Filomat 32:3 (2018), 985–990
https://doi.org/10.2298/FIL1803985S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Exact Solution of the One Phase Inverse Stefan Problem

Merey M. Sarsengeldin[a], Stanislav N. Kharin[b], Samat Kassabek[c], Zamanbek Mukambetkazin[c]

[a]*Satbayev University, Sigma LABS, Almaty, Kazakhstan, and Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan*
[b]*Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan, and Kazakh-British Technical University, Almaty, Kazakhstan*
[c]*Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan*

Abstract. Exact solution of inverse one phase Stefan problem is represented in the form of linear combination of integral error functions. Heat flux function is reconstructed and coefficients of solution function are found exactly. Test problem was considered for engineering purposes and it was shown that by collocation method the error for three points does not exceed 0.01%. Error estimate was calculated by maximum principle.

1. Introduction

In previous studies it was shown that Stefan type problems nicely fit and satisfy experimental data [4, 8] and can serve as mathematical model for arcing processes in electric contacts and diverse electric contact phenomena [3]. Worth to note that for some cases electric contact processes, for instance arcing [2, 9], is so rapid that experimental investigation of phenomenon is almost impossible therefore mathematical modeling plays vital role for understanding and analyzing electric contact phenomena.

This study is a sequel of series of studies [6, 7], whose ultimate purpose is to determine and investigate components of arc occurring during opening electrical contacts and it is an attempt to model arcing process on the base of inverse one phase Stefan problem.

A long bibliography of studies on free-moving boundary problems [10] and literature therein is devoted to qualitative and quantitative investigations of the Stefan type problems as well as their applications and this type of problems are recognized as most complicated problems in the theory of parabolic equations. For equations responsible for governing electric contact phenomena including nonlinear cases classical theory of heat potentials enables one to construct analytical solutions only for some cases and reduction to integral equations leads to singularity of integrals at initial time when domain degenerates therefore the method of successive approximations is inapplicable [1, 5]. Thus finding exact solution for such problems seems to be valuable as from theoretical point of view as well as for investigation and analyzing afore mentioned electric contact phenomena.

2010 *Mathematics Subject Classification*. Primary 80A22; Secondary 33B20
Keywords. Integral error function, inverse one phase Stefan problem, exact solution, electric contacts
Received: 31 December 2016; Revised: 03 March 2017; Accepted: 14 April 2017
Communicated by Allaberen Ashyralyev
This research is financially supported by a grant AP05133919 and by the target program BR05236656 from the Science Committee from the Ministry of Science and Education of the Republic of Kazakhstan.
Email addresses: merey@mail.ru (Merey M. Sarsengeldin), staskharin@yahoo.com (Stanislav N. Kharin), kassabek@gmail.com (Samat Kassabek), zamanbek.mukambetkazin@gmail.com (Zamanbek Mukambetkazin)

2. One Phase Inverse Stefan Problem

Under the assumption that liquid zone is well mixed and power balance is given as in Stefan's condition i.e. energy is consumed only for melting moving boundary the model of heat transfer in electric contact is based on the following one phase inverse Stefan problem.

Let us consider a heat equation

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2}, 0 < x < \alpha(t), t > 0, \tag{1}$$

subjected to following conditions:

$$-\lambda \left. \frac{\partial U}{\partial x} \right|_{x=0} = P(t), t > 0, \tag{2}$$

$$U(\alpha(t), t) = U_m, t > 0, \tag{3}$$

$$U(0, 0) = 0, \tag{4}$$

$$-\lambda_1 \left. \frac{\partial U}{\partial x} \right|_{x=\alpha(t)} = L\gamma \frac{d\alpha(t)}{dt}, t > 0, \tag{5}$$

where $P(t)$ is a heat flux which has to be found along with temperature function $U(x, t)$. U_m is a melting temperature and $\alpha(t)$ is a known moving boundary.

3. Method of Solution

Solution is represented in the form of integral error functions

$$U(x, t) = \sum_{n=0}^{\infty} (2a\sqrt{t})^n \left[A_n i^n erfc(\frac{x}{2a\sqrt{t}}) + B_n i^n erfc(-\frac{x}{2a\sqrt{t}}) \right], \tag{6}$$

where A_n, B_n and $P(t)$ has to be determined.

Let us suggest that known moving boundary can be expressed in Maclaurin's series

$$\alpha(t) = \sum_{n=0}^{\infty} \frac{\alpha^{(n)}(0)}{n!} \cdot t^{\frac{n}{2}} \tag{7}$$

and heat flux $P(t)$ can be expressed in the following form

$$P(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + ... + p_n t^n, \tag{8}$$

where $p_0, p_1, p_2, p_3 ... p_n$ has to be determined.

Making substitution $\sqrt{t} = \tau$, we rewrite the conditions (2), (3), (5) in terms of τ and then, we compare the equal powers in the left and right hand sides of equations using differentiation and putting $\tau = 0$.

From (3), we get

$$\sum_{n=0}^{\infty}(2a\tau)^n\left[A_n i^n erfc(\frac{\alpha(\tau)}{2a\tau}) + B_n i^n erfc(-\frac{\alpha(\tau)}{2a\tau})\right] = U_m, \tag{9}$$

when $n = 0$

$$A_0 i^0 erfc\left(\frac{\alpha_1}{2a}\right) + B_0 i^0 erfc\left(-\frac{\alpha_1}{2a}\right) = U_m. \tag{10}$$

For $k = 0, 1, 2, ...$, we utilize Leibniz rule for k-th derivative of product and Faa Di Bruno's formula for k-th derivative of composite function, thus

$$\left[\sum_{n=0}^{\infty}(2a\tau)^n\, i^n erfc\left(\pm\frac{\alpha(\tau)}{2a\tau}\right)\right]^{(k)} = \sum_{n=0}^{k}\frac{2^{\frac{n}{2}}k!}{(k-n)!}\left[i^n erfc(\pm\delta)\right]^{(k-n)}$$

$$= \sum_{n=0}^{k}\frac{2^{\frac{n}{2}}k!}{(k-n)!}\sum_{m=1}^{k-n}\left[i^n erfc(\pm\delta)\right]^{(m)}\Big|_{\delta=0}\cdot B_{k-n,m}\left((\pm\delta)', (\pm\delta)''^{(k-n-m+1)}\right)\Big|_{\tau=0},$$

where $B_{k-n,m}$ is Bell's polynomial

$$B_{k-n,m} = \sum\frac{(k-n)!}{j_1!j_2!...j_{k-n-m+1}!}(\pm\delta_1)^{j_1}(\pm\delta_2)^{j_2}...(\pm\delta_{k-n-m+1})^{j_{k-n-m+1}},$$

$\delta = \frac{\alpha(\tau)}{2a\tau}$ and $\delta_n = \frac{\alpha_n}{2a}, n = 1, 2, 3, ...$

$$j_1 + j_2 + ... + j_{k-n-m+1} = m,$$
$$j_1 + 2j_2 + ... + (k-n-m+1)j_{k-n-m+1} = k-n,$$
$$\left[i^n erfc(\pm\delta)\right]^{(m)}\Big|_{\delta=0} = (\pm1)^m\frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m)!\sqrt{n}}.$$

Ultimately, we get

$$\left[\sum_{n=0}^{\infty}(2a\tau)^n\left[A_n i^n erfc(\delta(\tau)) + B_n i^n erfc(-\delta(\tau))\right]\right]^{(k)}\Bigg|_{\tau=0} = U_m,$$

which yields

$$\begin{aligned}&\sum_{n=0}^{k}A_n\frac{2^{\frac{n}{2}}k!}{(k-n)!}\sum_{m=1}^{k-n}\frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m)!\sqrt{n}}\sum\frac{(k-n)!}{j_1!j_2!...j_{k-n-m+1}!}(\delta_1)^{j_1}(\delta_2)^{j_2}...(\delta_{k-n-m+1})^{j_{k-n-m+1}}\\ &= -\sum_{n=0}^{k}B_n\frac{2^{\frac{n}{2}}k!}{(k-n)!}\sum_{m=1}^{k-n}(-1)^m\frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m)!\sqrt{n}}\sum\frac{(k-n)!}{j_1!j_2!...j_{k-n-m+1}!}(-\delta_1)^{j_1}(-\delta_2)^{j_2}...(-\delta_{k-n-m+1})^{j_{k-n-m+1}}.\end{aligned} \tag{11}$$

Multiplying both sides of (5) by $2a\tau$, we get

$$-\lambda_1\sum_{n=0}^{\infty}(2a\tau)^n\left[-A_n i^{n-1}erfc\left(\frac{\alpha(\tau)}{2a\tau}\right) + B_n i^{n-1}erfc\left(-\frac{\alpha(\tau)}{2a\tau}\right)\right] = L\gamma 2a\tau\frac{d\alpha(\tau)}{d\tau}, \tag{12}$$

and

$$\begin{aligned}&\sum_{n=1}^{k} -A_n \frac{2^{\frac{n}{2}} k!}{(k-n)!} \sum_{m=1}^{k-n} \frac{\Gamma\left(\frac{n-m}{2}\right)}{(n-m-1)!\sqrt{n-1}} \sum \frac{(k-n)!}{j_1! j_2! \ldots j_{k-n-m+1}!} (\delta_1)^{j_1} (\delta_2)^{j_2} \ldots (\delta_{k-n-m+1})^{j_{k-n-m+1}} \\ &+ \sum_{n=1}^{k} B_n \frac{2^{\frac{n}{2}} k!}{(k-n)!} \sum_{m=1}^{k-n} (-1)^m \frac{\Gamma\left(\frac{n-m}{2}\right)}{(n-m-1)!\sqrt{n-1}} \sum \frac{(k-n)!}{j_1! j_2! \ldots j_{k-n-m+1}!} (-\delta_1)^{j_1} (-\delta_2)^{j_2} \ldots (-\delta_{k-n-m+1})^{j_{k-n-m+1}} \\ &= \frac{2aL\gamma \cdot k! \cdot k\alpha_k}{-\lambda_1}.\end{aligned} \tag{13}$$

For $n = 0$ and $k = 0$

$$-\lambda \left[-A_0 i^{-1} erfc\left(\frac{\alpha_1}{2a}\right) + B_0 i^{-1} erfc\left(-\frac{\alpha_1}{2a}\right) \right] = 0. \tag{14}$$

Thus, A_n, B_n can be determined from the system of equations (11), (13).
From (2)

$$-\lambda \sum_{n=0}^{\infty} (2a\tau)^{n-1} \left[B_n - A_n\right] i^{n-1} erfc\,(0) = P(\tau),$$

we get coefficients of heat flux P(t)

$$p_{n-1} = \lambda\,(2a)^{n-1} \left[A_n - B_n\right] i^{n-1} erfc\,(0)\,, n = 1, 2, 3, \ldots \tag{15}$$

Remark 3.1. Convergence of series (6) can be proved by the help of the same idea represented by second author in [8].

4. Test Problem

In this section, we show that it is possible to reach 0.01% error using only three points $t_0 = 0,\ t_1 = 0.5$ and $t_2 = 1$ by collocation method which seems to be very practical for engineers.

Solution is found both exactly and approximately.

For the problem $\alpha(t) = \alpha\sqrt{t}$ (moving boundary), solution should be sought by formula (6).

Coefficients A_n, B_n can be determined from (11), (13) and P_n from (15) or straightforwardly by satisfying conditions and equating like terms as following:

$$B_0 = \frac{U_m \exp\left(\frac{\alpha}{2a}\right)^2 - \frac{L\gamma\alpha a\sqrt{\pi}}{2\lambda} \cdot erfc\left(\frac{\alpha}{2a}\right)}{erfc\left(-\frac{\alpha}{2a}\right)\exp\left(\frac{\alpha}{2a}\right)^2 + erfc\left(\frac{\alpha}{2a}\right)\exp\left(-\left(\frac{\alpha}{2a}\right)^2\right)}, \tag{16}$$

$$A_0 = \frac{U_m - B_0 \cdot erfc\left(-\frac{\alpha}{2a}\right)}{erfc\left(\frac{\alpha}{2a}\right)}. \tag{17}$$

By straightforward substitution $\alpha(t) = \alpha\sqrt{t}$ conditions (3), (5) are reduced to the following system from which we determine A_n, B_n coefficients ($n = 1, 2, 3 \ldots$)

$$\left.\begin{aligned}&\lambda\,(2a)^{n-1}\left[A_n i^{n-1} erfc\left(\frac{\alpha}{2a}\right) - B_n i^{n-1} erfc\left(-\frac{\alpha}{2a}\right)\right] = 0\\ &A_n i^n erfc\left(\frac{\alpha}{2a}\right) - B_n i^n erfc\left(-\frac{\alpha}{2a}\right) = 0\end{aligned}\right\}. \tag{18}$$

Thus it's easy to see that $A_n = B_n = 0, n = 1, 2, 3, \cdots$.

Let $u_m = 0$, $\lambda = 1$, $a = 1$, $L = \alpha = \gamma = 1$. Then using Mathcad 15 for collocation method we get following approximate values for $A_0 = 0.579$, $A_1 = 0$, $A_2 = 0$, $B_0 = -0.183$, $B_1 = 0$, $B_2 = 0$. Exact values can be computed from (16), (17) and (18).

In Figure 1, the graphs of both reconstructed exact (**P(t)**) and approximate (**P_(t)**) flux functions are illustrated.

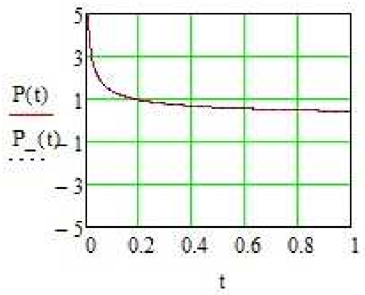

Figure 1: Graphs of both reconstructed exact and approximate flux functions

Computation of relative error of flux function is illustrated in Figures 2 and 3.

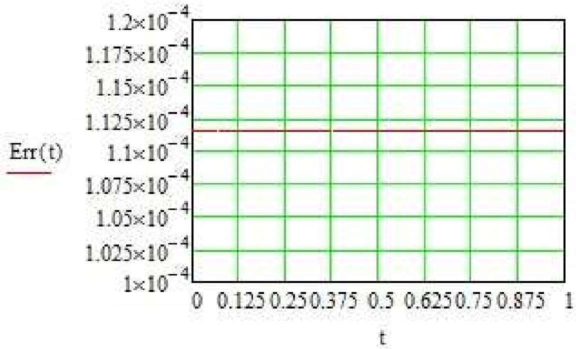

Figure 2: Computation of relative error of flux function

In Figure 3, we can observe that relative error doesn't exceed 0.01 for three points $t_0 = 0$, $t_1 = 0.5$ and $t_2 = 1$.

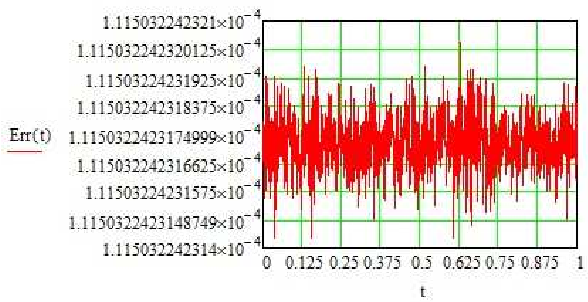

Figure 3: Scaled graph of relative error

5. Conclusion

Thus, coefficients A_n, B_n of the series (6) and function $P(t)$ can be determined from (11),(13) and (15) respectively. It was shown in the test problem that by maximum principle error estimate doesn't exceed 0.01%. Calculations were performed on Mathcad 15 for three points by collocation method. In test problem, we found exact solution by proposed method, however, it could be found by classical heat potential of single layer.

References

[1] A. Friedman, Free boundary problems for parabolic equations I. Melting of solids, J. Math. Mech. 8 (1959) 499–517
[2] R. Holm, Electric Contacts, (4-th edition), Springer Verlag, 1981.
[3] S.N. Kharin, H. Nouri, T. Davies, The mathematical models of welding dynamics in closed and switching electrical contacts, Proc. 49th IEEE Holm Conf. Electrical Contacts, Washington, USA, (2003) 107–123.
[4] S.N. Kharin, M. Sarsengeldin, Influence of contact materials on phenomena in a short electrical arc, Key Engineering Materials, Trans Tech Publications 510-511 (2012) 321–329.
[5] L.I. Rubinstein, The Stefan Problem, Transl. Math. Monogr. 27, AMS, Providence, RI, 1971.
[6] M.M. Sarsengeldin, A.S.Erdogan, T. Nauryz, H. Nouri, Exact solution of inverse spherical two phase Stefan problem, arXiv:1703.04946.
[7] M.M. Sarsengeldin, S. Guvercin, S. Kassabek, Solution of one phase inverse Stefan problem by Hartree functions method, J. Physics: Conference series (2014) 193–199.
[8] M.M. Sarsengeldin, S.N. Kharin, Method of the integral error functions for the solution of the one- and two-phase Stefan problems and its application, Filomat 31 (2017) 1017–1029.
[9] P.G. Slade, Electrical Contacts:Principles and Applications, CRC Press, 1999.
[10] D.A. Tarzia, A bibliography on moving-free boundary problems for the heat-diffusion equation. The Stefan and related problems, MAT-Ser. A 2 (2000) 1–297.