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Summation by Riesz Means of the Fourier-Laplace Series

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Abstract. In this work, we investigate conditions for summability of the Fourier-Laplace series of integrable functions by Riesz means. The kernel of Riesz means is estimated through comparison with the Cesaro means. Properties of D and D^* points are required in obtaining this estimation.

1. Introduction

Consider S^N to be a unit sphere in R^{N+1}. In spherical coordinates, it has the parametrization $x = r \cdot \theta \in R^{N+1}$, where $r > 0$ represents a positive and θ an element of the unit sphere S^N. The Laplace operator can then be written as

$$\Delta f = \frac{1}{r^N}\frac{\partial}{\partial r}\left(r^N \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta_s f,$$

where Δ_s is the Laplace-Beltrami operator on S^N, known also as the spherical Laplacian. The spherical Laplacian of a function defined on $S^N \subset R^{N+1}$ can be computed as the ordinary Laplacian of the function extended to $R^{N+1} \setminus \{0\}$ so that it is constant along that rays, i.e., homogeneous of degree zero.

The Laplace-Beltrami operator has a complete system of eigenfunctions in $L_2(S^N)$:

$$\left\{Y_1^{(k)}(x), Y_2^{(k)}(x), ..., Y_{a_k}^{(k)}(x)\right\}, \quad k = 0, 1, 2, ...$$

The Laplace-Beltrami operator $-\Delta$ has a distinct set of eigenvalues denoted by $\lambda_0, \lambda_1, ...$ and arranged in increasing order with H_k as the eigenspace corresponding to λ_k. Elements of H_k are referred as spherical harmonics of degree k. It is well known (Stein and Weiss [14]) that $dim\ H_k = a_k$:

$$a_k = \begin{cases} 1, & if \quad k = 0, \\ N, & if \quad k = 1, \\ \frac{(N+k)!}{N!k!} - \frac{(N+k-2)!}{N!(k-2)!}, & if \quad k \geq 2. \end{cases} \tag{1}$$

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Let $\hat{A}$ be a self-adjoint extension of the Laplace-Beltrami operator $-\Delta_S$ in $L_2(S^N)$ and if E_λ is the corresponding spectral resolution, then for all functions $f \in L_2(S^N)$ we have

$$\hat{A}f = \int_0^\infty \lambda dE_\lambda f.$$

It is easy to check that E_λ coincides with partial sums of Foruer-Laplace series of f.

$$E_\lambda f(x) = \sum_{\lambda_k<\lambda} \sum_{j=1}^{a_k} f_{j,k} Y_j^{(k)}(x), \tag{2}$$

where $f_{j,k}$ are Fourier coefficients of the function $f \in L_2\left(S^N\right)$ and can represented as follows

$$f_{j,k} = \int\limits_{S^N} f(y) Y_j^{(k)}(y) d\sigma(y), \quad j = 1, 2, ..., a_k, \quad k = 0, 1, 2, ...$$

The integral operators

$$E_n^\alpha f(x) = \int\limits_{S^N} f(y) \Theta^\alpha(x, y, n) d\sigma(y), \tag{3}$$

are singular integrals with singularity at the point x. The behavior of such integrals depend on the local properties of $f(x)$. In the current work, it is shown that the convergence of singular integrals depend not only on x, but also on its diametrical opposite, x^*.

The main purpose of this work is to investigate convergence problems of the spectral expansions (2). The behavior of the latter corresponding to the Laplace-Beltrami operator is closely connected with the asymptotical behavior of the spectral function of the Laplace-Beltrami operator, which is defined by

$$\Theta(x, y, n) = \sum_{k=0}^{n} \sum_{j=1}^{a_k} Y_j^{(k)}(x) Y_j^{(k)}(y). \tag{4}$$

The spectral expansions of the function f can be rewritten as follows:

$$E_\lambda f(x) = \int\limits_{S^N} f(y) \Theta(x, y, \lambda) d\sigma(y). \tag{5}$$

The Riesz means of order $\alpha \geq 0$ of the spectral function $\Theta(x, y, \lambda)$:

$$\Theta^\alpha(x, y, \text{ł}) = \sum_{\lambda_k<\lambda} \left(1 - \frac{\lambda_k}{\lambda}\right)^\alpha \sum_{j=1}^{a_k} Y_j^{(k)}(x) Y_j^{(k)}(y), \tag{6}$$

will assist in defining the Riesz means of the spectral expansions (2) by

$$E_\lambda^s f(x) = \int\limits_{S^N} \Theta^s(x, y, \lambda) f(y) d\sigma(y).$$

Convergence problems of multiple trigonometric series and spectral decompositions studied by Sh.A. Alimov, V.A. Il'in and E.M. Nikishin in [3]. The summability problems of Fourier-Laplace series on the sphere described in the book [15] by L.V. Zhizhiashvili and S.B. Topuriya. In this work, we investigate conditions for summability of Fourier-Laplace series of integrable functions by Riesz means. Properties of D^* point are used in estimating the Riesz means kernel in comparison with Cesaro means. In cases related to spectral expansions of integrable functions on the unit sphere, problems of summability of the Cezaro means studied in the papers by Bonami and Clerc [4], Pulatov [8, 9], Meaney [7], Bastis [5] and Rakhimov [10–12]. The Riesz means of these expansions studied in the papers by Ahmedov [1, 2] and Rasedee [13].

2. Preliminaries

Let $\gamma(x, y)$ denote the spherical distance between x and y on S^N. By $D(x, h)$ we denote a spherical surface segment centered at the point $x \in S^N$ and spherical radius $0 < h \leq \pi$:

$$D(x, h) = \{y : y \in S^N, (x, y) > \cos h, 0 < h \leq \pi\}.$$

The surface of the latter is

$$|D(x, h)| = \frac{2\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)} \int_0^h \sin^{N-1} t dt.$$

If $\gamma(x, y) = \pi$, we refer y as the diametrically opposite point to x.

Definition 2.1. The point $x \in S^N$ is called a D- point of f if

$$\lim_{h\to 0} \frac{1}{h^N} \int_{D(x,h)} [f(y) - f(x)] d\sigma(y) = 0.$$

Definition 2.2. If $x \in S^N$ and its diametrical opposite $x^* \in S^N$ are D-points of f, then x is said to be a D^*-point of f.

Using Gegenbauer polynomials

$$P_k^{\left(\frac{N-1}{2}\right)}(t) = \frac{(-2)^k \Gamma\left(k + \frac{N-1}{2}\right) \Gamma(k + N - 1)}{\Gamma\left(\frac{N-1}{2}\right) \Gamma(2k + N - 1)} (1 - t^2)^{-\frac{N-2}{2}} \frac{d^k}{dt^k} \left[(1 - t^2)^{k + \frac{N-2}{2}}\right],$$

we can represent the Riesz means of the spectral function as follows:

$$\Theta^\alpha(x, y, n) = \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\alpha \frac{\Gamma\left(k + \frac{N-1}{2}\right) \Gamma(k + N - 1)}{\pi^{\frac{N+1}{2}}} P_k^{\frac{N-1}{2}}(\cos\gamma).$$

Since $\Theta^\alpha(x, y, n)$ depends only on the spherical distance between x and y, we can write : $\Theta^\alpha(x, y, n) = \Theta^\alpha(\gamma)$.
Consider the sequence of integral operators:

$$E_n^\alpha f(x) = \int_{S^N} f(y) \Theta^\alpha(x, y, n) d\sigma(y) = \int_{S^N} f(y) \Theta_n^\alpha(\gamma) d\sigma(y).$$

Lemma 2.3. *For $n \to \infty$, the function $\Theta_n^\alpha(\gamma)$ has the estimation*

$$\Theta_n^\alpha(\gamma) \leq \begin{cases} C & 0 \leq \gamma \leq \frac{1}{n}, \\ C \cdot \frac{1}{n^{\alpha - \frac{N-1}{2}}} \cdot \frac{1}{\left(\sin\frac{\gamma}{2}\right)^{\alpha+1} (\sin\gamma)^{\frac{N-1}{2}}}, & \frac{1}{n} < \gamma < \pi - \frac{1}{n}, \\ C \cdot n^{N-1-\alpha}, & \pi - \frac{1}{n} \leq \gamma \leq \pi, \end{cases} \tag{7}$$

for a constant C.

Proof. Use the following asymptotic formula (see for the Cezaro means in [6] and in [13] for the Riesz means): if $\left|\frac{\pi}{2} - \gamma(x,y)\right| < \frac{\pi n}{2(n+1)}$, $n \to \infty$, then

$$\Theta^{\alpha}(x,y,n) = n^{\frac{N-1}{2}-\alpha}(N-1)\frac{\sin\left[(n+N/2+\alpha/2)\gamma - \pi(N-1+2\alpha)/4\right]}{(2\sin\gamma)^{\frac{N-1}{2}}\left(2\sin\frac{\gamma}{2}\right)^{1+\alpha}} + n^{\frac{N-3}{2}-\alpha}\frac{\eta_n(\gamma)}{(\sin\gamma)^{\frac{N+1}{2}}\left(\sin\frac{\gamma}{2}\right)^{1+\alpha}} + \frac{\varepsilon_n(\gamma)}{(n+1)\left(\sin\frac{\gamma}{2}\right)^{1+N}},$$

where $|\eta_n(\gamma)| < C$ and $|\varepsilon_n(\gamma)| < C$.

Then for $0 \le \gamma \le \frac{1}{n}$, we have

$$\Theta^{\alpha}(\gamma) = O(1)\, n^{N}.$$

When $\pi - \frac{1}{n} \le \gamma < \pi$, we obtain

$$\Theta(\gamma) = O(1) n^{N-1-\alpha}, \quad \pi - \frac{1}{n} \le \gamma \le \pi.$$

Finally, if $\frac{1}{n} \le \gamma \le \pi - \frac{1}{n}$ then

$$\Theta^{\alpha}(\gamma) = \frac{O(1)\, n^{\frac{N-1}{2}-\alpha}}{(\sin\gamma)^{\frac{N-1}{2}}\left(\sin\frac{\gamma}{2}\right)^{1+\alpha}}. \tag{8}$$

Lemma 2.3 is proved. □

3. Main Result

The following is the main result of this research.

Theorem 3.1. *Let $f \in L_1\left(S^N\right)$, $N \ge 2$. If $\alpha > \frac{N-1}{2}$, then*

$$\lim_{n\to\infty} E_n^{\alpha} f(x) = f(x),$$

for every D^* - point of the function f.

Proof. The Riesz means of the spectral expansion of the Laplace-Beltrami operator can be represented in the form of the following integral operator:

$$E_n^{\alpha} f(x) = \int_{S^N} \Theta^{\alpha}(x,y,n) f(y)\, d\sigma(y),$$

with the kernel

$$\Theta^{\alpha}(x,\ y,\ n) = \sum_{k=0}^{n}\left(1 - \frac{\lambda_k}{\lambda_n}\right)^{\alpha} \sum_{j=1}^{a_k} Y_j^{(k)}(x) Y_j^{(k)}(y).$$

For the kernel $\Theta^{\alpha}(x,y,n) = \Theta_n^{\alpha}(\gamma)$ we have (7).

Consider the difference

$$E_n^{\alpha} f(x) - f(x) = \int_{S^N} [f(y) - f(x)]\Theta_n^{\alpha}(\gamma)\, d\sigma(y),$$

which can be written as

$$E_n^\alpha f(x) - f(x) = \int_0^\pi \Theta_n^\alpha(\gamma) \int_{(x,y)=\cos\gamma} [f(y) - f(x)] dt(y).$$

Writing

$$\psi_x(\gamma) = \frac{1}{\left|S^{N-1}\right| (\sin\gamma)^{N-1}} \int_{(x,y)=\cos\gamma} [f(y) - f(x)]\, dt(y), \tag{9}$$

where $f(y)$ are elements of the surface

$$\left\{y : y \in S^N : (x, y) = \cos\gamma,\ 0 < \gamma < \pi\right\},$$

we have

$$E_n^\alpha f(x) - f(x) = \left|S^{N-1}\right| \int_0^\pi \Theta_n^\alpha(\gamma)\psi_x(\gamma)(\sin\gamma)^{N-1}\, d\gamma.$$

For any $\delta \in (0, \pi)$ the latter integral can be divided as follows:

$$\left\{\int_0^\delta + \int_\delta^{\pi-\delta} + \int_{\pi-\delta}^\pi\right\} \Theta_n^\alpha(\gamma)\psi_x(\gamma)(\sin\gamma)^{N-1}\, d\gamma = I_1 + I_2 + I_3.$$

Let us estimate I_1:

$$I_1 = \int_0^\delta \Theta_n^\alpha(\gamma)\psi_x(\gamma)(\sin\gamma)^{N-1}\, d\gamma.$$

Any large n can be expressed as the following (we fix $n > \frac{1}{\delta}$)

$$I_1 = \int_0^{\frac{1}{n}} \Theta_n^\alpha(\gamma)\psi_x(\gamma)\left(\sin\gamma\right)^{N-1}\, d\gamma + \int_{\frac{1}{n}}^\delta \Theta_n^\alpha(\gamma)\psi_x(\gamma)\left(\sin\gamma\right)^{N-1}\, d\gamma = I_1' + I_1''.$$

Using $\Theta_n^\alpha(\gamma) = O\left(n^N\right)$ for

$$I_1' = \int_0^{\frac{1}{n}} \Theta_n^\alpha(\cos\gamma)\psi_x(\gamma)\left(\sin\gamma\right)^{N-1}\, d\gamma,$$

gives

$$\left|I_1'\right| \leq C \int_0^{\frac{1}{n}} n^N |\psi_x(\gamma)| \left(\sin\gamma\right)^{N-1}\, d\gamma.$$

To estimate the inequality directly above, we have the following lemma.

Lemma 3.2. *Let* $f(t)(\sin t)^{N-1} \in L_1(0, \pi)$ *and*

$$M(h) = \sup_{0<h\leq\pi} \left\{\frac{1}{h^N}\left|\int_0^h f(t)(\sin t)^{N-1}\, dt\right|\right\} < \infty, \tag{10}$$

then for any non-negative decreasing function $g \in L_1(0,\pi)$, we have

$$\left|\int_0^h g(t)f(t)(\sin t)^{N-1}\,dt\right| \le NM(h)\int_0^h t^{N-1}g(t)\,dt. \tag{11}$$

Proof. Let $0 < \kappa < h$. On the closed interval $[\kappa, h]$, the function $g(t)$ is bounded and the integral

$$\int_\kappa^h f(t)g(t)\sin^{N-1} t\,dt \tag{12}$$

exists. Denoting $F(t)$ by

$$F(t) = \int_0^t f(u)\sin^{N-1} u\,du,$$

(12) can be expressed as the following Stieltjes integral:

$$\int_\kappa^h f(t)g(t)\sin^{N-1} t\,dt = \int_\kappa^h g(t)\,dF(t).$$

Use of integration by parts gives

$$\int_\kappa^h f(t)g(t)\sin^{N-1} t\,dt = [F(h)g(h) - F(\kappa)g(\kappa)] + \int_\kappa^h F(t)\,d[-g(t)].$$

Without the loss of generality, we can assume $g(h) = 0$. This gives us

$$\int_\kappa^h f(t)g(t)\sin^{N-1} t\,dt = -F(\kappa)g(\kappa) + \int_\kappa^h F(t)\,d[-g(t)].$$

$$|F(t)| = \left|\int_0^t f(u)\sin^{N-1} u\,du\right| =$$

$$\le t^N \sup_{0<t<\pi}\left\{\frac{1}{t^N}\left|\int_0^t f(u)\sin^{N-1} u\,du\right|\right\} = M(h)t^N.$$

This gives the inequality

$$|F(t)| \le M(h)t^N. \tag{13}$$

Let us estimate

$$g(\kappa)\kappa^N = g(\kappa)N\int_0^\kappa t^{N-1}dt.$$

As $g(t)$ is decreasing,

$$g(\kappa)N\int_0^\kappa t^{N-1}dt \le N\int_0^\kappa t^{N-1}g(t)dt,$$

hence

$$g(\kappa)\kappa^N \leq N\int_0^\kappa t^{N-1}g(t)dt. \tag{14}$$

From the latter we obtain:

$$|F(\kappa)g(\kappa)| \leq M(h)\kappa^N g(\kappa) \leq M(h)N\int_0^\kappa t^{N-1}g(t)\,dt.$$

As $-g(t)$ is increases (13) gives

$$\left|\int_\kappa^h F(t)\,d[-g(t)]\right| \leq M(h)\int_\kappa^h t^N\,d[-g(t)].$$

Then

$$\int_\kappa^h t^N\,d[-g(t)] = g(\kappa)\kappa^N + N\int_\kappa^h g(t)t^{N-1}\,dt.$$

together with (14) gives

$$\left|\int_\kappa^h t^N\,d[-g(t)]\right| \leq \left[N\int_0^\kappa t^{N-1}g(t)dt + N\int_\kappa^h t^{N-1}g(t)\,dt\right].$$

By combining the above, we obtain

$$\left|\int_\kappa^h f(t)g(t)\sin^{N-1}tdt\right| \leq M(h)N\left\{\int_0^\kappa t^{N-1}g(t)dt + \int_\kappa^h t^{N-1}g(t)dt\right\}. \tag{15}$$

Under the assumption $g(h) = 0$, the limit h can be replaced by β, where $\kappa < \beta < h$. And allowing κ and β tend to 0, is can be seen that

$$\lim\int_\kappa^\beta f(t)g(t)\sin^{N-1}t\,dt = 0,$$

which proves the existence of (12). As $\kappa \to 0$ in (15) we obtain (11), which completes the proof of the lemma. □

Resuming with main estimation, from Lemma 3.2 we obtain

$$I_1' \leq C\int_0^{\frac{1}{n}} t^{N-1}n^N\,dt = C\,n^N\frac{t^N}{N}\bigg|_0^{\frac{C}{n}} = \frac{1}{N}$$

$$\leq C\sup_{0<h\leq\pi}\left\{\frac{1}{h^N}\left|\int_0^{\frac{1}{n}}\psi_x(\gamma)\,(\sin\gamma)^{N-1}\;d\gamma\right|\right\} < C\varepsilon,$$

by using the property of D^*-point.

For the second part of the equation,

$$I_1'' = \int_{\frac{1}{n}}^{\delta} \Theta_n^\alpha(\gamma)|\psi_x(\gamma)| (\sin\gamma)^{N-1} \; d\gamma.$$

Applying $\left|\Theta_n^\alpha(\cos\gamma)\right| \le \frac{C}{n^{\alpha-\frac{N-1}{2}}} \frac{1}{\sin\left(\frac{\gamma}{2}\right)^{\alpha+1}(\sin\gamma)^{\frac{N-1}{2}}}, \;\; \gamma \in (\frac{1}{n}, \delta]$ gives the following estimate

$$\left|I_1''\right| \le \frac{C}{n^{\alpha-\frac{N-1}{2}}} \int_{\frac{1}{n}}^{\delta} \frac{1}{\left(\sin\frac{\gamma}{2}\right)^{\alpha+1}(\sin\gamma)^{\frac{N-1}{2}}} |\psi_x(\gamma)|(\sin\gamma)^{N-1} \, d\gamma.$$

By Lemma 3.2,

$$\left|I_1''\right| \;\le\; M(\delta)\frac{C}{n^{\alpha-\frac{N-1}{2}}} \int_{\frac{1}{n}}^{\delta} \frac{\gamma^{N-1}}{\left(\frac{\gamma}{2}\right)^{\alpha+1}\gamma^{\frac{N-1}{2}}} d\gamma \le CM(\delta)\left((\delta n)^{\frac{N-1}{2}-\alpha}+1\right), \quad \alpha > \frac{N-1}{2}.$$

It is clear that $\lim_{n\to\infty} I_2 = 0$. This follows from

Lemma 3.3. *Let $\{\varphi_n(t)\} \in L_1[a,b]$. If there exists $K > 0$, such that $\left|\varphi_n(t)\right| < K$, $\forall n$, $\forall t \in [a,b]$ and for any $c \in [a,b]$.*

$$\lim_{n\to\infty} \int_a^c \varphi_n(t)\, dt = 0,$$

then for any $f \in L_1[a,b]$

$$\lim_{n\to\infty} \int_a^b f(t)\varphi_n(t)\, dt = 0.$$

Using the fact that $\lim \int_{\delta}^{\pi-\delta} \Theta_n^\delta(\gamma)\, d\gamma = 0$, I_3 is given by the following estimation:

$$I_3 = \int_{\pi-\delta}^{\pi} \Theta_n^\alpha(\gamma) \int_{(x,y)=\cos\gamma} [f(y) - f(x)]dt(y),$$

which can be written as

$$\begin{aligned} I_3 &= \int_{\pi-\delta}^{\pi} \Theta_n^\alpha(\gamma)d\gamma \int_{(x,y)=\cos\gamma} [f(y) - f(x^*)]dt(y) \;+\; \int_{\pi-\delta}^{\pi} \Theta_n^\alpha(\gamma) \int_{(x,y)=\cos\gamma} [f(x^*) - f(x)]dt(y) \\ &= \int_{\pi-\delta}^{\pi} \Theta_n^\alpha(\gamma)d\gamma \int_{(x,y)=\cos\gamma} [f(y) - f(x^*)]dt(y) \;+\; [f(x^*) - f(x)] \int_{\pi-\delta}^{\pi} \Theta_n^\alpha(\gamma)d\gamma \int_{(x,y)=\cos\gamma} dt(y) \\ &= I_3' + I_3'', \end{aligned}$$

where

$$\lim_{n\to\infty} I_3'' = 0.$$

Next, we proceed with estimating I_3' by

$$I_3' = \int_{\pi-\delta}^{\pi} \Theta_n^{\alpha}(\gamma)\, d\gamma \int_{(x,y)=\cos\gamma} [f(y) - f(x^*)]\, dt(y) = \int_{\pi-\delta}^{\pi} \Theta_n^{\alpha}(\gamma)\psi_{x^*}(\gamma)(\sin\gamma)^{N-1}\, d\gamma.$$

The limis of integration can rewritten as

$$\begin{aligned} I_3' &= \int_0^{\delta} \Theta_n^{\alpha}(\cos(\pi-\tau))\psi_{x^*}(\tau)(\sin\tau)^{N-1}\, d\tau \\ &= \int_0^{\frac{1}{n}} \Theta_n^{\alpha}(\pi-\tau)\psi_{x^*}(\tau)(\sin\tau)^{N-1}\, d\tau + \int_{\frac{1}{n}}^{\delta} \Theta_n^{\alpha}(\pi-\tau)\psi_{x^*}(\tau)(\sin\tau)^{N-1}\, d\tau. \end{aligned}$$

I_3' is then estimated as the following inequality:

$$\begin{aligned} \left|I_3'\right| \le & \int_0^{\frac{1}{n}} Cn^{N-1-\alpha}|\psi_{x^*}(\tau)|(\sin\tau)^{N-1}\, d\tau \\ & + \int_{\frac{1}{n}}^{\delta} \frac{C'}{n^{\alpha-\frac{N-1}{2}}} \frac{1}{\left(\sin\frac{\tau}{2}\right)^{\alpha+1}(\sin\tau)^{\frac{N-1}{2}}} |\psi_{x^*}(\tau)|(\sin\tau)^{N-1}\, d\tau = A + B, \end{aligned}$$

where it is divided into two integrals by choosing $\delta > \frac{1}{n}$.

From Lemma 3.2, the first integral of the inequality can be written as follows:

$$A = C\int_0^{\frac{1}{n}} n^{N-1-\alpha}\tau^{N-1}\, d\tau = Cn^{N-1-\alpha} \left.\frac{\tau^N}{N}\right|_0^{\frac{1}{n}} = \frac{Cn^{-1-\alpha}}{N},$$

where $n^{-1-\alpha} \to 0$ when $n \to \infty$.

Finally, the second integral is estimated by

$$\begin{aligned} B &= \int_{\frac{1}{n}}^{\delta} \frac{C'}{n^{\alpha-\frac{N-1}{2}}} \frac{1}{\left(\sin\frac{\tau}{2}\right)^{\alpha+1}(\sin\tau)^{\frac{N-1}{2}}} |\psi_{x^*}(\tau)|(\sin\tau)^{N-1}\, d\tau \\ &\le M(\delta)\int_{\frac{1}{n}}^{\delta} \frac{C}{n^{\alpha-\frac{N-1}{2}}} \frac{\tau^{N-1}}{\left(\frac{\tau}{2}\right)^{\alpha+1}\tau^{\frac{N-1}{2}}}\, d\tau \\ &= \frac{M(\delta)n^{\frac{N-1}{2}-\alpha}}{\frac{N-1}{2}-\alpha}\left(\delta^{-\alpha+\frac{N-1}{2}} - \left(\frac{1}{n}\right)^{-\alpha+\frac{N-1}{2}}\right) \\ &\le CM(\delta)\left((\delta n)^{\frac{N-1}{2}-\alpha} + 1\right), \end{aligned}$$

for $\alpha > \frac{N-1}{2}$ and as $n \to \infty$, $(\delta n)^{\frac{N-1}{2}-\alpha} \to 0$.

This completes the proof of Theorem 3.1. □

4. Conclusion

Conditions for summability are established in the class of integrable functions. The convergence of integrable functions of the Fourier-Laplace series at one point is dependant not only on the behavior of the function at a given point, but also its diametrical opposite.

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