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Height of Prime Hyperideals in Krasner Hyperrings

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Abstract. Extending the notion of dimension of a hyperring, we introduce the concept of height of a prime hyperideal of a hyperring, similarly as in ring theory, and we present some basic properties and relations with the dimension notion. In the second part of the article we illustrate some results concerning the height of prime hyperideals in Noetherian/Artinian hyperrings.

1. Introduction

Hyperrings, multirings or superrings are all hyperstructures endowed with two binary (hyper)operations, namely the addition and the multiplication, satisfying similar properties. For the first time, hyperrings have been introduced by Krasner [9] in connection with the theory of valued fields, like hypercompositional algebraic structures with the addition being a hyperoperation, while the multiplication remains a binary operation. The additive hyperstructure was defined like a canonical hypergroup and the multiplicative structure as a semigroup. This is, till now, the most well known and studied type of hyperrings, called Krasner hyperring. If the addition is a binary operation and the multiplication a binary hypercomposition, then we talk about the multiplicative hyperrings defined by Rota [17], that lately attracted the attention of several researchers (for example, see the papers published in 2014 [2, 5, 8]). Moreover, in 1973 Mittas [14] introduced the superrings, where both, the addition and the multiplication, are hyperoperations, with the additive hyperstructure being a canonical hypergroup. If we consider the additive hyperstructure as an arbitrary hypergroup, we obtain a new kind of hyperring, the most general one, investigated by Spartalis [18] in the context of P-hyperrings. In a multiring, introduced and studied for the first time by Marshall [10], the additive part is a canonical hypergroup, the multiplicative one a commutative monoid with the absorbing element 0, while just the left distributivity holds. Thus it is clear that a Krasner hyperring is a multiring, where the right distributivity holds, too. But, as shown in [10], there are many interesting examples of multirings that are not hyperrings. For a general overview of hyperring and hyperfield theory till the 90's, it is worth to read the expository paper of Nakassis [15], while a recent survey on the hyperring theory, completed with a series of applications, is provided in the book [4] of Davvaz and Leoreanu-Fotea.

Like in the ring theory, one may define the hyperideals, as a natural generalizations of ideals (and therefore, in particular, Jacobson radical and nilradical based on the notions of prime and maximal hyperideals), Artinian and Noetherian hyperrings, as well as the dimension of a hyperring. All these concepts

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are well explained and described, together with suitable examples, in the PhD thesis recently defended by N. Ramaruban [16] to the University of Cincinnati, and most of them are recalled in Section 2 of this note. The central notion of this paper, the height of a prime hyperideal, is presented in Section 3, while Section 4 is dedicated for the study of it in Noetherian/Artinian hyperrings (see Theorem 4.10). The paper ends with Section 5, including some conclusions and new lines of research.

2. Preliminaries

In this section, we gather some results and definitions related to hyperideals of hyperrings, which will be used in the next sections. For more details regarding this theme, the readers are refereed to [4].

Throughtout this paper, unless otherwise stated, R denotes a Krasner hyperring, called by short hyperring.

Definition 2.1. ([4]) A subhyperring I of a hyperring R is a *left* (respectively *right*) hyperideal of R, if $r \cdot a \in I$ (respectively $a \cdot r \in I$), for all $r \in R$ and $a \in I$. It is called a *hyperideal* of R if it is both a left and a right hyperideal of R.

A proper hyperideal M of R is called a *maximal* hyperideal of R, if the only hyperideals of R that contain M are M itself and R.

A proper hyperideal P of a hyperring R is called a *prime* hyperideal of R if, for every pair of hyperideals A and B of R such that $AB \subseteq P$, it follows that $A \subseteq P$ or $B \subseteq P$.

It is well known that, in a commutative unitary hyperring, there exists a maximal hyperideal of R containing I, for any proper hyperideal I of R. Moreover, in a such hyperring, each maximal hyperideal is a prime hyperideal, thereby there exists at least one prime hyperideal in R.

The intersection of all maximal hyperideals of R is called the *Jacobson radical* of R and denoted by $J(R)$, while the intersection of all prime hyperideals of R is called the *nilradical* of R, denoted by $\mathcal{N}(R)$.

A nonzero hyperring R having a unique maximal hyperideal is called a *local hyperring*.

The *radical* of a hyperideal I of a hyperring R, denoted by $r(I)$, is defined as $r(I) = \{x \mid x^n \in I, \quad \text{for some } n \in \mathbb{N}\}$. It can be proved that, the radical of I is the intersection of all prime hyperideals of R containing I.

Definition 2.2. A prime hyperideal P of R is said to be *minimal prime hyperideal over a hyperideal I of R* if it is minimal among all prime hyperideals of R containing I. A prime hyperideal P is said to be *minimal prime hyperideal* if it is a minimal prime hyperideal over the zero hyperideal of R.

Theorem 2.3. ([4]) *Let R be a Krasner hyperring, $a \in R$ and $X \subset R$.*

(*i*) *The principal hyperideal $< a >$ generated by a is equal to the set*

$$\{t \mid \ t \in ra + as + na + k(a-a) + \sum_{i=1}^{m} r_i a s_i, \ r, s, r_i, s_i \in R, m \in \mathbb{Z}^+ \text{ and } n, k \in \mathbb{Z}\}.$$

(*ii*) *If R has a unit element, then*

$$< a >= \{t \mid \ t \in k(a-a) + \sum_{i=1}^{m} r_i a s_i, \ r_i, s_i \in R, m, k \in \mathbb{Z}^+\}.$$

(*iii*) *If a is in the center of R, then*

$$< a >= \{t \mid \ t \in ra + na + k(a-a), r \in R, n \in \mathbb{Z}^+\},$$

where the center of R is the set $\{x \in R \mid xy = yx, \ for \ all \ y \in R\}$.

(*iv*) *$Ra = \{ra \mid r \in R\}$ is a left hyperideal in R and $aR = \{ar \mid r \in R\}$ is a right hyperideal in R. If R has a unit element, then $a \in aR \cap Ra$.*

(*v*) *If R has a unit element and a is in the center of R, then* $Ra =< a >= aR$.

(*vi*) *If R has a unit element and X is included in the center of R, then*

$$< X >= \{t \mid t \in \sum_{i=1}^{m} r_i x_i,\ r_i \in R, x_i \in X, m \in \mathbb{Z}^+\}.$$

In the following we recall the concepts of extension and contraction of hyperideals and some of their properties.

Definition 2.4. ([16]) Let $f : R \to S$ be a hyperring homomorphism, I be a hyperideal of R and J be a hyperideal of S.

(*i*) The hyperideal $< f(I) >$ of S generated by the set $f(I)$ is called the *extension* of I and it is denoted by I^e. Explicitly, we have

$$< f(I) >= \{x \in S \mid x \in \sum_{i=1}^{n} f(a_i)b_i,\ where\ a_i \in I;\ b_i \in S;\ n \in \mathbb{N}\}.$$

(*ii*) The hyperideal $f^{-1}(J) = \{a \in R \mid f(a) \in J\}$ is called the *contraction* of J and it is denoted by J^c. It is known that, if J is a prime hyperideal in S, then J^c is a prime hyperideal in R.

Proposition 2.5. ([16]) *Let* $f : R \to S$ *be a hyperring homomorphism, I and J be hyperideals of R and S, respectively. Then it follows that:*

(*i*) $I \subseteq I^{ec}$ *and* $J \supseteq J^{ce}$.

(*ii*) $J^c = J^{cec}$ *and* $I^e = I^{ece}$.

(*iii*) *If I is a prime hyperideal of R, then it is the contraction of a prime hyperideal of S if and only if* $I^{ec} = I$.

Starting with a hyperring R and defining an external operation, one gets a new hyperstructure, called R-hypermodule.

Definition 2.6. ([16]) Let R be a hyperring. A *left R-hypermodule M* is a commutative hypergroup with respect to addition, together with a map $R \times M \to M$, given by $(r, m) \mapsto r \cdot m = rm \in M$, such that for all $a, b \in R$ and $m_1, m_2 \in M$ we have:

i) $(a + b)m_1 = am_1 + bm_1$;

ii) $a(m_1 + m_2) = am_1 + am_2$;

iii) $(ab)m_1 = a(bm_1)$;

iv) $a0_M = 0_R m_1 = 0_M$, where $0_M, 0_R$ are the zero elements of M, R respectively;

v) $1m_1 = m_1$, where 1 is the multiplicative identity in R.

If R is a hyperfield, then M is called a *hypervectorspace*.

Definition 2.7. ([16]) A left R-hypermodule M is called *finitely generated*, if there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of M such that $M = \{z \mid \exists r_1, \ldots, r_n \in R, n \in \mathbb{N}\ such\ that\ z \in \sum_{i=1}^{n} r_i x_i\}$. The set $\{x_1, x_2, \ldots, x_n\}$ is called the *generating set*.

The next result has been investigated for the first time in the framework of hyperrings by Zahedi et al. [22].

Theorem 2.8. ([16, 22]; Nakayama's Lemma) *Let M be a finitely generated R-hypermodule and I a hyperideal of R contained by the Jacobson radical of R. Then $M = IM$ implies that $M = \{0\}$.*

Corollary 2.9. ([16]) *Let M be a finitely generated R-hypermodule, N a subhypermodule of M, and I a hyperideal contained by the Jacobson radical of R. Then $M = IM + N$ implies that $M = N$.*

In the following we recall the construction of the *hyperring of fractions* [3]. Let R be any hyperring and let S be any multiplicatively closed subset of R with $1 \in S$. Define a relation $\equiv$ on $R \times S$ by $(a, s) \equiv (b, t)$ if and only if $0 \in (at - bs)u$, for some $u \in S$. Denote the equivalence class of (a, s) with $\frac{a}{s}$ and let $S^{-1}R$ denote the set of all equivalence classes. We endow the set $S^{-1}R$ with a hyperring structure, by defining the addition and the multiplication between fractions as follows:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \quad and \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

We know that $S^{-1}R$ forms a hyperring under these operations.

Remark 2.10. If P is a prime hyperideal of a hyperring R, then $S = R \setminus P$ is a multiplicatively closed subset of R. In this case, we denote $S^{-1}R = R_P$. As proved in [16], the elements $\frac{a}{s}$, with $a \in P$, form a hyperideal M in R_P, that is the only maximal hyperideal of R_P; therefore R_P is a local hyperring.

The main properties of the hyperring of fractions $S^{-1}R$ are grouped in the following result.

Proposition 2.11. ([16]) *Let S be a multiplicatively closed subset of a hyperring R.*

i) *Every hyperideal in $S^{-1}R$ is an extended hyperideal.*

ii) *If I is a hyperideal in R, then $I^e = S^{-1}R$ if and only if $I \cap S = \emptyset$.*

iii) *A hyperideal I is a contracted hyperideal of R if and only if no element of S is a zero divisor in R/I.*

iv) *The prime hyperideals of $S^{-1}R$ are in one-to-one correspondence with the prime hyperideals of R that don't meet S, with the correspondence given by $P \leftrightarrow S^{-1}P$.*

Similarly, one constructs the *hypermodule of fractions*. Let M be an R-hypermodule and S be a multiplicatively closed subset of R. Define a relation $\equiv$ on $M \times S$ by $(m, s) \equiv (m_1, s_1)$ if and only if there exists $t \in S$ such that $0 \in t(ms_1 - m_1 s)$, that is $ms_1 t = m_1 st$. This is clearly an equivalence relation. Let $\frac{m}{s}$ denote the equivalence class of the pair (m, s), and let $S^{-1}M$ denote the set of all such fractions. Then $S^{-1}M$ is an $S^{-1}R$-hypermodule.

If P is a prime hyperideal of R and M is an R-hypermodule, then the R_P-hypermodule $(R \setminus P)^{-1}M$ is simply denoted by M_P.

We conclude this section with some definitions regarding the primary hyperideals and the primary decompositions of hyperideals.

Definition 2.12. ([16]) A hyperideal Q in a hyperring R is called *primary* if $Q \neq R$ and if whenever $xy \in Q$ either $x \in Q$ or $y^n \in Q$, for some $n \in \mathbb{N}$. If $P = r(Q)$, then Q is said to be a *P-primary hyperideal* of R.

It is clear that every prime hyperideal is also primary.

Definition 2.13. ([16]) A *primary decomposition* of a hyperideal I in the hyperring R is an expression of I as a finite intersection of primary hyperideals, say $I = \bigcap_{i=1}^{n} Q_i$, where each Q_i is primary. If, moreover

i) the radicals $r(Q_i)$ are all distinct, and

ii) $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$, $1 \leq i \leq n$,

then the primary decomposition is called *minimal*. We say that any hyperideal I of R is *decomposable*, if it has a primary decomposition.

Theorem 2.14. ([16]; First Uniqueness Theorem) *Let I be a decomposable hyperideal of R and let $I = \bigcap_{i=1}^{n} Q_i$ be a minimal primary decomposition of I. Let $P_i = r(Q_i)$, $1 \leq i \leq n$. Then P_i are precisely the prime hyperideals which occur in the set of hyperideals $r(I : x)$, $x \in R$, and hence are independent of the particular decomposition of I. The notation $(I : x)$ means the quotient hyperideal $\{a \in R \mid ax \subset I\}$.*

Definition 2.15. ([16]) The prime hyperideals P_i in Theorem 2.14 are said *to belong to I*. The minimal elements of the set $\{P_1, P_2, \ldots, P_n\}$ are called the *minimal or isolated prime hyperideals* of I. The others are called *embedded prime hyperideals*.

3. Height of a Prime Hyperideal

In this section, we define the height of a prime hyperideal of a hyperring, presenting connections with the notion of dimension of a hyperring.

Definition 3.1. Let R be a non-trivial commutative hyperring.

(*i*) An expression of the type

$$P_0 \subset P_1 \subset \ldots \subset P_n$$

(note the strict inclusion), where $P_0, \ldots, P_n$ are prime hyperideals of R, is called a *chain* of prime hyperideals of R; the length of such a chain is the number of the "links" between the terms of the chain, that is, 1 less than the number of prime hyperideals in the sequence. Thus the above displayed chain has length n. Note that, for a prime hyperideal P, we consider P to be a chain, with just one prime hyperideal of R, of length 0. Since R is non-trivial, it contains at least one prime hyperideal, so there certainly exists at least one chain of prime hyperideals of R of length 0.
The supremum of the lengths of all chains of prime hyperideals of R is called the *dimension* of R, denoted by $dim(R)$.

(*ii*) A chain of the type

$$P_0 \subset P_1 \subset \ldots \subset P_n$$

of prime hyperideals of R is said to be *saturated* precisely when, for every $i \in \mathbb{N}$, with $1 \leq i \leq n$, there is no prime hyperideal P such that $P_{i-1} \subset P \subset P_i$, that is, if and only if we cannot make a chain of length $n+1$ by inserting an additional prime hyperideal of R between two terms in the chain.

(*iii*) A chain of the type

$$P_0 \subset P_1 \subset \ldots \subset P_n$$

of prime hyperideals of R is said to be *maximal*, when it is saturated, P_n is a maximal prime hyperideal of R and P_0 is a minimal prime hyperideal of the zero hyperideal 0 of R.

Example 3.2. The dimension of a hyperfield is 0. Indeed, let R be a non-trivial hyperring. It is known that, R is a hyperfield if and only if the only hyperideals of R are 0 and R itself, so it contains just one chain of prime hyperideals of length 0.

Example 3.3. The dimension of a non-trivial Artinian hyperring is 0, since in a such hyperring each prime hyperideal is also maximal.

In the following example, we first recall Krasner's construction of hyperrings and hyperfields from rings and fields, respectively, and then, based on it, we get a hyperring of dimension 1.

Example 3.4. ([16]) If $(R, +, \cdot)$ is a ring and G a subset of R such that $(G, \cdot)$ is a group, then we can define an equivalence relation $\cong$ on R as follows: $r \cong s$ if and only if $rG = sG$. The equivalence class represented by r is $P(r) = \{s \in R \mid sG = rG\} = rG$. Define now a hyperoperation $\oplus$ on the set of all equivalence classes R/G as follows: $P(r) \oplus P(s) = \{P(t) \mid P(t) \cap (P(r) + P(s)) \neq \emptyset\} = \{tG \mid \exists g_1, g_2 \in G \text{ such that } t = rg_1 + sg_2\} = \{tG \mid tG \subseteq rG + sG\}$, and define a binary operation on R/G as $rG \cdot sG = rsG(P(r) \cdot P(s) = P(rs))$. Then $(R/G, \oplus, \cdot)$ forms a hyperring. Moreover, if we choose R to be a field, then we get that $(R/G, \oplus, \cdot)$ is a hyperfield.

Now, consider the set of integers $\mathbb{Z}$ and its multiplicative subgroup $G = \{-1, 1\}$. The Krasner construction $\frac{\mathbb{Z}}{G}$ is a principal hyperideal hyperdomain, i.e. it is a hyperring with no zero divisors and whose hyperideals are generated by a single element. Also, the prime (also maximal) hyperideals of $\frac{\mathbb{Z}}{G}$ are of the form $< pG >$, where p is a prime number. In $\frac{\mathbb{Z}}{G}$, we have $< 0\mathbb{Z} > \subset < 2\mathbb{Z} >$, a chain of prime hyperideals of length 1. Since every nonzero prime hyperideal of $\frac{\mathbb{Z}}{G}$ is maximal, there does not exist a chain of prime hyperideals of $\frac{\mathbb{Z}}{G}$ of length 2, therefore $dim\frac{\mathbb{Z}}{G} = 1$.

Definition 3.5. Let P be a prime hyperideal of a non-trivial commutative hyperring R. The *height* of P, denoted by $ht_R P$, is defined to be the supremum of the lengths of all chains

$$P_0 \subset P_1 \subset \ldots \subset P_n$$

of prime hyperideals of R, for which $P_n = P$, if this supremum exists, and it is ∞, otherwise.

Example 3.6. Suppose that $\mathbb{R}[x, y]$ is the ring of polynomials of two variables over the field of real numbers and $G = \{-1, 1\}$ the set of constant polynomials -1 and 1. Let $R = \mathbb{R}[x, y]/G$ be the hyperring obtained with Krasner's construction in Example 3.4. Since $< x > /G$ and $< x, y > /G$ are prime hyperideals of R and $< x > /G \subseteq < x, y > /G$, it follows that $ht_R(< x, y > /G) = 1$. Similarly, in the hyperring $R = \mathbb{R}[x_1, x_2, \ldots, x_n]/G$, since $< x_1 > /G$, $< x_1, x_2 > /G$,..., $< x_1, x_2, \ldots, x_n > /G$ are all prime hyperideals of R with

$$(< x_1 > /G) \subseteq (< x_1, x_2 > /G) \subseteq (< x_1, x_2, ..., x_n > /G),$$

we conclude that $ht_R(< x_1, x_2, ..., x_n > /G) = n - 1$, for any positive integer n.

Lemma 3.7. *Let P be a prime hyperideal of the commutative hyperring R and I be a hyperideal of R such that $I \subseteq P$. Then the set*

$$\Theta = \{P' \mid P' \textit{is a prime hyperideal and } I \subseteq P' \subseteq P\}$$

has a minimal element with respect to the inclusion.

Proof. Using Zorn's lemma, the proof is straightforward. □

Note that a minimal element of Θ is a minimal prime hyperideal over I, and so we deduce that there exists a minimal prime hyperideal P'' over I, with $P'' \subseteq P$.

Remark 3.8. Let R be a non-trivial commutative hyperring. By Corollary 3.9 [16], every prime hyperideal of R is contained in a maximal hyperideal of R (and every maximal hyperideal is prime). Moreover, every prime hyperideal of R contains a minimal prime hyperideal. It follows that $dimR$ is equal to the supremum of lengths of chains $P_0 \subset P_1 \subset \ldots \subset P_n$ of prime hyperideals of R, with P_n maximal and P_0 a minimal prime hyperideal. Indeed, if we have an arbitrary chain of prime hyperideals of R with length h, like: $P'_0 \subset P'_1 \subset \ldots \subset P'_h$, then it is bounded above by the length of a special chain as it follows. If P'_0 is not a minimal prime hyperideal, then another prime hyperideal can be inserted before it. On the other hand, if P'_h is not a maximal hyperideal of R, then another prime hyperideal can be inserted above it. Thus, if $dimR$ is finite, then

$$dimR = sup\{ht_R M \mid M \textit{ is a maximal hyperideal of } R\}$$

$$= sup\{ht_R P \mid P \textit{ is a prime hyperideal of } R\}$$

From Remark 3.8 it clearly follows the following assertion.

Corollary 3.9. *If R is a local commutative hyperring with the maximal hyperideal M, then $dimR = ht_R M$.*

Theorem 3.10. *Let S be a multiplicatively closed subset of R and P be a prime hyperideal of R such that $P \cap S = \emptyset$. Then $ht_R P = ht_{S^{-1}R} S^{-1}P$.*

Proof. By Proposition 2.11, it follows that $S^{-1}P$ is a prime hyperideal of $S^{-1}R$. Let

$$P_0 \subset P_1 \subset \ldots \subset P_n = P$$

be a chain of prime hyperideals of R. Again by Proposition 2.11, it follows that

$$P_0^e \subset P_1^e \subset \ldots \subset P_n^e$$

is a chain of prime hyperideals of $S^{-1}R$ with $P_n^e = P^e = S^{-1}P$, and therefore $ht_RP \leq ht_{S^{-1}R}S^{-1}P$. On the other side, if

$$Q_0 \subset Q_1 \subset \ldots \subset Q_n$$

is a chain of prime hyperideals of $S^{-1}R$ with $Q_n = P^e$, then using Propositions 2.11 and 2.5, we get that

$$Q_0^c \subset Q_1^c \subset \ldots \subset Q_n^c$$

is a chain of prime hyperideals of R with $Q_n^c = P^{ec} = P$. So we have $ht_{S^{-1}R}S^{-1}P \leq ht_RP$. Therefore, it follows that $ht_RP = ht_{S^{-1}R}S^{-1}P$. □

Combining the previous results, we get now the following important consequence.

Corollary 3.11. *For a prime hyperideal P of a commutative hyperring R, it follows that*

$$ht_RP = ht_{R_P}S^{-1}P = dim(R_P).$$

Proof. According to Remark 2.10, we know that $R_P = S^{-1}R$ is a local hyperring with the maximal hyperideal $S^{-1}P$. Hence, by Corollary 3.9 and Theorem 3.10, it follows that $ht_RP = ht_{R_P}S^{-1}P = dim(R_P)$. □

Theorem 3.12. *Let R be a commutative hyperring and I be a hyperideal of R. Then the dimension $dim(R/I)$ of the quotient hyperring R/I is equal to the supremum of the lengths of chains $P_0 \subset P_1 \subset \ldots \subset P_n$ of prime hyperideals of R, where all P_i, $0 \leq i \leq n$, contain I, if this supremum exists, and it is ∞, otherwise.*

Proof. It is well known that there exists a biunivocal correspondence between chains of prime hyperideals in a quotient hyperring of R and chains of prime hyperideals of R. In particular, there is a biunivocal correspondence between chains of prime hyperideals of the quotient hyperring R/I, where I is a proper hyperideal of R, and chains of prime hyperideals of R containing I. It means that a chain of prime hyperideals of R/I has the form

$$P_0/I \subset P_1/I \subset \ldots \subset P_n/I,$$

where $P_0 \subset P_1 \subset \ldots \subset P_n$ is a chain of prime hyperideals of R, with $I \subseteq P_0$.
Now we can conclude that, $dim(R/I)$ is equal to the supremum of lengths of chains $P_0 \subset P_1 \subset \ldots \subset P_n$ of prime hyperideals of R, where all P_i, $0 \leq i \leq n$, contain I. □

Theorem 3.13. *Let R be a commutative hyperring and P be a prime hyperideal of R. Then*

$$ht_RP + dim(R/P) \leq dimR.$$

Proof. First, we notice that, if $P_0 \subset P_1 \subset \ldots \subset P_n$ is a chain of prime hyperideals of R, with $P_n = P$, and $P'_0 \subset P'_1 \subset \ldots \subset P'_h$ is another chain of prime hyperideals of R, with $P'_0 = P$, then

$$P_0 \subset P_1 \subset \ldots \subset P_n \subset P'_1 \subset P'_2 \subset \ldots \subset P'_h$$

is a chain of prime hyperideals of R of length $n + h$. Based on this note and on Theorem 3.12, it follows that $ht_RP + dim(R/P) \leq dimR$. □

4. Height of Prime Hyperideals in Noetherian/Artinian Hyperrings

After recalling some fundamental results concerning Noetherian/Artinian hyperrings, we present some auxiliary results regarding the prime hyperideal and we conclude with one theorem that gives some information about the height of a minimal prime hyperideal over a principal hyperideal in a Noetherian hyperring.

Definition 4.1. ([16]) A hyperring R is said to be *Noetherian* if it satisfies the *ascending chain condition* on hyperideals of R: for every ascending chain of hyperideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ there is $N \in \mathbb{N}$ such that $I_n = I_N$, for every natural number $n \geq N$ (this is equivalent with saying that, every ascending chain of hyperideals has a maximal element).

Definition 4.2. ([16]) A hyperring R is said to be *Artinian* if it satisfies the *descending chain condition* on hyperideals of R: for every descending chain of hyperideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ there is $N \in \mathbb{N}$ such that $I_n = I_N$, for every natural number $n \geq N$ (this is equivalent with saying that, every descending chain of hyperideals has a minimal element).

Corollary 4.3. ([16]) *Let R be a hyperring in which the zero hyperideal is a product $M_1M_2\ldots M_n$ of (not necessarily distinct) maximal hyperideals. Then R is Noetherian if and only if it is Artinian.*

Proposition 4.4. ([16]) *In a Noetherian hyperring R, the following assertions hold.*

i) *Every hyperideal has a primary decomposition.*

ii) *Every hyperideal contains a power of its radical.*

iii) *The nilradical is nilpotent.*

Proposition 4.5. ([16]) *Let R be an Artinian hyperring.*

i) *Every prime hyperideal of R is maximal.*

ii) *R has only a finite number of maximal hyperideals.*

A connection between Noetherian and Artinian hyperrings is expressed in the following result.

Theorem 4.6. ([16]) *A hyperring R is Artinian if and only if it is Noetherian and $dim(R) = 0$.*

Lemma 4.7. *Let R be a commutative Noetherian hyperring and let P be a minimal prime hyperideal over a proper hyperideal I of R. Let S be a multiplicatively closed subset of R such that $P \cap S = \emptyset$. Then $S^{-1}P$ is a minimal prime hyperideal over the hyperideal $S^{-1}I$ of $S^{-1}R$.*

Proof. We use the extension and contraction notations together with the natural hyperring homomorphism $R \to S^{-1}R$. By Proposition 2.11, we know that $S^{-1}P$ is a prime hyperideal of $S^{-1}R$ and $S^{-1}I \subseteq S^{-1}P$.

Suppose that $S^{-1}P$ is not a minimal prime hyperideal over $S^{-1}I$. Then, by Lemma 3.7, there exists a prime hyperideal Q' of $S^{-1}R$ such that $S^{-1}I \subseteq Q' \subsetneq S^{-1}P$. Again, according with Proposition 2.11, there exists a prime hyperideal Q of R such that $Q \cap S = \emptyset$ and $S^{-1}Q = Q^e = Q'$. Now using Proposition 2.5, we obtain that

$$I \subseteq I^{ec} \subseteq (Q')^c = Q^{ec} = Q \subset P^{ec} = P,$$

meaning that there exists another prime hyperideal Q of R over I, contrary to the fact that P is a minimal prime hyperideal over I. Thus, the supposition is false. □

Lemma 4.8. *Let I be a decomposable hyperideal of a commutative hyperring R and P be a prime hyperideal of R. Then $I \subseteq P$ and P is a prime hyperideal, which is a minimal (with respect to inclusion) prime hyperideal containing I among all prime hyperideals of R if and only if P is a minimal prime hyperideal over I.*
In particular, the minimal prime hyperideal P, which contains I, belongs to I.

Proof. Let

$$I = Q_1 \cap Q_2 \cap \ldots \cap Q_n,\ with\ r(Q_i) = P_i,\ i = 1, 2, \ldots, n$$

be a minimal primary decomposition of I, where all hyperideals P_i belong to I. Hence $I \subseteq P$ if and only if $P = r(P) \supseteq r(I) = \cap_{i=1}^n r(Q_i) = \cap_{i=1}^n P_i$, equivalently with $P_j \subseteq P$, for some j, $1 \leq j \leq n$.

Now suppose that P is minimal among all prime hyperideals that contain I. Then by the above argument, $P' \subseteq P$, for some P' that belongs to I. But P is minimal with respect to inclusion, thus $P = P'$.

Conversely, suppose that P is a minimal prime hyperideal over I. Then there exists a minimal (with respect to inclusion) prime hyperideal P' containing I and $P' \subseteq P$. Again, by the above argument, there exists P'' that belongs to I such that $P'' \subseteq P'$. Then $P'' \subseteq P' \subseteq P$. Since P is a minimal prime hyperideal over I, we must have $P = P' = P''$. Therefore $P = P'$ is a minimal prime hyperideal containing I. □

Theorem 4.9. *Let R be a commutative Noetherian hyperring in which every prime hyperideal is maximal. Then*

i) *R contains a finite number of maximal hyperideals.*

ii) *R is an Artinian hyperring.*

Proof. *i*) Let M be a maximal hyperideal of R. Since every prime hyperideal of R is maximal, M must be a minimal prime hyperideal containing 0 and based on Lemma 4.8, M belongs to 0. Hence

$$\{prime\ hyperideals\ of\ R\} \subseteq \{prime\ hyperideals\ belonging\ to\ 0\} \subseteq \{prime\ hyperideals\ of\ R\}.$$

Therefore $\{prime\ hyperideals\ belonging\ to\ 0\} = \{prime\ hyperideals\ of\ R\}$ and it is a finite set, because R is a Noetherian hyperring.

ii) Let $M_1, M_2, \ldots, M_n$ be all maximal hyperideals of R (that are also minimal prime hyperideals of R). It follows that $r(0) = \bigcap_{i=1}^n M_i$. By Proposition 4.4, the zero hyperideal contains a power of its radical, so there exists $t \in \mathbb{N}$ such that $r(0)^t = 0$. Hence

$$M_1^t M_2^t \ldots M_n^t \subseteq (\bigcap_{i=1}^{n} M_i)^t = r(0)^t = 0,$$

meaning that $M_1^t M_2^t \ldots M_n^t R = 0$. Since R is a Noetherian hyperring, according with Corollary 4.3, we deduce that R is also an Artinian hyperring. □

Based on all these auxiliary results, we are now in the position to prove main theorem of this section, similarly to the Krull's principal ideal theorem in rings theory.

Theorem 4.10. *Let R be a commutative Noetherian hyperring and let $a \in R$ be a non-unit element. Let P be a minimal prime hyperideal over the principal hyperideal $< a >$ of R. Then $ht_R P \leq 1$.*

Proof. Take $S = R \setminus P$ as a multiplicatively closed subset of R. Then by Remark 2.10, $R_P = S^{-1}R$ is a local hyperring with the maximal hyperideal $S^{-1}P$. Since P is a minimal prime hyperideal over $< a >$, it follows that $S^{-1}P$ is a minimal prime hyperideal of $< a > R_P = S^{-1} < a >$. On the other hand, we know by Corollary 3.11 that $ht_{R_P} S^{-1}P = ht_R P$.

It is therefore enough to prove this theorem, assuming the additional hypotheses that R is a local hyperring, having M as the unique maximal hyperideal, and $P = M$.

Suppose, by absurd, that $ht_R M > 1$. Thereby there exists a chain $Q' \subset Q \subset M$ of prime hyperideals of R of length 2. Since M is a minimal prime hyperideal over $< a >$ and also the unique maximal hyperideal of R, it follows that $M/ < a >$ is the unique prime hyperideal of $R/ < a >$. Therefore, by Theorem 4.9 part *ii*), the hyperring $R/ < a >$ is an Artinian local hyperring.

By using the extension and contraction notations together with the natural hyperring homomorphism $R \to S^{-1}Q = R_Q$, it is easy to see that $(Q^n)^{ec}$ is a Q-primary hyperideal of R and, for each $n \in \mathbb{N}$, we have $(Q^{n+1})^{ec} \subseteq (Q^n)^{ec}$. Hence

$$(Q^1)^{ec} + < a > / < a > \supseteq (Q^2)^{ec} + < a > / < a > \supseteq \ldots \supseteq (Q^n)^{ec} + < a > / < a > \supseteq \ldots$$

is a descending chain of hyperideals in the Artinian hyperring $R/ < a >$. Thus there exists $m \in \mathbb{N}$ such that $(Q^m)^{ec}+ < a >= (Q^{m+1})^{ec}+ < a >$.

Now set $r \in (Q^m)^{ec}$; using Theorem 2.3, we have $r = s + ac$, for some $s \in (Q^{m+1})^{ec}$ and $c \in R$. Then $ac = r - s \in (Q^m)^{ec}$, which is a Q-primary hyperideal of R. Since $a \notin Q$ and M is a minimal prime hyperideal over $< a >$, it follows that $c \in (Q^m)^{ec}$. We get then that

$$(Q^m)^{ec} = (Q^{m+1})^{ec} + a(Q^m)^{ec}.$$

Hence, by Corollary 2.9, we have $(Q^m)^{ec} = (Q^{m+1})^{ec}$. Now if we turn back to the hyperring R_Q and use Proposition 2.5 and the property of the extension saying that $(I_1I_2)^e = I_1^eI_2^e$, we obtain that

$$(Q^e)^m = (Q^m)^e = (Q^m)^{ece} = ((Q^m)^{ec})^e = (Q^{m+1})^{ec})^e = (Q^e)^{m+1}.$$

Using now Nakayama's Lemma, we conclude that $(Q^e)^m = 0$.

Thus, in the local hyperring R_Q, the maximal hyperideal Q^e is nilpotent, so it is clear that Q^e is contained in every prime hyperideal of R_Q. But this contradicts the fact that $Q'^e \subset Q^e$ is a chain of prime hyperideals of R_Q. So the assumption is faulse, meaning that $ht_RP \leq 1$. □

5. Conclusions and Future Works

Theory of hyperrings has its origin in the years 50's, when the French mathematician Mark Krasner [9] used them as a tool in the theory of approximation of valued fields. Later on, the Greek school represented by Stratigopoulos [20] and Mittas [13, 14] have initiated the general study of these algebraic hyperstructures, investigated later on and nowadays also by Massouros [11], Spartalis [18, 19], Vougiouklis [21], Jančić-Rašović [1, 6, 7], Mirvakili et al. [12], etc.

As for the corresponding algebraic structures-the rings, in this paper we define and present several properties of the height of a prime hyperideal in a Krasner hyperring. The height of a proper prime hyperideal P of a hyperring R is the maximum (or ∞ if a such number does not exist) of the lengths of the chains of distinct prime hyperideals contained in P. After presenting some results connecting the notions of dimension of a hyperring and the height of a prime hyperideal, our study has foccused on prime hyperideals in Noetherian (Artinian) hyperrings, concluded with a generalization of the Krull's principal ideal theorem in rings theory.

Our future work will include new results regarding the height of a hyperideal, besides them an extension of Theorem 4.10 in the case of a finitely generated hyperideal with more than one generator.

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