Filomat 32:2 (2018), 589–598
https://doi.org/10.2298/FIL1802589L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Forward Order Laws for $\{1,2,3\}$- and $\{1,2,4\}$-inverses of a Three Matrix Products

Zhongshan Liu[a], Zhiping Xiong[a]

[a] *School of Mathematics and Computational Science, Wuyi University, Jiangmen 529020, P.R.China*

Abstract. In this article, we study the forward order laws for $\{1,2,3\}$- and $\{1,2,4\}$-inverses of a product of three matrices by using the maximal and minimal ranks of the generalized Schur complement. The necessary and sufficient conditions for $A_1\{1,2,3\}A_2\{1,2,3\}A_3\{1,2,3\} \subseteq (A_1A_2A_3)\{1,2,3\}$ and $A_1\{1,2,4\}A_2\{1,2,4\}A_3\{1,2,4\} \subseteq (A_1A_2A_3)\{1,2,4\}$ are presented.

1. Introduction

Throughout this paper $\mathbb{C}^{m\times n}$ and $\mathbb{C}^m$ denote the set of $m \times n$ complex matrices and m-dimensional complex vectors, respectively. The identity matrix in $\mathbb{C}^{n\times n}$ is denoted by I_n and $O_{m\times n}$ is the $m \times n$ matrix of all zero entries (if no confusion occurs, we will drop the subscript). For any matrix $A \in \mathbb{C}^{m\times n}$, let $r(A)$, A^*, $R(A)$ and $N(A)$ denote the rank, the conjugate transpose, the range space (or column space) and the null space of A, respectively.

The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ denoted by $A^\dagger$, is the unique element $X \in \mathbb{C}^{n\times m}$ which satisfies the following four Penrose equations [6] :

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA. \tag{1.1}$$

Let $\emptyset \neq \zeta \subseteq \{1,2,3,4\}$. Then A_ζ denotes the set of all matrices X which satisfy (i) for $i \in \zeta$. Any $X \in A_\zeta$ is called an ζ-inverse of A. As usually, X is called a $\{1,3\}$-inverse or a least squares g-inverse of A if it is an element of $A\{1,3\}$ and X is called a $\{1,4\}$-inverse or a minimum norm g-inverse of A if it is an element of $A\{1,4\}$. similarly, X is called a $\{1,2,3\}$-inverse of A if it is an element of $A\{1,2,3\}$ and X is called a $\{1,2,4\}$-inverse of A if it is an element of $A\{1,2,4\}$. The unique $\{1,2,3,4\}$-inverse of A is called the Moore-Penrose inverse of A. We refer the reader to [1, 13] for basic results on the generalized inverses.

Theory and computations of the reverse order laws for generalized inverses of matrix product are important subjects in many branches of applied science, such as non-linear control theory, matrix theory, matrix algebra, see [6, 8, 9, 13]. One of the core problems in reverse order laws is to find the necessary

2010 *Mathematics Subject Classification*. 47A05; 15A09; 15A24
Keywords. Forward order law, Generalized inverse, Maximal and minimal ranks, Matrix product, Generalized Schur complement.
Received: 04 February 2017; Revised: 06 October 2017; Accepted: 30 October 2017
Communicated by Dragana Cvetković Ilić
Corresponding author: Zhiping Xiong
This work was supported by the NSFC (Grant No:11571004, 11771159) and the Guangdong Natural Science Fund of China (Grant No: 2014A030313625) and the Training plan for the Outstanding Young Teachers in Higher Education of Guangdong (Grant No: SYq2014002) and the Student Innovation Training Program of Guangdong province, P.R.China (No. 201511349071).
Email address: xzpwhere@163.com (Zhiping Xiong)

and sufficient conditions for the reverse order laws for the generalized inverse of matrix product and it has attracted considerable attention, see [1, 2, 7, 9, 18, 19].

In 1996, Grevill [3] first gave a necessary and sufficient condition for the reverse order law $(AB)^\dagger = B^\dagger A^\dagger$. Since then, more equivalent conditions for the reverse order laws for generalized inverses of matrix product have been derived. Hartwig [4] and Tian [10, 11] studied the reverse order laws for Moore-Penrose inverse of three and multiple matrix product, respectively. Using the Product Singular Value Decomposition (PSVD), Wei [14] and De Pierro and Wei [2], studied necessary and sufficient conditions for $B\{1\}A\{1\} \subseteq (AB)\{1\}$ and $B\{1,2\}A\{1,2\} \subseteq (AB)\{1,2\}$. With the same method, Wei [2, 15], Wei and Guo [16] studied the equivalent conditions for the reverse order laws of $\{1\}$-inverses, $\{1,2\}$-inverses, $\{1,3\}$-inverses and $\{1,4\}$-inverses of multiple matrix products. During the recent years, Zheng and Xiong [18, 19] studied the reverse order laws for $\{1,2,3\}$-inverses and $\{1,2,4\}$-inverses of multiple products. For other interesting results on this subject see [1, 2, 8, 9, 14].

In 2007, Xiong and Zheng [17] studied the forward order law for the generalized inverse of multiple matrix products, by using the maximal rank of generalized Schur complement. With the same thread, in this paper we obtain the necessary and sufficient conditions for one side inclusion

$$A_1\{1,2,3\}A_2\{1,2,3\}A_3\{1,2,3\} \subseteq (A_1A_2A_3)\{1,2,3\} \tag{1.2}$$

and

$$A_1\{1,2,4\}A_2\{1,2,4\}A_3\{1,2,4\} \subseteq (A_1A_2A_3)\{1,2,4\}. \tag{1.3}$$

To our knowledge, there is no article yet discussing the forward order laws for these two generalized inverses in the literature.

The main tools of the later discussion are the following lemmas.

Lemma 1.1 *[12] Let $A \in \mathbb{C}^{m\times n}$,$B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$. Then*

$$\max_{A^{(1,2,3)}} r(D - CA^{(1,2,3)}B) = \min\left\{ r\begin{pmatrix} A^*A & A^*B \\ C & D \end{pmatrix} - r(A),\ r\begin{pmatrix} A^*B \\ D \end{pmatrix} \right\}, \tag{1.4}$$

$$\min_{A^{(1,2,3)}} r(D - CA^{(1,2,3)}B) = r\begin{pmatrix} A^*A & A^*B \\ C & D \end{pmatrix} + r\begin{pmatrix} A^*B \\ D \end{pmatrix} - r\begin{pmatrix} A & O \\ O & A^*B \\ C & D \end{pmatrix}. \tag{1.5}$$

Lemma 1.2 *[1] Let $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$, then*

$$X \in A\{1,2,3\} \Longleftrightarrow A^*AX = A^* \quad and \quad r(X) = r(A), \tag{1.6}$$

$$X \in A\{1,2,4\} \Longleftrightarrow XAA^* = A^* \quad and \quad r(X) = r(A). \tag{1.7}$$

Lemma 1.3 *[5] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{p\times n}$, then*

$$r\begin{pmatrix} A, & B \end{pmatrix} = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{1.8}$$

$$r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C). \tag{1.9}$$

where the projectors $E_A = I - AA^\dagger$, $E_B = I - BB^\dagger$, $F_A = I - A^\dagger A$, $F_C = I - C^\dagger C$.

2. The necessary and sufficient conditions for inclusion (1.2).

In this section, we will present some necessary and sufficient conditions for one side include (1.2), by using the maximal and minimal ranks of some generalized Schur complement forms. Let

$$S_{(A_1A_2A_3)^*} = S_{\mu^*} = (A_1A_2A_3)^* - (A_1A_2A_3)^*A_1A_2A_3X_1X_2X_3 = \mu^* - \mu^*\mu X_1X_2X_3 \tag{2.10}$$

where $A_i \in \mathbb{C}^{m\times m}$, $X_i \in A_i\{1,2,3\}$, $i = 1,2,3$, and $\mu = A_1A_2A_3$. For the convenience of readers, we first give a brief outline of the basic idea. From the formula (1.6) in Lemma 1.2, we know that the inclusion (1.2) holds if and only if

$$\mu^*\mu X_1X_2X_3 = \mu^* \text{ and } r(X_1X_2X_3) = r(\mu)$$

hold for any $X_i \in A_i\{1,2,3\}$, $i = 1,2,3$, and $\mu = A_1A_2A_3$, which are respectively equivalent to the following two identities

$$\max_{X_1,X_2,X_3} r(\mu^* - \mu^*\mu X_1X_2X_3) = 0 \tag{2.11}$$

and

$$\max_{X_1,X_2,X_3} r(X_1X_2X_3) = \min_{X_1,X_2,X_3} r(X_1X_2X_3) = r(\mu). \tag{2.12}$$

To begin with, some useful results are introduced, which presenting the necessary and sufficient conditions for inclusion (1.2).

Lemma 2.1 *Let $A_i \in \mathbb{C}^{m\times m}$, $X_i \in A_i\{1,2,3\}$, $i = 1,2,3$, and $\mu = A_1A_2A_3$. Then*

$$\begin{aligned}
&\max_{X_1,X_2,X_3} r(\mu^* - \mu^*\mu X_1X_2X_3) \\
= \; & r\begin{pmatrix}\mu^*\mu - \mu^*A_3A_2A_1, & \mu^*A_3A_2E_{A_1}, & \mu^*A_3E_{A_2}, & \mu^*E_{A_3}\end{pmatrix} \\
= \; & r\begin{pmatrix}\mu^*\mu - \mu^*A_3A_2A_1 & \mu^*A_3A_2 & \mu^*A_3 & \mu^* \\ O & A_1^* & O & O \\ O & O & A_2^* & O \\ O & O & O & A_3^*\end{pmatrix} - \sum_{i=1}^{3} r(A_i).
\end{aligned} \tag{2.13}$$

Proof. According to Lemma 1.3 and Lemma 1.1 (1.4) with $A = A_3$, $B = I_m$, $C = \mu^*\mu X_1X_2$ and $D = \mu^*$, we have

$$\begin{aligned}
&\max_{X_3} r(\mu^* - \mu^*\mu X_1X_2X_3) \\
= \; & \min\left\{ r\begin{pmatrix}A_3^*A_3 & A_3^* \\ \mu^*\mu X_1X_2 & \mu^*\end{pmatrix} - r(A_3), \; r\begin{pmatrix}A_3^* \\ \mu^*\end{pmatrix}\right\} \\
= \; & \min\left\{ r\begin{pmatrix}O & A_3^* \\ \mu^*\mu X_1X_2 - \mu^*A_3 & \mu^*\end{pmatrix} - r(A_3), \; r\begin{pmatrix}A_3^* \\ \mu^*\end{pmatrix}\right\} \\
= \; & r\begin{pmatrix}\mu^*\mu X_1X_2 - \mu^*A_3, & \mu^*E_{A_3}\end{pmatrix} \\
= \; & r\Big(\mu^*\mu X_1X_2\begin{pmatrix}I_m, & O\end{pmatrix} - \begin{pmatrix}\mu^*A_3, & -\mu^*E_{A_3}\end{pmatrix}\Big),
\end{aligned} \tag{2.14}$$

where the third equality hold as

$$r\begin{pmatrix}A_3^*A_3 & A_3^* \\ \mu^*\mu X_1X_2 & \mu^*\end{pmatrix} = r\begin{pmatrix}O & A_3^* \\ \mu^*\mu X_1X_2 - \mu^*A_3 & \mu^*\end{pmatrix} \le r\begin{pmatrix}O \\ \mu^*\mu X_1X_2 - \mu^*A_3\end{pmatrix} + r\begin{pmatrix}A_3^* \\ \mu^*\end{pmatrix}$$

and

$$r\begin{pmatrix}O \\ \mu^*\mu X_1X_2 - \mu^*A_3\end{pmatrix} = r(\mu^*\mu X_1X_2 - \mu^*A_3) \le r(\mu) \le r(A_3)$$

and

$$r\begin{pmatrix} O & A_3^* \\ \mu^*\mu X_1X_2 - \mu^*A_3 & \mu^* \end{pmatrix} = r\left(\mu^*\mu X_1X_2 - \mu^*A_3, \quad \mu^*E_{A_3}\right) + r(A_3).$$

Again by Lemma 1.3 and Lemma 1.1 (1.4) with $A = A_2, B = \left(I_m, \quad O\right), C = \mu^*\mu X_1$ and $D = \left(\mu^*A_3, \quad -\mu^*E_{A_3}\right)$, we obtain

$$\begin{aligned}
& \max_{X_2,X_3} r(\mu^* - \mu^*\mu X_1X_2X_3) \\
= \; & \max_{X_2} r\Big(\mu^*\mu X_1X_2\left(I_m, \quad O\right) - \left(\mu^*A_3, \quad -\mu^*E_{A_3}\right)\Big) \\
= \; & \min\left\{ r\begin{pmatrix} A_2^*A_2 & A_2^* & O \\ \mu^*\mu X_1 & \mu^*A_3 & -\mu^*E_{A_3} \end{pmatrix} - r(A_2), \; r\begin{pmatrix} A_2^* & O \\ \mu^*A_3 & -\mu^*E_{A_3} \end{pmatrix} \right\} \\
= \; & r\begin{pmatrix} O & A_2^* & O \\ \mu^*\mu X_1 - \mu^*A_3A_2 & \mu^*A_3 & -\mu^*E_{A_3} \end{pmatrix} - r(A_2) \\
= \; & r\Big(\left(\mu^*\mu X_1 - \mu^*A_3A_2, \quad \mu^*A_3, \quad -\mu^*E_{A_3}\right) \cdot F_{\left(O, \quad A_2^*, \quad O\right)}\Big) \\
= \; & r\left(\mu^*\mu X_1 - \mu^*A_3A_2, \quad \mu^*A_3E_{A_2}, \quad -\mu^*E_{A_3}\right) \\
= \; & r\Big(\mu^*\mu X_1\left(I_m, \quad O, \quad O\right) - \left(\mu^*A_3A_2, \quad -\mu^*A_3E_{A_2}, \quad \mu^*E_{A_3}\right)\Big),
\end{aligned} \tag{2.15}$$

where the third equality hold as

$$r\begin{pmatrix} A_2^*A_2 & A_2^* & O \\ \mu^*\mu X_1 & \mu^*A_3 & -\mu^*E_{A_3} \end{pmatrix} = r\begin{pmatrix} O & A_2^* & O \\ \mu^*\mu X_1 - \mu^*A_3A_2 & \mu^*A_3 & -\mu^*E_{A_3} \end{pmatrix} \le r\begin{pmatrix} O \\ \mu^*\mu X_1 - \mu^*A_3A_2 \end{pmatrix} + r\begin{pmatrix} A_2^* & O \\ \mu^*A_3 & -\mu^*E_{A_3} \end{pmatrix}$$

and

$$r\begin{pmatrix} O \\ \mu^*\mu X_1 - \mu^*A_3A_2 \end{pmatrix} = r(\mu^*\mu X_1 - \mu^*A_3A_2) \le r(\mu) \le r(A_2)$$

and

$$r\begin{pmatrix} O & A_2^* & O \\ \mu^*\mu X_1 - \mu^*A_3A_2 & \mu^*A_3 & -\mu^*E_{A_3} \end{pmatrix} = r(A_2) + r\Big(\left(\mu^*\mu X_1 - \mu^*A_3A_2, \quad \mu^*A_3, \quad -\mu^*E_{A_3}\right) \cdot F_{\left(O, \quad A_2^*, \quad O\right)}\Big).$$

Again by Lemma 1.3 and Lemma 1.1 (1.4) with $A = A_1$, $B = \left(I_m, \quad O, \quad O\right)$, $C = \mu^*\mu$ and $D = \left(\mu^*A_3A_2, \quad -\mu^*A_3E_{A_2}, \quad \mu^*E_{A_3}\right)$, we have

$$\begin{aligned}
& \max_{X_1,X_2,X_3} r(\mu^* - \mu^*\mu X_1X_2X_3) \\
= \; & \max_{X_1} r\Big(\mu^*\mu X_1\left(I_m, \quad O, \quad O\right) - \left(\mu^*A_3A_2, \quad -\mu^*A_3E_{A_2}, \quad \mu^*E_{A_3}\right)\Big) \\
= \; & \min\left\{ r\begin{pmatrix} A_1^*A_1 & A_1^* & O & O \\ \mu^*\mu & \mu^*A_3A_2 & -\mu^*A_3E_{A_2} & \mu^*E_{A_3} \end{pmatrix} - r(A_1), \right. \\
& \qquad\qquad\qquad \left. r\begin{pmatrix} A_1^* & O & O \\ \mu^*A_3A_2 & -\mu^*A_3E_{A_2} & \mu^*E_{A_3} \end{pmatrix} \right\} \\
= \; & r\begin{pmatrix} O & A_1^* & O & O \\ \mu^*\mu - \mu^*A_3A_2A_1 & \mu^*A_3A_2 & -\mu^*A_3E_{A_2} & \mu^*E_{A_3} \end{pmatrix} - r(A_1) \\
= \; & r\Big(\left(\mu^*\mu - \mu^*A_3A_2A_1, \quad \mu^*A_3A_2, \quad -\mu^*A_3E_{A_2}, \quad \mu^*E_{A_3}\right) \cdot F_{\left(O, \quad A_1^*, \quad O, \quad O\right)}\Big) \\
= \; & r\left(\mu^*\mu - \mu^*A_3A_2A_1, \quad \mu^*A_3A_2E_{A_1}, \quad \mu^*A_3E_{A_2}, \quad \mu^*E_{A_3}\right),
\end{aligned} \tag{2.16}$$

where the third equality hold as

$$r\begin{pmatrix} A_1^*A_1 & A_1^* & O & O \\ \mu^*\mu & \mu^*A_3A_2 & -\mu^*A_3E_{A_2} & \mu^*E_{A_3} \end{pmatrix} = r\begin{pmatrix} O & A_1^* & O & O \\ \mu^*\mu-\mu^*A_3A_2A_1 & \mu^*A_3A_2 & -\mu^*A_3E_{A_2} & \mu^*E_{A_3} \end{pmatrix}$$

and

$$r\begin{pmatrix} O & A_1^* & O & O \\ \mu^*\mu-\mu^*A_3A_2A_1 & \mu^*A_3A_2 & -\mu^*A_3E_{A_2} & \mu^*E_{A_3} \end{pmatrix} \le r\begin{pmatrix} O \\ \mu^*\mu-\mu^*A_3A_2A_1 \end{pmatrix} + r\begin{pmatrix} A_1^* & O & O \\ \mu^*A_3A_2 & -\mu^*A_3E_{A_2} & \mu^*E_{A_3} \end{pmatrix}$$

and

$$r\begin{pmatrix} O \\ \mu^*\mu-\mu^*A_3A_2A_1 \end{pmatrix} = r(\mu^*\mu-\mu^*A_3A_2A_1) \le r(\mu) \le r(A_1).$$

Combing (2.16) with the formula (1.9) in Lemma 1.3, we finally have

$$\begin{aligned} & \max_{X_1,X_2,X_3} r(\mu^*-\mu^*\mu X_1X_2X_3) \\ = \; & r\left(\mu^*\mu-\mu^*A_3A_2A_1, \;\; \mu^*A_3A_2E_{A_1}, \;\; \mu^*A_3E_{A_2}, \;\; \mu^*E_{A_3}\right) \\ = \; & r\begin{pmatrix} \mu^*\mu-\mu^*A_3A_2A_1 & \mu^*A_3A_2 & \mu^*A_3 & \mu^* \\ O & A_1^* & O & O \\ O & O & A_2^* & O \\ O & O & O & A_3^* \end{pmatrix} - \sum_{i=1}^{3} r(A_i). \quad \square \end{aligned}$$

Next Lemma gives the expression in the ranks of the known matrices for

$$\max_{\substack{X_1,X_2,X_3 \\ X_i\in A_i\{1,2,3\}}} r(X_1X_2X_3).$$

Lemma 2.2 *Let $A_i \in \mathbb{C}^{m\times m}$, $X_i \in A_i\{1,2,3\}$, $i = 1,2,3$. Then*

$$\max_{X_1,X_2,X_3} r(X_1X_2X_3) = \min\left\{ r(A_1), \;\; r(A_2), \;\; r(A_3)\right\}. \tag{2.17}$$

Proof. By the formula (1.4) in Lemma 1.1 with $A = A_3$, $B = I_m$, $C = X_1X_2$ and $D = O$, we have

$$\begin{aligned} & \max_{X_3} r(X_1X_2X_3) \\ = \; & \min\left\{ r\begin{pmatrix} A_3^*A_3 & A_3^* \\ X_1X_2 & O \end{pmatrix} - r(A_3), \;\; r\begin{pmatrix} A_3^* \\ O \end{pmatrix} \right\} \\ = \; & \min\left\{ r\begin{pmatrix} O & A_3^* \\ X_1X_2 & O \end{pmatrix} - r(A_3), \;\; r(A_3) \right\} \\ = \; & \min\left\{ r(X_1X_2), \;\; r(A_3)\right\}. \end{aligned} \tag{2.18}$$

Again by the formula (1.4) in Lemma 1.1 with $A = A_2$, $B = I_m$, $C = X_1$ and $D = O$, we have

$$
\begin{aligned}
&\max_{X_2,X_3} r(X_1X_2X_3) \\
= \ & \min\left\{ \max_{X_2} r(X_1X_2),\ \ r(A_3)\right\} \\
= \ & \min\left\{ \min\left\{ r\begin{pmatrix} A_2^*A_2 & A_2^* \\ X_1 & O \end{pmatrix} - r(A_2),\ \ r\begin{pmatrix} A_2^* \\ O \end{pmatrix}\right\},\ \ r(A_3)\right\} \\
= \ & \min\left\{ \min\left\{ r\begin{pmatrix} O & A_2^* \\ X_1 & O \end{pmatrix} - r(A_2),\ \ r\begin{pmatrix} A_2^* \\ O \end{pmatrix}\right\},\ \ r(A_3)\right\} \\
= \ & \min\left\{ \min\left\{ r(X_1),\ \ r(A_2)\right\},\ \ r(A_3)\right\} \\
= \ & \min\left\{ r(X_1),\ \ r(A_2),\ \ r(A_3)\right\}.
\end{aligned}
\tag{2.19}
$$

Since $X_1 \in A_1\{1,2,3\}$, we have $r(X_1) = r(A_1)$. Then by (2.19), we have

$$
\max_{X_1,X_2,X_3} r(X_1X_2X_3) = \min\left\{ r(A_1),\ \ r(A_2),\ \ r(A_3)\right\}. \qquad \square
$$

We now give the expression in the ranks of the known matrices for the following minimal rank problem:

$$
\min_{\substack{X_1,X_2,X_3 \\ X_i \in A_i\{1,2,3\}}} r(X_1X_2X_3).
$$

Lemma 2.3 *Let* $A_i \in \mathbb{C}^{m\times m}$, $X_i \in A_i\{1,2,3\}$, $i = 1,2,3$. *Then*

$$
\begin{aligned}
&\min_{X_1,X_2,X_3} r(X_1X_2X_3) \\
= \ & r(A_3) - r\begin{pmatrix} A_3E_{A_2}, & A_3A_2E_{A_1} \end{pmatrix} \\
= \ & \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_3 & A_3A_2 \\ A_2^* & O \\ O & A_1^* \end{pmatrix}.
\end{aligned}
\tag{2.20}
$$

Proof. Using the formula (1.5) in Lemma 1.1 with $A = A_1$, $B = X_2X_3$, $C = I_m$ and $D = O$, we have

$$
\begin{aligned}
&\min_{X_1} r(X_1X_2X_3) \\
= \ & r\begin{pmatrix} A_1^*A_1 & A_1^*X_2X_3 \\ I_m & O \end{pmatrix} + r(A_1^*X_2X_3) - r\begin{pmatrix} A_1 & O \\ O & A_1^*X_2X_3 \\ I_m & O \end{pmatrix} \\
= \ & r(A_1^*X_2X_3).
\end{aligned}
\tag{2.21}
$$

By Lemma 1.3 and the formula (1.5) in Lemma 1.1 with $A = A_2$, $B = X_3$, $C = A_1^*$ and $D = O$, we have

$$\begin{aligned}
&\min_{X_2,X_1} r(X_1X_2X_3)\\
=\ &\min_{X_2} r(A_1^*X_2X_3)\\
=\ & r\begin{pmatrix} A_2^*A_2 & A_2^*X_3 \\ A_1^* & O \end{pmatrix} + r\begin{pmatrix} A_2^*X_3 \\ O \end{pmatrix} - r\begin{pmatrix} A_2 & O \\ O & A_2^*X_3 \\ A_1^* & O \end{pmatrix}\\
=\ & r\left(\begin{pmatrix} A_2^*A_2, & A_2^*X_3 \end{pmatrix} \cdot F_{\begin{pmatrix} A_1^*, & O \end{pmatrix}}\right) + r(A_1) - r\begin{pmatrix} A_2 \\ A_1^* \end{pmatrix}\\
=\ & r\begin{pmatrix} A_2^*A_2E_{A_1}, & A_2^*X_3 \end{pmatrix} + r(A_1) - r\begin{pmatrix} A_2 \\ A_1^* \end{pmatrix}\\
=\ & r\left(A_2^*X_3\begin{pmatrix} I_m, & O \end{pmatrix} - \begin{pmatrix} O, & -A_2^*A_2E_{A_1} \end{pmatrix}\right) + r(A_1) - r\begin{pmatrix} A_2 \\ A_1^* \end{pmatrix}.
\end{aligned} \tag{2.22}$$

Again by Lemma 1.3 and the formula (1.5) in Lemma 1.1 with $A = A_3$, $B = \begin{pmatrix} I_m, & O \end{pmatrix}$, $C = A_2^*$ and $D = \begin{pmatrix} O, & -A_2^*A_2E_{A_1} \end{pmatrix}$, we have

$$\begin{aligned}
&\min_{X_3,X_2,X_1} r(X_1X_2X_3)\\
=\ &\min_{X_3} r\left(A_2^*X_3\begin{pmatrix} I_m, & O \end{pmatrix} - \begin{pmatrix} O, & -A_2^*A_2E_{A_1} \end{pmatrix}\right) + r(A_1) - r\begin{pmatrix} A_2 \\ A_1^* \end{pmatrix}\\
=\ & r\begin{pmatrix} A_3^*A_3 & A_3^* & O \\ A_2^* & O & -A_2^*A_2E_{A_1} \end{pmatrix} + r\begin{pmatrix} A_3^* & O \\ O & -A_2^*A_2E_{A_1} \end{pmatrix} - r\begin{pmatrix} A_3 & O & O \\ O & A_3^* & O \\ A_2^* & O & -A_2^*A_2E_{A_1} \end{pmatrix} + r(A_1) - r\begin{pmatrix} A_2 \\ A_1^* \end{pmatrix}\\
=\ & r(A_1) + r(A_2) + r(A_3) + r(A_2E_{A_1}) - r\begin{pmatrix} A_2 \\ A_1^* \end{pmatrix} - r\begin{pmatrix} A_3 & O \\ A_2^* & -A_2^*A_2E_{A_1} \end{pmatrix}\\
=\ & r(A_2) + r(A_3) - r\begin{pmatrix} A_3 & A_3A_2E_{A_1} \\ A_2^* & O \end{pmatrix}\\
=\ & r(A_3) - r\begin{pmatrix} A_3E_{A_2}, & A_3A_2E_{A_1} \end{pmatrix}.
\end{aligned} \tag{2.23}$$

Combing (2.23) with the formula (1.9) in Lemma 1.3, we finally have

$$\begin{aligned}
&\min_{X_3X_2,X_1} r(X_1X_2X_3)\\
=\ & r(A_3) - r\begin{pmatrix} A_3E_{A_2}, & A_3A_2E_{A_1} \end{pmatrix}\\
=\ & \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_3 & A_3A_2 \\ A_2^* & O \\ O & A_1^* \end{pmatrix}. \qquad \square
\end{aligned}$$

From Lemmas 2.1, 2.2 and 2.3, we immediately obtain the following theorem by equations (2.11) and (2.12).

Theorem 2.1 *Let $A_i \in \mathbb{C}^{m\times m}$, $i = 1, 2, 3$, and $\mu = A_1A_2A_3$. Then the following statements are equivalent:*

(1) $A_1\{1, 2, 3\}A_2\{1, 2, 3\}A_3\{1, 2, 3\} \subseteq (A_1A_2A_3)\{1, 2, 3\}$;

(2) $r\begin{pmatrix} \mu^*\mu - \mu^*A_3A_2A_1, & \mu^*A_3A_2E_{A_1}, & \mu^*A_3E_{A_2}, & \mu^*E_{A_3} \end{pmatrix} = 0$ and

$$r(\mu) = \min\left\{r(A_1),\ r(A_2),\ r(A_3)\right\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_3 & A_3A_2 \\ A_2^* & O \\ O & A_1^* \end{pmatrix}.$$

We now state more equivalent conditions for one side inclusion (1.2) without proofs since they are easy.

Corollary 2.1 *Let $A_i \in \mathbb{C}^{m\times m}$, $i = 1,2,3$, and $\mu = A_1A_2A_3$. Then the following statements are equivalent:*

(1) $A_1\{1,2,3\}A_2\{1,2,3\}A_3\{1,2,3\} \subseteq (A_1A_2A_3)\{1,2,3\}$;

(2) $\mu^*\mu - \mu^*A_3A_2A_1 = O$ *and* $\mu^*A_3A_2E_{A_1} = O$ *and* $\mu^*A_3E_{A_2} = O$ *and* $\mu^*E_{A_3} = O$ *and*

$$r(\mu) = \min\left\{r(A_1),\ r(A_2),\ r(A_3)\right\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_3 & A_3A_2 \\ A_2^* & O \\ O & A_1^* \end{pmatrix};$$

(3) $\mu^*\mu = \mu^*A_3A_2A_1$ *and* $N(\mu^*A_3A_2) \supset N(A_1^*)$ *and* $N(\mu^*A_3) \supset N(A_2^*)$ *and* $N(\mu^*) \supset N(A_3^*)$ *and*

$$r(\mu) = \min\left\{r(A_1),\ r(A_2),\ r(A_3)\right\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_3 & A_3A_2 \\ A_2^* & O \\ O & A_1^* \end{pmatrix};$$

(4) $$r\begin{pmatrix} \mu^*\mu - \mu^*A_3A_2A_1 & \mu^*A_3A_2 & \mu^*A_3 & \mu^* \\ O & A_1^* & O & O \\ O & O & A_2^* & O \\ O & O & O & A_3^* \end{pmatrix} = \sum_{i=1}^{3} r(A_i) \text{ and}$$

$$r(\mu) = \min\left\{r(A_1),\ r(A_2),\ r(A_3)\right\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_3 & A_3A_2 \\ A_2^* & O \\ O & A_1^* \end{pmatrix}.$$

Corollary 2.2 *Let $A_i \in \mathbb{C}^{m\times m}$, $i = 1,2$. Then the following statements are equivalent:*

(1) $A_1\{1,2,3\}A_2\{1,2,3\} \subseteq (A_1A_2)\{1,2,3\}$;

(2) $r\Big((A_1A_2)^*A_1A_2 - (A_1A_2)^*A_2A_1,\ \ (A_1A_2)^*A_2E_{A_1},\ \ (A_1A_2)^*E_{A_2}\Big) = 0$ *and*

$$r(A_1A_2) = \min\left\{r(A_1),\ r(A_2)\right\} = \sum_{i=1}^{2} r(A_i) - r\begin{pmatrix} A_2 \\ A_1^* \end{pmatrix};$$

(3) $(A_1A_2)^*A_1A_2 - (A_1A_2)^*A_2A_1 = O$ *and* $(A_1A_2)^*A_2E_{A_1} = O$ *and* $(A_1A_2)^*E_{A_2} = O$ *and*

$$r(A_1A_2) = \min\left\{r(A_1),\ r(A_2)\right\} = \sum_{i=1}^{2} r(A_i) - r\begin{pmatrix} A_2 \\ A_1^* \end{pmatrix};$$

(4) $(A_1A_2)^*A_1A_2 = (A_1A_2)^*A_2A_1$ *and* $N\Big((A_1A_2)^*A_2\Big) \supset N\Big(A_1^*\Big)$ *and* $N\Big((A_1A_2)^*\Big) \supset N\Big(A_2^*\Big)$ *and*

$$r(A_1A_2) = \min\left\{r(A_1),\ r(A_2)\right\} = \sum_{i=1}^{2} r(A_i) - r\begin{pmatrix} A_2 \\ A_1^* \end{pmatrix};$$

(5) $$r\begin{pmatrix} (A_1A_2)^*A_1A_2 - (A_1A_2)^*A_2A_1 & (A_1A_2)^*A_2 & (A_1A_2)^* \\ O & A_1^* & O \\ O & O & A_2^* \end{pmatrix} = \sum_{i=1}^{2} r(A_i) \text{ and}$$

$$r(A_1A_2) = \min\left\{r(A_1),\ r(A_2)\right\} = \sum_{i=1}^{2} r(A_i) - r\begin{pmatrix} A_2 \\ A_1^* \end{pmatrix}.$$

Notice that $XAA^* = A^*$ and $r(X) = r(A)$ are equivalent to the equations $AA^*X^* = A^*$ and $r(X^*) = r(A^*)$, respectively. This implies that, by the formula (1.6) and (1.7) in Lemma 1.2, $X \in A\{1,2,4\}$ if and only if $X^* \in A\{1,2,3\}$. So we can get the necessary and sufficient conditions for (1.3) by a similar approach in the previous section and hence provide the following results without the proof.

Theorem 2.2 *Let $A_i \in \mathbb{C}^{m\times m}$, $i = 1,2,3$, and $\mu = A_1A_2A_3$. Then the following statements are equivalent:*

(1) $A_1\{1,2,4\}A_2\{1,2,4\}A_3\{1,2,4\} \subseteq (A_1A_2A_3)\{1,2,4\}$;

(2) $r\begin{pmatrix} A_3A_2A_1\mu^* - \mu\mu^* \\ F_{A_3}A_2A_1\mu^* \\ F_{A_2}A_1\mu^* \\ F_{A_1}\mu^* \end{pmatrix} = 0$ *and*

$$r(\mu) = \min\left\{r(A_3),\ r(A_2),\ r(A_1)\right\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_1 & O & A_2^* \\ A_2A_1 & A_3^* & O \end{pmatrix};$$

(3) $A_3A_2A_1\mu^* - \mu\mu^* = O$ *and* $F_{A_3}A_2A_1\mu^* = O$ *and* $F_{A_2}A_1\mu^* = O$ *and* $F_{A_1}\mu^* = O$ *and*

$$r(\mu) = \min\left\{r(A_3),\ r(A_2),\ r(A_1)\right\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_1 & O & A_2^* \\ A_2A_1 & A_3^* & O \end{pmatrix};$$

(4) $A_3A_2A_1\mu^* = \mu\mu^*$ *and* $R(A_2A_1\mu^*) \subseteq R(A_3^*)$ *and* $R(A_1\mu^*) \subseteq R(A_2^*)$ *and* $R(\mu^*) \subseteq R(A_1^*)$ *and*

$$r(\mu) = \min\left\{r(A_3),\ r(A_2),\ r(A_1)\right\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_1 & O & A_2^* \\ A_2A_1 & A_3^* & O \end{pmatrix};$$

(5) $r\begin{pmatrix} A_3A_2A_1\mu^* - \mu\mu^* & O & O & O \\ A_2A_1\mu^* & O & O & A_3^* \\ A_1\mu^* & O & A_2^* & O \\ \mu^* & A_1^* & O & O \end{pmatrix} = \sum_{i=1}^{3} r(A_i)$ *and*

$$r(\mu) = \min\left\{r(A_3),\ r(A_2),\ r(A_1)\right\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_1 & O & A_2^* \\ A_2A_1 & A_3^* & O \end{pmatrix}.$$

Corollary 2.3 *Let* $A_i \in \mathbb{C}^{m\times m}$, $i = 1, 2$. *Then the following statements are equivalent:*

(1) $A_1\{1,2,4\}A_2\{1,2,4\} \subseteq (A_1A_2)\{1,2,4\}$;

(2) $r\begin{pmatrix} A_2A_1(A_1A_2)^* - A_1A_2(A_1A_2)^* \\ F_{A_2}A_1(A_1A_2)^* \\ F_{A_1}(A_1A_2)^* \end{pmatrix} = 0$ *and*

$$r(A_1A_2) = \min\left\{r(A_2),\ r(A_1)\right\} = \sum_{i=1}^{2} r(A_i) - r\left(A_1,\quad A_2^*\right);$$

(3) $A_2A_1(A_1A_2)^* - A_1A_2(A_1A_2)^* = O$ *and* $F_{A_2}A_1(A_1A_2)^* = O$ *and* $F_{A_1}(A_1A_2)^* = O$ *and*

$$r(A_1A_2) = \min\left\{r(A_2),\ r(A_1)\right\} = \sum_{i=1}^{2} r(A_i) - r\left(A_1,\quad A_2^*\right);$$

(4) $A_2A_1(A_1A_2)^* = A_1A_2(A_1A_2)^*$ *and* $R\left(A_1(A_1A_2)^*\right) \subseteq R\left(A_2^*\right)$ *and* $R\left((A_1A_2)^*\right) \subseteq R\left(A_1^*\right)$ *and*

$$r(A_1A_2) = \min\left\{r(A_2),\ r(A_1)\right\} = \sum_{i=1}^{2} r(A_i) - r\left(A_1,\quad A_2^*\right);$$

(5) $r\begin{pmatrix} A_2A_1(A_1A_2)^* - A_1A_2(A_1A_2)^* & O & O \\ A_1(A_1A_2)^* & O & A_2^* \\ (A_1A_2)^* & A_1^* & O \end{pmatrix} = \sum_{i=1}^{2} r(A_i)$ *and*

$$r(A_1A_2) = \min\left\{r(A_2),\ r(A_1)\right\} = \sum_{i=1}^{2} r(A_i) - r\left(A_1,\quad A_2^*\right).$$

3. Acknowledgements

The authors would like to thank Professor Dragana Cvetkovic Ilic and the anonymous referees for their very detailed comments and constructive suggestions, which greatly improved the presentation of this paper.

References

[1] A. Ben-Israel and T. N. E. Greville. *Generalized Inverse: Theory and Applications.* Wiley-Interscience, 1974; 2nd Edition, Springer-Verlag, New York, 2002.

[2] A. R. Depierro and M. Wei. *Reverse order laws for recive generalized inverse of products of matrices*. Linear Algebra Appl., 277 (1996) 299-311.

[3] T. N. E. Greville. *Note on the generalized inverses of a matrix products*. SIAM Review, 8 (1966) 518-521.

[4] R. E. Hartwing. *The reverse order law revisited*. Linear Algebra Appl., 76 (1986) 241–246.

[5] G. Marsaglia and G. P. H. S. Tyan. *Equalities and inequalities for ranks of matrices*. Linear and Multilinear Algebra, 2 (1974) 269-292.

[6] R. Penrose. *A generalized for matrix*. Proc. Cambridge Philos Soc., 51 (1955) 406-413.

[7] J. Nikolov Radenković. *Reverse order law for multiple operator product*. Linear Multilinear Algebra., 64:7 (2016) 1266-1282.

[8] P. Stanimirovic and M. Tasic. *Computing generalized inverses using LU factorrization of Matrix product*. Int. J. Comp. Math., 85 (2008) 1865-1878.

[9] Y. Tian. *Upper and lower bounds for ranks of matrix expressions using generalized inverses*. Linear Algebra Appl., 355 (2002) 187-214.

[10] Y. Tian. *The Moore-Penrose inverse order of a triple matrix product*. Math. Pract. Theory., 1 (1992) 64-70.

[11] Y. Tian. *Reverse order laws for generalized inverse of multiple materix products*. Linear Algebra Appl., 211 (1994) 85-100.

[12] Y. Tian. *More on maximal and minimal ranks of Schur complements with applications*. Appl. Math. Comput., 152 (2004) 675-692.

[13] G. Wang, Y. Wei and S. Qiao. *Generalized inverses: Theory and Computations*. Science Press, Beijing, 2004.

[14] M. Wei. *Equivalent conditions for generalized inverses of products*. Linear Algebra Appl., 266 (1997) 347-363.

[15] M. Wei. *Reverse order laws for generalized inverse of multiple matrix products*. Linear Algebra Appl., 293 (1999) 273-288.

[16] G. Wang and W. Gao. *Reverse order laws for lest squares g-inverses and minimum-norm g-inverses of products of two matrices*. Linear Algebra Appl., 342 (2002) 117-132.

[17] Z. Xiong and B. Zheng. *Forward order law for the generalized inverses of multiple matrix products*. J. Appl. Math. Comput., 25 (2007) 415-424.

[18] B. Zheng and Z. Xiong. *The reverse order laws for* $\{1,2,3\}$*-,* $\{1,2,4\}$*-inverses of multiple matrix products*. Linear Multilinear Algebra., 58 (2010) 765-782.

[19] Z. Xiong and B. Zheng. *The reverse order laws for* $\{1,2,3\}$*-,* $\{1,2,4\}$*-inverses of a two-matrix products*. Appl. Math. Let., 21 (2008) 649-655.