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Sufficient Conditions for Carathéodory Functions

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Abstract. In the present paper, we obtain several sufficient conditions for Carathéodory functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We also obtain sufficient conditions for p-valent or starlike functions. Moreover, we improve some results due to Nunokawa [Tsukuba J. Math. 13 (1989), 453–455] as some special cases of main results.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A} \equiv \mathcal{A}(1)$. A function $f \in \mathcal{A}(p)$ is called p-valent in $\mathbb{U}$ if f satisfies the following two conditions:

(i) for $w \in \mathbb{C}$, the equation $f(z) = w$ has at most p roots in $\mathbb{U}$;

(ii) there exists a $w_0 \in \mathbb{C}$ such that the equation $f(z) = w_0$ has exactly p roots in $\mathbb{U}$.

A function $f \in \mathcal{A}(p)$ is said to be p-valent starlike if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

If a function $f \in \mathcal{A}$ is 1-valent starlike, then it is called starlike. It is known that that p-valent starlike function in $\mathcal{A}(p)$ is p-valent.

Let $\mathcal{P}$ be the class of functions p which are analytic in the unit disk $\mathbb{U}$, with $p(0) = 1$ and $\Re\{p(z)\} > 0$ in $\mathbb{U}$. If $p \in \mathcal{P}$, then we say that p is a Carathéodory function. It is well-known that if $f \in \mathcal{A}$ with $f' \in \mathcal{P}$, then

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the function f is univalent in $\mathbb{U}$ (*cf.* [1, 10]). In 1935, Ozaki [9] extended the above result as follows: if f is analytic in a convex domain D and

$$\Re\left\{\exp(\mathrm{i}\alpha)f^{(p)}(z)\right\} > 0 \quad (z \in D), \tag{1}$$

where α is a real constant, then f is at most p-valent in D. This shows that if $f \in \mathcal{A}(p)$ with

$$\Re\left\{f^{(p)}(z)\right\} > 0 \quad (z \in \mathbb{U}),$$

then f is at most p-valent in $\mathbb{U}$. Nunokawa [3] (see also [4]) improved the above result to the following.

Theorem A ([3, Nunokawa]) *Let $p \geq 2$. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ is analytic in $\mathbb{U}$ and*

$$\left|\arg\left\{f^{(p)}(z)\right\}\right| < \frac{3}{4}\pi \quad (z \in \mathbb{U}),$$

then f is p-valent in $\mathbb{U}$.

Recently, Nunokawa et al. [6] found some sufficient conditions for function to be p-valent by improving Ozaki's condition given by (1). Also, in [7] and [8], Nunokawa and Sokół obtained another p-valent conditions by using geometric properties of functions in $\mathcal{A}(p)$.

The purpose of the present paper is to investigate some sufficient conditions for Carathéodory functions and to find some conditions for p-valent functions or starlike functions. And we improve Theorem A obtained by Nunokawa [3].

The following lemmas will be required for our results.

Lemma 1.1. ([5, Nunokawa]) *Let p be analytic in $\mathbb{U}$, $p(z) \neq 0$ in $\mathbb{U}$, $p(0) = 1$ and suppose that there exists a $z_0 \in \mathbb{U}$ such that*

$$\left|\arg p(z)\right| < \frac{\pi}{2}\alpha \quad \textit{for} \quad |z| < |z_0|$$

and

$$\left|\arg p(z_0)\right| = \frac{\pi}{2}\alpha, \quad \alpha > 0.$$

Then

$$\frac{z_0 p'(z_0)}{p(z_0)} = \mathrm{i}k\alpha,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right), \quad \textit{when} \quad \arg\{p(z_0)\} = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right), \quad \textit{when} \quad \arg\{p(z_0)\} = -\frac{\pi}{2}\alpha,$$

with

$$p(z_0)^{1/\alpha} = \pm\mathrm{i}a.$$

Lemma 1.2. ([2, Nunokawa]) *Let $f \in \mathcal{A}(p)$. If there exists a $(p-k+1)$-valent starlike function $g(z) = z^{p-k+1} + \sum_{n=p-k+2}^{\infty} b_n z^n$ that satisfies*

$$\Re\left\{\frac{zf^{(k)}(z)}{g(z)}\right\} > 0 \quad (z \in \mathbb{U}),$$

then f is p-valent in $\mathbb{U}$.

2. Main Results

Theorem 2.1. *Let p be analytic in $\mathbb{U}$, $p(z) \neq 0$ in $\mathbb{U}$, $p(0) = 1$ and suppose that*

$$\left|\arg\{p(z) + zp'(z) - \alpha\}\right| < \frac{\pi}{2} + \arctan(\sqrt{1+2\alpha}) \quad (z \in \mathbb{U}), \tag{2}$$

where $0 \leq \alpha < 1$. Then, we have

$$\left|\arg\{p(z)\}\right| < \frac{\pi}{2} \quad (z \in \mathbb{U}),$$

or

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

Proof. If there exists a point z_0 ($|z_0| < 1$) such that

$$|\arg\{p(z)\}| < \frac{\pi}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2},$$

then, by Lemma 1.1 with $\alpha = 1$, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \mathrm{i}k.$$

For the case $\arg\{p(z_0)\} = \pi/2$, $p(z_0) = \mathrm{i}a$ and $a > 0$, we have

$$\begin{aligned}
&\arg\{p(z_0) + z_0 p'(z_0) - \alpha\} \\
&= \arg\{p(z_0)\} + \arg\left\{1 + \frac{z_0 p'(z_0)}{p(z_0)} - \frac{\alpha}{p(z_0)}\right\} \\
&= \frac{\pi}{2} + \arg\left\{1 + \mathrm{i}k + \mathrm{i}\frac{\alpha}{a}\right\} \\
&\geq \frac{\pi}{2} + \arg\left\{1 + \frac{\mathrm{i}}{2}\left(a + \frac{1+2\alpha}{a}\right)\right\} \\
&\geq \frac{\pi}{2} + \arctan(\sqrt{1+2\alpha}),
\end{aligned}$$

which contradicts the hypothesis (2).
For the case $\arg\{p(z_0)\} = -\pi/2$, applying the same method as the above, we have

$$\arg\{p(z_0) + z_0 p'(z_0) - \alpha\} \leq -\left(\frac{\pi}{2} + \arctan(\sqrt{1+2\alpha})\right).$$

This also contradicts the hypothesis (2) and therefore, it completes the proof of Theorem 2.1. □

Example 2.2. Consider a function $p_1 : \mathbb{U} \to \mathbb{C}$ defined by

$$p_1(z) = -\frac{1}{z}\log(1-z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1}. \tag{3}$$

Then we have

$$p_1(z) + zp_1'(z) - \frac{1}{2} = \frac{1+z}{2(1-z)}.$$

Hence p_1 satisfies the condition (2) with $\alpha = 1/2$. Therefore, by Theorem 2.1, we have $\Re\{p_1(z)\} > 0$ in $\mathbb{U}$. Actually, the function p_1 satisfies that $\Re\{p_1(z)\} > \log 2 = 0.693147\cdots$ in $\mathbb{U}$ (See Figure 1 below)

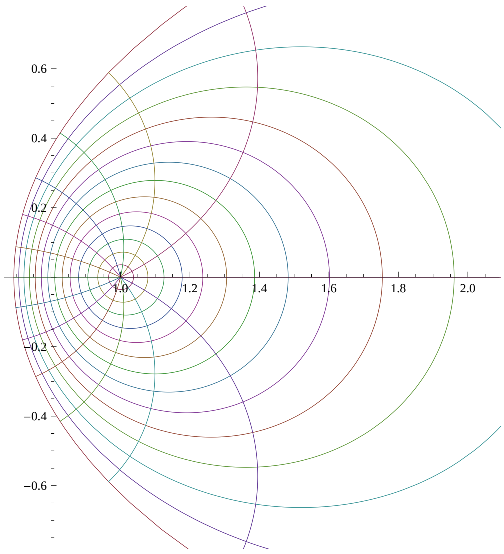

Figure 1: the image of p_1 on $\mathbb{U}$

Applying Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let $p \geq 2$. If $f \in \mathcal{A}(p)$ satisfies $f^{(p-1)} \neq 0$ in $\mathbb{U}$ and*

$$\left|\arg\left\{f^{(p)}(z) - \alpha \cdot p!\right\}\right| < \frac{\pi}{2} + \arctan(\sqrt{1+2\alpha}) \quad (z \in \mathbb{U}),$$

where $0 \leq \alpha < 1$, then f is p-valent in $\mathbb{U}$.

Proof. Let us put

$$p(z) = \frac{f^{(p-1)}(z)}{p!z}, \quad p(0) = 1.$$

Then it follows that

$$\begin{aligned} &\left|\arg\left\{p(z) + zp'(z) - \alpha\right\}\right| \\ &= \left|\arg\left\{\frac{f^{(p)}(z)}{p!} - \alpha\right\}\right| \\ &= \left|\arg\left\{f^{(p)}(z) - \alpha \cdot p!\right\}\right| \\ &< \frac{\pi}{2} + \arctan(\sqrt{1+2\alpha}). \end{aligned}$$

From Theorem 2.1, we have $\Re\{p(z)\} > 0$ in $\mathbb{U}$, or equivalently,

$$\Re\left\{\frac{f^{(p-1)}(z)}{z}\right\} > 0 \quad (z \in \mathbb{U}).$$

This shows that f is p-valent in $\mathbb{U}$. □

Example 2.4. Consider a function $f_1 : \mathbb{U} \to \mathbb{C}$ defined by

$$f_1(z) = 2[z + (1-z)\log(1-z)] = z^2 + \frac{1}{3}z^3 + \frac{1}{6}z^4 + \frac{1}{10}z^5 + \cdots.$$

Then, we have

$$\left|\arg\left\{f_1''(z)-1\right\}\right| = \left|\arg\left\{p_1(z)+zp_1'(z)-\frac{1}{2}\right\}\right| < \frac{\pi}{2},$$

where p_1 is the function defined by (3). Therefore, by Corollary 2.3 with $p = 2$ and $\alpha = 1/2$, the function f_1 is 2-valent in $\mathbb{U}$.

Remark 2.5. For the case $\alpha = 0$ in Corollary 2.3, we have Theorem A as aforementioned.

Theorem 2.6. *Let p be analytic in $\mathbb{U}$, $p(0) = 1$, $p(z) \neq 0$ in $\mathbb{U}$ and suppose that*

$$\left|\arg\left\{p(z)+\frac{zp'(z)}{p(z)}+\alpha\right\}\right| < \frac{\pi}{2} - \arctan\left(\frac{\alpha}{\sqrt{3}}\right) \quad (z \in \mathbb{U}), \tag{4}$$

where $0 \leq \alpha < \infty$. Then we have

$$\left|\arg\left\{p(z)\right\}\right| < \frac{\pi}{2} \quad (z \in \mathbb{U}).$$

Proof. If there exists a point z_0 ($|z_0| < 1$) such that

$$|\arg\{p(z)\}| < \frac{\pi}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2},$$

then, by Lemma 1.1 with $\alpha = 1$, we have

$$\frac{z_0p'(z_0)}{p(z_0)} = \mathrm{i}k.$$

For the case $\arg\{p(z_0)\} = \pi/2$, $p(z_0) = \mathrm{i}a$ and $a > 0$, we have

$$\begin{aligned}
&\arg\left\{p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}+\alpha\right\} \\
&= \arg\{p(z_0)\} + \arg\left\{1+\frac{z_0p'(z_0)}{p(z_0)^2}+\frac{\alpha}{p(z_0)}\right\} \\
&= \frac{\pi}{2} + \arg\left\{1+\frac{k}{a}-\mathrm{i}\frac{\alpha}{a}\right\} \\
&= \frac{\pi}{2} - \arctan\left\{\frac{\alpha}{a+k}\right\} \\
&\geq \frac{\pi}{2} - \arctan\left(\frac{\alpha}{\sqrt{3}}\right),
\end{aligned}$$

which contradicts the hypothesis (4).
For the case $\arg\{p(z_0)\} = -\pi/2$, applying the same method as the above, we have

$$\arg\left\{p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}+\alpha\right\} \leq -\left(\frac{\pi}{2} - \arctan\left(\frac{\alpha}{\sqrt{3}}\right)\right).$$

This also contradicts the hypothesis (4) and therefore, it completes the proof of Theorem 2.6. □

Corollary 2.7. *Let $f \in \mathcal{A}$ and suppose that*

$$\left|\arg\left\{1+\frac{zf''(z)}{f'(z)}+\alpha\right\}\right| < \frac{\pi}{2}-\arctan\left(\frac{\alpha}{\sqrt{3}}\right) \quad (z \in \mathbb{U}),$$

where $0 \leq \alpha < \infty$. Then f is starlike in $\mathbb{U}$.

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1.$$

Then it follows that

$$\begin{aligned} &\left|\arg\left\{p(z)+\frac{zp'(z)}{p(z)}+\alpha\right\}\right| \\ &= \left|\arg\left\{1+\frac{zf''(z)}{f'(z)}+\alpha\right\}\right| \\ &< \frac{\pi}{2}-\arctan\left(\frac{\alpha}{\sqrt{3}}\right). \end{aligned}$$

From Theorem 2.6, we have $\Re\{p(z)\} > 0$ in $\mathbb{U}$ and

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

This shows that f is starlike in $\mathbb{U}$. □

Theorem 2.8. *Let p be analytic in $\mathbb{U}$, $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$ and suppose that*

$$\Re\left\{\sqrt{p(z)+zp'(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{5}$$

Then we have

$$\left|\arg\{p(z)\}\right| < \frac{\pi}{2}\alpha_1 \quad (z \in \mathbb{U}),$$

where α_1 is the positive root of the equation

$$\alpha + \frac{2}{\pi}\arctan(\alpha) = 2 \tag{6}$$

and $1.39 < \alpha_1 < 1.40$.

Proof. If there exists a point z_0 ($|z_0| < 1$) such that

$$|\arg\{p(z)\}| < \frac{\pi}{2}\alpha_1 \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2}\alpha_1,$$

then, by Lemma 1.1 with $\alpha = \alpha_1$, we have

$$\frac{z_0p'(z_0)}{p(z_0)} = \mathrm{i}\alpha_1 k.$$

For the case $\arg p(z_0) = \pi\alpha_1/2$, we have

$$\begin{aligned}
&\arg\left\{\sqrt{p(z_0)+z_0p'(z_0)}\right\}\\
&= \frac{1}{2}\left(\arg\{p(z_0)\} + \arg\left\{1+\frac{z_0p'(z_0)}{p(z_0)}\right\}\right)\\
&= \frac{1}{2}\left(\frac{\pi}{2}\alpha_1 + \arg\{1+\mathrm{i}\alpha_1 k\}\right)\\
&\geq \frac{1}{2}\left(\frac{\pi}{2}\alpha_1 + \arctan(\alpha_1)\right)\\
&= \frac{\pi}{2},
\end{aligned}$$

which implies that

$$\Re\left\{\sqrt{p(z_0)+z_0p'(z_0)}\right\} \leq 0.$$

And this contradicts the hypothesis (5).

For the case $\arg p(z_0) = -\pi\alpha_1/2$, applying the same method as the above, we have

$$\arg\left\{\sqrt{p(z_0)+z_0p'(z_0)}\right\} \leq -\frac{1}{2}\pi, \quad \text{or} \quad \Re\left\{\sqrt{p(z_0)+z_0p'(z_0)}\right\} \leq 0.$$

This also contradicts the hypothesis (5) and therefore it completes the proof of Theorem 2.8. □

Example 2.9. Consider a function $p_2 : \mathbb{U} \to \mathbb{C}$ defined by

$$\begin{aligned}
p_2(z) &= \frac{5-z}{1-z} + \frac{4}{z}\log(1-z)\\
&= 1 + 2z + \frac{8}{3}z^2 + 3z^3 + \frac{16}{5}z^4 + \frac{10}{3}z^5 + \cdots .
\end{aligned}$$

A simple calculation leads us to the equation

$$p_2(z) + zp_2'(z) = \left(\frac{1+z}{1-z}\right)^2.$$

Therefore the function p_2 satisfy the inequality (5) and it follows from Theorem 2.8 that

$$\left|\arg\{p_2(z)\}\right| < \frac{\pi}{2}\alpha_1 \quad (z \in \mathbb{U}).$$

Let us put

$$\begin{aligned}
f(\theta) &:= \Re\left\{p_2\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}\\
&= 3 + 2\cos\theta\log(2-2\cos\theta) - 4\sin\theta\arctan\left(\frac{\sin\theta}{1-\cos\theta}\right) \quad (\theta \in (0,\pi))
\end{aligned}$$

and

$$\begin{aligned}
g(\theta) &:= \Im\left\{p_2\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}\\
&= \frac{3\sin\theta}{1-\cos\theta} - 2\sin\theta\log(2-2\cos\theta) - 4\cos\theta\arctan\left(\frac{\sin\theta}{1-\cos\theta}\right) \quad (\theta \in (0,\pi)).
\end{aligned}$$

Then we have

$$\left|\arg\left\{p_2\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}\right| \leq \left|\arg\left\{p_2\left(\mathrm{e}^{\mathrm{i}\theta_0}\right)\right\}\right| < 2.022 \quad (\theta \in (0,\pi)),$$

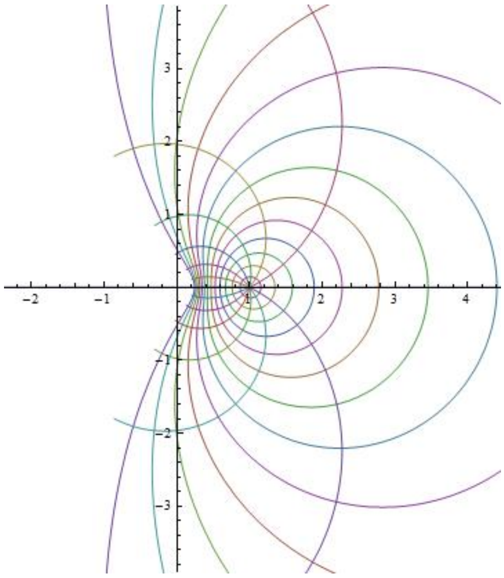

Figure 2: the image of p_2 on $\mathbb{U}$

where θ_0 $(0.804 < \theta_0 < 0.805)$ is the root of the equation $g'(\theta)f(\theta) = f'(\theta)g(\theta)$ (See figure 2 above). Thus, this implies that

$$\left|\arg\{p_2(z)\}\right| < \frac{\pi}{2}\alpha_1 \quad (z \in \mathbb{U}).$$

Applying Theorem 2.8, we have the following corollary.

Corollary 2.10. *Let $p \geq 4$. Let $f \in \mathcal{A}(p)$ satisfy $f^{(k)} \neq 0$ for $k = p-1, p-2$ and $p-3$ in $\mathbb{U}$. If*

$$\left|\arg\left\{f^{(p)}(z)\right\}\right| < \pi \quad (z \in \mathbb{U}),$$

then f is p-valent in $\mathbb{U}$.

Proof. Let us put

$$q_1(z) = \frac{f^{(p-1)}(z)}{p!z}, \quad q_1(0) = 1.$$

Then it follows that

$$q_1(z) + zq_1'(z) = \frac{f^{(p)}(z)}{p!}.$$

Applying Theorem 2.8, we have

$$\left|\arg\left\{\frac{f^{(p-1)}}{z}\right\}\right| = \left|\arg\{q_1(z)\}\right| < \frac{\pi}{2}\alpha_1 \quad (z \in \mathbb{U}), \tag{7}$$

where α_1 $(1.39 < \alpha_1 < 1.40)$ is the positive root of the equation given by (6).

Next, let us put

$$q_2(z) = \frac{2f^{(p-2)}}{p!z^2}, \quad q_2(0) = 1.$$

Then it follows that

$$2q_2(z) + zq_2'(z) = q_2(z)\left(2 + \frac{zq_2'(z)}{q_2(z)}\right) = \frac{2f^{(p-1)}}{p!z}.$$

Let α_2 be the positive root of the equation

$$\alpha + \frac{2}{\pi}\arctan\left(\frac{\alpha}{2}\right) = \alpha_1$$

and

$$1.08 < \alpha_2 < 1.09.$$

If there exists a point z_1, $|z_1| < 1$ such that

$$\left|\arg\{q_2(z)\}\right| < \frac{\pi}{2}\alpha_2 \quad \text{for} \quad |z| < |z_1|$$

and

$$\left|\arg\{q_2(z_1)\}\right| = \frac{\pi}{2}\alpha_2,$$

then we have

$$\frac{z_1 q_2'(z_1)}{q_2(z_1)} = \mathrm{i}\alpha_2 k.$$

For the case $\arg q_2(z_1) = \pi\alpha_2/2$, we have

$$\begin{aligned}
&\arg\left\{2q_2(z_1) + z_1q_2'(z_1)\right\} = \arg\left\{\frac{f^{(p-1)}(z_1)}{z_1}\right\}\\
&= \arg q_2(z_1) + \arg\left\{2 + \frac{z_1q_2'(z_1)}{q_2(z_1)}\right\}\\
&= \frac{\pi}{2}\alpha_2 + \arg\{2 + \mathrm{i}\alpha_2 k\}\\
&\geq \frac{\pi}{2}\alpha_2 + \arctan\frac{\alpha_2}{2} = \frac{\pi}{2}\alpha_1,
\end{aligned}$$

which contradicts (7)
For the case $\arg q_2(z_1) = -\pi\alpha_2/2$, we have

$$\begin{aligned}
&\arg\left\{2q_2(z_1) + z_1q_2'(z_1)\right\}\\
&= \arg\left\{\frac{2f^{(p-1)}(z_1)}{p!z_1}\right\} = \arg\left\{\frac{f^{(p-1)}(z_1)}{z_1}\right\}\\
&\leq -\frac{\pi}{2}\alpha_1.
\end{aligned}$$

This also contradicts (7) and therefore, we have

$$\left|\arg\{q_2(z)\}\right| = \left|\arg\left\{\frac{f^{(p-2)}(z)}{z^2}\right\}\right| < \frac{\pi}{2}\alpha_2 \quad (z \in \mathbb{U}),$$

where

$$\alpha_2 + \frac{2}{\pi}\arctan\frac{\alpha_2}{2} = \alpha_1$$

and

$$1.08 < \alpha_2 < 1.09.$$

Let

$$q_3(z) = \frac{6f^{(p-3)}(z)}{p!z^3}, \quad q_3(0) = 1.$$

Then it follows that

$$3q_3(z) + zq_3'(z) = \frac{6f^{(p-2)}(z)}{p!z^2}.$$

Applying the same method as the above, we have

$$\begin{aligned}
&\left|\arg\left\{3q_3(z) + zq_3'(z)\right\}\right| \\
&= \left|\arg\{q_3(z)\} + \arg\left\{3 + \frac{zq_3'(z)}{q_3(z)}\right\}\right| \\
&= \left|\arg\left\{\frac{6f^{(p-2)}(z)}{p!z^2}\right\}\right| \\
&= \left|\arg\left\{\frac{f^{(p-2)}(z)}{z^2}\right\}\right| \\
&< \frac{\pi}{2}\left(\alpha_3 + \frac{2}{\pi}\arctan\left(\frac{\alpha_3}{3}\right)\right) = \frac{\pi}{2}\alpha_2,
\end{aligned}$$

where

$$0.903 < \alpha_3 < 0.904.$$

This shows that

$$\left|\arg\left\{\frac{zf^{(p-3)}(z)}{z^4}\right\}\right| = \left|\arg\left\{\frac{f^{(p-3)}(z)}{z^3}\right\}\right| < \frac{\pi}{2}\alpha_3 < \frac{\pi}{2} \quad (z \in \mathbb{U}),$$

or

$$\Re\left\{\frac{zf^{(p-3)}(z)}{z^4}\right\} > 0 \quad (z \in \mathbb{U}). \tag{8}$$

It is trivial that $g(z) = z^4$ is 4-valent starlike function in $\mathbb{U}$. Therefore, from (8) and Lemma 1.2, we see that f is p-valent in $\mathbb{U}$. This completes our proof of Corollary 2.10. □

Remark 2.11. We remark that Corollary 2.10 improves Theorem A for the case $p \geq 4$.

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