w = 0$ in Theorem 5.10, then we have the following inequality for Riemann-Liouville fractional integral*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(I_{b^-}^{\alpha} f\right)(a) + \left(I_{a^+}^{\alpha} f\right)(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\frac{\alpha+1}{(s+1)(\alpha+s+2)} - \beta(s+1,\alpha+2)\right]^{\frac{1}{q}}\left[\left|f''(a)\right|^q + \left|f''(b)\right|^q\right]^{\frac{1}{q}}.$$

Corollary 5.13. *Choosing $s = 1$ in Corollary 5.12, we obtain*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\left(I_{b^-}^{\alpha} f\right)(a) + \left(I_{a^+}^{\alpha} f\right)(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\frac{1}{\alpha+3}\left(\frac{\alpha+1}{2} - \frac{1}{\alpha+2}\right)\right]^{\frac{1}{q}}\left[\left|f''(a)\right|^q + \left|f''(b)\right|^q\right]^{\frac{1}{q}}.$$

6. Concluding Remarks

In this paper, we established the Hermite-Hadamard and trapezoid type inequalities for mappings whose first and second derivatives in absolute value are s-convex and related results to present new type inequalities involving generalized fractional integral operator. The results presented in this paper would provide generalizations of those given in earlier works. The findings of this study have several significant implications for future applications.

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