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A New Type of q-Szász-Mirakjan Operators

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Abstract. In this paper we introduce a new q-Szász-Mirakjan operator based on a new q-exponential function. We derive various formulae for the moments, prove the uniform convergence of the sequence of operators to the identity operator on compact intervals and show a Voronovskaja type result.

1. Introduction

Throughout this paper we use the notations that Thomas Ernst introduced in [8]. Assume that $q \in \mathbb{R}^+ \backslash \{1\}$. The q-analogues of a real number a and of an integer n are defined by

$$\{a\}_q := \frac{1-q^a}{1-q},\ a \in \mathbb{R}, \tag{1}$$

$$\{n\}_q := \begin{cases} \sum_{k=1}^{n} q^{k-1} = 1+q+q^2+\cdots+q^{n-1} & \text{if } \ n \in \mathbb{N}^* \\ 0 & \text{if } \ n = 0 \end{cases}. \tag{2}$$

Jan Cieśliński [6] introduced another q-integer, which is motivated by the form of the q-exponential $\mathcal{E}_q^x$ defined below

$$[n]_q := \begin{cases} \{n\}_q \frac{2}{1+q^{n-1}} = \frac{1-q^n}{1-q} \cdot \frac{2}{1+q^{n-1}} & \text{if } \ n \in \mathbb{N}^* \\ 0 & \text{if } \ n = 0 \end{cases}. \tag{3}$$

The corresponding q-factorials are defined by

$$\{n\}_q! := \begin{cases} \prod_{k=1}^{n} \{k\}_q = (1+q)(1+q+q^2)\cdots(1+\cdots+q^{n-1}) & \text{if } \ n \in \mathbb{N}^* \\ 1 & \text{if } \ n = 0 \end{cases}, \tag{4}$$

$$[n]_q! := \begin{cases} \prod_{k=1}^{n} [k]_q = \prod_{k=1}^{n} \left(\{k\}_q \frac{2}{1+q^{k-1}}\right) = \{n\}_q! \frac{2^n}{\prod_{k=0}^{n-1}(1+q^k)} & \text{if } \ n \in \mathbb{N}^* \\ 1 & \text{if } \ n = 0 \end{cases}. \tag{5}$$

We assume from now on that $q \in (0,1)$. In the standard approach to q-calculus there are two exponential functions

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{\{j\}_q!},\ |x| < \frac{1}{1-q} \tag{6}$$

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and

$$E_q^x = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{x^j}{\{j\}_q!},\ x \in \mathbb{R}. \tag{7}$$

Both q-exponential functions can be represented by infinite products

$$e_q^x = \prod_{k=0}^{\infty} \frac{1}{1 - q^k(1-q)x}, \quad E_q^x = \prod_{k=0}^{\infty} (1 + q^k(1-q)x). \tag{8}$$

A new q-exponential function $\mathcal{E}_q^x$ is defined by, see [6],

$$\mathcal{E}_q^x := e_q^{\frac{x}{2}} E_q^{\frac{x}{2}} = \prod_{k=0}^{\infty} \frac{1 + q^k(1-q)\frac{x}{2}}{1 - q^k(1-q)\frac{x}{2}} \tag{9}$$

and it can be represented as

$$\mathcal{E}_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!},\ |x| < \frac{2}{1-q}. \tag{10}$$

It was proved in [6] that $\mathcal{E}_q^x$ has better qualitative properties than the standard q-exponential functions. Therefore, we will use $\mathcal{E}_q^x$ to introduce a new q-analogue of the Szász-Mirakjan operators.

We recall that the Szász-Mirakjan operators have the following form

$$S_n(f)(x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{11}$$

for $n \geq 1$, $x \geq 0$ and all functions $f : [0, +\infty) \to \mathbb{R}$ such that the series at the right-hand side is absolutely convergent, see [3].

Remark 1.1. *This function space includes in particular the function subspace of all $f : [0, +\infty) \to \mathbb{R}$ such that $|f(x)| \leq Me^{\alpha x}$, where $x \geq 0$, $M \geq 0$ and $\alpha \in \mathbb{R}$, see [3].*

Recently more q-analogues of these operators were introduced, see for example [4], [11] and [10]. We introduce the following definition of the q-Szász-Mirakjan operator.

Definition 1.2. *Let $f \in C[0, \infty)$ be a function. Then*

$$S_{n,q}(f)(x) = \frac{1}{\mathcal{E}_q^{[n]_q x}} \sum_{k=0}^{\infty} \frac{[n]_q^k x^k}{[k]_q!} f\left(\frac{\{k\}_q}{[n]_q}\right), \tag{12}$$

where $n \in \mathbb{N}^*$, $0 < q < 1$ and $0 \leq x < \frac{2}{(1-q)[n]_q} = \frac{2}{1-q} \cdot \frac{(1-q)(1+q^{n-1})}{2(1-q^n)} = \frac{1+q^{n-1}}{1-q^n}$.

Remark 1.3. *The interval fixed for the variable x is chosen such that the q-exponential $\mathcal{E}_q^{[n]_q x}$ is convergent.*

Remark 1.4. *If we calculate*

$$\frac{\{k\}_q}{[n]_q} = \frac{1-q^k}{1-q} \cdot \frac{1-q}{1-q^n} \cdot \frac{1+q^{n-1}}{2} = \frac{1-q^k}{2} \cdot \frac{1+q^{n-1}}{1-q^n}, \tag{13}$$

then we note that

$$0 \leq \frac{\{k\}_q}{[n]_q} < \frac{1+q^{n-1}}{1-q^n},\ \forall\, k \in \mathbb{N}, \tag{14}$$

which means that the interval fixed for the variable x also suits the domain of the function f.

We set

$$s_{nk}(q;x) = \frac{1}{\mathcal{E}_q^{[n]_q x}} \cdot \frac{[n]_q^k x^k}{[k]_q!}, \tag{15}$$

which has the following properties

$$\sum_{k=0}^{\infty} s_{nk}(q;x) = \frac{1}{\mathcal{E}_q^{[n]_q x}} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q!} = 1, \tag{16}$$

$$s_{nk}(q;x) \geq 0,\ \forall\, n \in \mathbb{N}^*,\ \forall\, 0 < q < 1,\ \forall\, 0 \leq x < \frac{2}{(1-q)[n]_q}. \tag{17}$$

From (16) it follows that $S_{n,q}(\mathbf{1}) = \mathbf{1}$, where $\mathbf{1}$ is the constant function with constant value 1.

Remark 1.5. *As their classical analogues, the q-Szasz-Mirakjan operators $S_{n,q}$ are also linear and positive.*

2. Moments

In this section we compute the moments $S_{n,q}(t^m)(x)$, $m = 0, 1, 2 \ldots$. We begin to state some recurrence formulae useful in the sequel.

We consider the following test functions

$$e_m(t) = t^m, \tag{18}$$

where $m \in \mathbb{N}$.

For the reader's convenience we recall formula (13) from [6]

$$\mathcal{E}_q^{qz} = \frac{1-(1-q)\frac{z}{2}}{1+(1-q)\frac{z}{2}} \mathcal{E}_q^z.$$

Lemma 2.1. *We consider $n \in \mathbb{N}^*$, $m \in \mathbb{N}$ and $0 < q < 1$. Then the following simple recurrence formulae hold true*

$$S_{n,q}(t^{m+1})(x) = \frac{1}{[n]_q(1-q)}\left[S_{n,q}(t^m)(x) - \frac{1-(1-q)\frac{[n]_q x}{2}}{1+(1-q)\frac{[n]_q x}{2}} \cdot S_{n,q}(t^m)(xq)\right],$$

$$S_{n,q}(t^{m+1})(x) = \sum_{j=0}^{m} \binom{m}{j} \frac{xq^j}{2[n]_q^{m-j}}\left[S_{n,q}(t^j)(x) + \frac{1-(1-q)\frac{[n]_q x}{2}}{1+(1-q)\frac{[n]_q x}{2}} \cdot S_{n,q}(t^j)(xq)\right]$$

$$S_{n,q}(t^{m+1})(x) = \frac{1}{2\mathcal{E}_q^{[n]_q x}} \sum_{j=0}^{m} \binom{m}{j} \frac{x}{[n]_q^{m-j}} [\mathcal{E}_q^{[n]_q x q^{m-j}} \cdot S_{n,q}(t^j)(xq^{m-j}) +$$

$$+\mathcal{E}_q^{[n]_q x q^{m-j+1}} \cdot S_{n,q}(t^j)(xq^{m-j+1})].$$

Proof We have

$$S_{n,q}(t^{m+1})(x) = \frac{1}{\mathcal{E}_q^{[n]_q x}} \sum_{k=0}^{\infty} \frac{[n]_q^k \cdot x^k}{[k]_q!} \cdot \frac{\{k\}_q^{m+1}}{[n]_q^{m+1}} = \frac{1}{\mathcal{E}_q^{[n]_q x}} \sum_{k=0}^{\infty} \frac{[n]_q^k \cdot x^k}{[k]_q!} \cdot \frac{\{k\}_q^m}{[n]_q^m} \cdot \frac{1}{[n]_q} \cdot \frac{1-q^k}{1-q} =$$

$$= \frac{1}{\mathcal{E}_q^{[n]_q x}} \cdot \frac{1}{[n]_q(1-q)}\left(\sum_{k=0}^{\infty} \frac{[n]_q^k \cdot x^k}{[k]_q!} \cdot \frac{\{k\}_q^m}{[n]_q^m} - \sum_{k=0}^{\infty} \frac{[n]_q^k \cdot (xq)^k}{[k]_q!} \cdot \frac{\{k\}_q^m}{[n]_q^m}\right) =$$

$$= \frac{1}{[n]_q(1-q)}\left[\frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{k=0}^{\infty}\frac{[n]_q^k\cdot x^k}{[k]_q!}\cdot\frac{\{k\}_q^m}{[n]_q^m}-\frac{\mathcal{E}_q^{[n]_q xq}}{\mathcal{E}_q^{[n]_q x}}\cdot\frac{1}{\mathcal{E}_q^{[n]_q xq}}\sum_{k=0}^{\infty}\frac{[n]_q^k\cdot(xq)^k}{[k]_q!}\cdot\frac{\{k\}_q^m}{[n]_q^m}\right]=$$

$$= \frac{1}{[n]_q(1-q)}\left[S_{n,q}(t^m)(x)-\frac{1-(1-q)\frac{[n]_q x}{2}}{1+(1-q)\frac{[n]_q x}{2}}\cdot S_{n,q}(t^m)(xq)\right]$$

The second recurrence formula is derived by means of the following formula

$$\{k\}_q = 1+q\cdot\{k-1\}_q = 1+q\cdot\frac{1-q^{k-1}}{1-q}=\frac{1-q^k}{1-q}. \tag{19}$$

In fact

$$S_{n,q}(t^{m+1})(x) = \frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{k=0}^{\infty}\frac{[n]_q^k\cdot x^k}{[k]_q!}\cdot\frac{\{k\}_q^{m+1}}{[n]_q^{m+1}} = \frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{k=0}^{\infty}\frac{[n]_q^{k-1}\cdot[n]_q\cdot x^k}{[k-1]_q!\cdot[k]_q}\cdot\frac{\{k\}_q^m}{[n]_q^m\cdot[n]_q}\cdot[k]_q\cdot$$

$$\cdot\frac{1+q^{k-1}}{2} = \frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{k=1}^{\infty}\frac{[n]_q^{k-1}\cdot x^k}{[k-1]_q!}\cdot\frac{1}{[n]_q^m}\cdot\frac{1+q^{k-1}}{2}\cdot\sum_{j=0}^{m}\binom{m}{j}\{k-1\}_q^j\cdot q^j =$$

$$= \frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\frac{x}{2[n]_q^{m-j}}\cdot\sum_{k=1}^{\infty}\frac{[n]_q^{k-1}\cdot x^{k-1}}{[k-1]_q!}\cdot\frac{\{k-1\}_q^j}{[n]_q^j}\cdot q^j(1+q^{k-1}) =$$

$$= \frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\frac{xq^j}{2[n]_q^{m-j}}\cdot\left[\sum_{k=1}^{\infty}\frac{[n]_q^{k-1}x^{k-1}}{[k-1]_q!}\cdot\frac{\{k-1\}_q^j}{[n]_q^j}+\sum_{k=1}^{\infty}\frac{[n]_q^{k-1}(xq)^{k-1}}{[k-1]_q!}\cdot\frac{\{k-1\}_q^j}{[n]_q^j}\right]=$$

$$= \sum_{j=0}^{m}\binom{m}{j}\frac{xq^j}{2[n]_q^{m-j}}\cdot\left[\frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{k=0}^{\infty}\frac{[n]_q^k x^k}{[k]_q!}\cdot\frac{\{k\}_q^j}{[n]_q^j}+\frac{\mathcal{E}_q^{[n]_q xq}}{\mathcal{E}_q^{[n]_q x}}\cdot\frac{1}{\mathcal{E}_q^{[n]_q xq}}\sum_{k=0}^{\infty}\frac{[n]_q^k(xq)^k}{[k]_q!}\cdot\frac{\{k\}_q^j}{[n]_q^j}\right]$$

The proof of the third recurrence formula is based on, see [10],

$$\{k\}_q = \{k-1\}_q + q^{k-1} = \frac{1-q^{k-1}}{1-q}+q^{k-1}=\frac{1-q^k}{1-q}. \tag{20}$$

Hence

$$S_{n,q}(t^{m+1})(x) = \frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{k=0}^{\infty}\frac{[n]_q^k\cdot x^k}{[k]_q!}\cdot\frac{\{k\}_q^{m+1}}{[n]_q^{m+1}} = \frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{k=0}^{\infty}\frac{[n]_q^{k-1}\cdot[n]_q\cdot x^k}{[k-1]_q!\cdot[k]_q}\cdot\frac{\{k\}_q^m}{[n]_q^m\cdot[n]_q}\cdot[k]_q\cdot$$

$$\cdot\frac{1+q^{k-1}}{2} = \frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{k=1}^{\infty}\frac{[n]_q^{k-1}\cdot x^k}{[k-1]_q!}\cdot\frac{1}{[n]_q^m}\cdot\frac{1+q^{k-1}}{2}\cdot\sum_{j=0}^{m}\binom{m}{j}\{k-1\}_q^j\cdot q^{(k-1)(m-j)} =$$

$$= \frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\frac{x}{2[n]_q^{m-j}}\sum_{k=1}^{\infty}\frac{[n]_q^{k-1}\cdot x^{k-1}}{[k-1]_q!}\cdot\frac{\{k-1\}_q^j}{[n]_q^j}(1+q^{k-1})\cdot q^{(k-1)(m-j)} =$$

$$= \frac{1}{\mathcal{E}_q^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\frac{x}{2[n]_q^{m-j}}\left(\sum_{k=1}^{\infty}\frac{[n]_q^{k-1}\cdot(xq^{m-j})^{k-1}}{[k-1]_q!}\cdot\frac{\{k-1\}_q^j}{[n]_q^j}+\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{[n]_q^{k-1}\cdot(xq^{m-j+1})^{k-1}}{[k-1]_q!}\cdot\frac{\{k-1\}_q^j}{[n]_q^j}\right)=$$

$$= \frac{1}{\mathcal{E}_q^{[n]_q x}} \sum_{j=0}^{m} \binom{m}{j} \frac{x}{2[n]_q^{m-j}} \left(\frac{\mathcal{E}_q^{[n]_q x q^{m-j}}}{\mathcal{E}_q^{[n]_q x q^{m-j}}} \sum_{k=1}^{\infty} \frac{[n]_q^{k-1} \cdot (xq^{m-j})^{k-1}}{[k-1]_q!} \cdot \frac{\{k-1\}_q^j}{[n]_q^j} + \right.$$

$$\left. + \frac{\mathcal{E}_q^{[n]_q x q^{m-j+1}}}{\mathcal{E}_q^{[n]_q x q^{m-j+1}}} \sum_{k=1}^{\infty} \frac{[n]_q^{k-1} \cdot (xq^{m-j+1})^{k-1}}{[k-1]_q!} \cdot \frac{\{k-1\}_q^j}{[n]_q^j} \right) \blacksquare$$

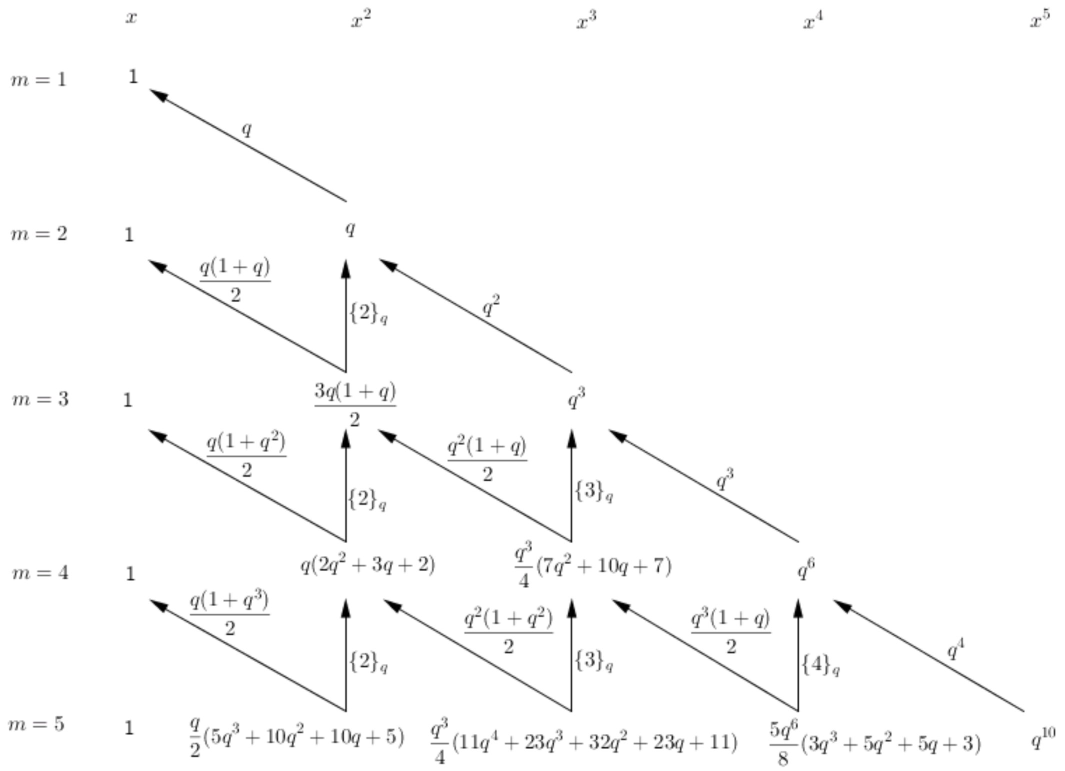
Lemma 2.2. *Let $n \in \mathbb{N}^*$, $0 < q < 1$ and $m \in \mathbb{N}^*$. The following formula holds true*

$$S_{n,q}(t^m)(x) = \frac{1}{\prod_{k=0}^{m-1} \left[1 + q^k(1-q)\frac{[n]_q x}{2}\right]} \sum_{j=1}^{m} a_{m,j}(q) \frac{x^j}{[n]_q^{m-j}}, \tag{21}$$

where

$$a_{m+1,j}(q) = \{j\}_q \cdot a_{m,j}(q) + \frac{q^{j-1}(1+q^{m-j+1})}{2} \cdot a_{m,j-1}(q),\ m \geq 0,\ j \geq 1 \text{ and} \tag{22}$$

$$a_{0,0}(q) = 1,\ a_{m,0}(q) = 0,\ m > 0,\ a_{m,j}(q) = 0,\ j > m.$$



Proof We use the first recurrence formula to facilitate the induction by m.

$$S_{n,q}(t^{m+1})(x) = \frac{1}{[n]_q(1-q)} \left[S_{n,q}(t^m)(x) - \frac{1-(1-q)\frac{[n]_q x}{2}}{1+(1-q)\frac{[n]_q x}{2}} \cdot S_{n,q}(t^m)(xq) \right] =$$

$$= \frac{1}{\prod_{k=0}^{m} \left[1 + q^k(1-q)\frac{[n]_q x}{2}\right]} \sum_{j=1}^{m} a_{m,j}(q) \frac{x^j}{[n]_q^{m-j}} \cdot \frac{1}{[n]_q(1-q)} \left(1 + q^m(1-q)\frac{[n]_q x}{2} - q^j + \right.$$

$$+q^j(1-q)\frac{[n]_q x}{2}\Bigg) = \frac{1}{\prod_{k=0}^{m}\left[1+q^k(1-q)\frac{[n]_q x}{2}\right]}\left[\sum_{j=1}^{m} a_{m,j}(q)\{j\}_q \frac{x^j}{[n]_q^{m+1-j}} + \sum_{j=2}^{m+1} a_{m,j-1}(q)\cdot\right.$$

$$\left.\cdot\frac{q^{j-1}(1+q^{m-j+1})}{2}\cdot\frac{x^j}{[n]_q^{m+1-j}}\right] = \frac{1}{\prod_{k=0}^{m}\left[1+q^k(1-q)\frac{[n]_q x}{2}\right]}\left[a_{m,1}(q)\frac{x}{[n]_q^m} + \sum_{j=2}^{m}\Bigg(a_{m,j}(q)\{j\}_q+\right.$$

$$\left.+a_{m,j-1}(q)\frac{q^{j-1}(1+q^{m-j+1})}{2}\Bigg)\frac{x^j}{[n]_q^{m+1-j}} + a_{m,m}(q)q^m x^{m+1}\right] \blacksquare$$

Lemma 2.3. *Let $n \in \mathbb{N}^*$ and $0 < q < 1$. The following formulae hold true*

- $S_{n,q}(1)(x) = 1,$ (23)
- $S_{n,q}(t)(x) = \frac{1}{1+(1-q)\frac{[n]_q x}{2}} \cdot x,$ (24)
- $S_{n,q}(t^2)(x) = \frac{1}{\prod_{k=0}^{1}\left[1+q^k(1-q)\frac{[n]_q x}{2}\right]}\left(x^2 q + \frac{x}{[n]_q}\right),$ (25)
- $S_{n,q}(t^3)(x) = \frac{1}{\prod_{k=0}^{2}\left[1+q^k(1-q)\frac{[n]_q x}{2}\right]}\left(x^3 q^3 + \frac{x^2}{[n]_q}\cdot\frac{3q(1+q)}{2} + \frac{x}{[n]_q^2}\right),$ (26)
- $S_{n,q}(t^4)(x) = \frac{1}{\prod_{k=0}^{3}\left[1+q^k(1-q)\frac{[n]_q x}{2}\right]}\left(x^4 q^6 + \frac{x^3}{[n]_q}\cdot\frac{q^3}{4}(7q^2+10q+7) + \frac{x^2}{[n]_q^2}\cdot q\cdot\right.$

  $\left.\cdot(2q^2+3q+2) + \frac{x}{[n]_q^3}\right).$ (27)

Proof We use the previous lemma and the formulae result from the scheme above. ■

Remark 2.4. *The quantities $Q_{n,k} = 1 + q^k(1-q)\frac{[n]_q x}{2}$, where $n \in \mathbb{N}^*$, $k \in \mathbb{N}$ and $0 < q < 1$ can be rewritten as $Q_{n,k} = 1 + q^k \frac{1-q^n}{1+q^{n-1}} x$.*

We want to study the convergence of the sequence $S_{n,q}$ to the identity operator. To this end we state the following preliminary result.

Lemma 2.5. *Let $(q_n)_{n\in\mathbb{N}^*}$ be a sequence such that $q_n \in (0,1)$, $\forall n \in \mathbb{N}^*$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$, $0 < a \le 1$. The following formulae hold true for $x \in \left[0, \frac{1+q_n^{n-1}}{1-q_n^n}\right)$*

$$\lim_{n\to\infty} S_{n,q_n}(t)(x) = \frac{1+a}{1+a+x(1-a)}x,$$

$$\lim_{n\to\infty} S_{n,q_n}(t^2)(x) = \frac{(1+a)^2}{[1+a+x(1-a)]^2}x^2,$$

$$\lim_{n\to\infty} S_{n,q_n}(t^3)(x) = \frac{(1+a)^3}{[1+a+x(1-a)]^3}x^3,$$

$$\lim_{n\to\infty} S_{n,q_n}(t^4)(x) = \frac{(1+a)^4}{[1+a+x(1-a)]^4}x^4.$$

Proof The formulae are derived from the ones in Lemma 2.3 and the fact that $\lim_{n\to+\infty}\frac{1}{[n]_{q_n}} = 0$.

3. Uniform Approximation on Compact Intervals

In order to prove the uniform convergence of the sequence S_{n,q_n} on compact intervals to the identity operator, we will use the following result from [1, Theorem 3.5].

Consider a metric space (X, d). Below the symbol $F(X)$ stands for the linear space of all real-valued functions defined on X. For any $x \in X$ we denote by $d_x \in C(X)$ the function $d_x(y) := d(x, y), y \in X$.

Theorem 3.1. *Let (X, d) be a locally compact metric space and consider a lattice subspace E of $F(X)$ containing the constant function 1 and all the functions $d_x^2 (x \in X)$. Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators from E into $F(X)$ and assume that*

- $\lim_{n\to\infty} L_n(1) = 1$ *uniformly on compact subsets of X;*
- $\lim_{n\to\infty} L_n(d_x^2)(x) = 0$ *uniformly on compact subsets of X.*

Then for every $f \in E \cap C_b(X)$

$$\lim_{n\to\infty} L_n(f) = f \text{ uniformly on compact subsets of } X.$$

Define

$$C_2[0,\infty) = \{f \in C[0,\infty) | \exists M > 0 : |f(x)| \leq M(1 + x^2), \forall x \geq 0\}$$

In our case $X = [0,\infty)$, $E = C_2[0,\infty)$, $d(x, y) = |x - y|$. Note that $d_x^2 = e_2 - 2xe_1 + x^2 1$.

Our main result is the following theorem.

Theorem 3.2. *Let $(q_n)_{n\in\mathbb{N}^*}$ be a sequence such that $q_n \in (0, 1)$, $\forall n \in \mathbb{N}^*$ and $q_n^n \to 1$ as $n \to \infty$. Then the sequence $S_{n,q_n}(f)$ converges uniformly to f, on any compact interval in $[0,\infty)$, for any $f \in C_2[0,\infty) \cap C_b([0,\infty))$.*

Proof The calculations hold true for $x \in \left[0, \frac{2}{(1-q_n)[n]_{q_n}}\right)$. Since the expression $\frac{2}{(1-q_n)[n]_{q_n}} \to \infty$ as $n \to \infty$, there exists a rank N_0 such that if $n \geq N_0$ then $[0, a] \subset \left[0, \frac{2}{(1-q_n)[n]_{q_n}}\right)$, where $a \in \mathbb{R}_+$. From Lemma 2.5 with $a = 1$ we have

$$\lim_{n\to\infty} S_{n,q_n}(1)(x) = 1,$$
$$\lim_{n\to\infty} S_{n,q_n}(t)(x) = x,$$
$$\lim_{n\to\infty} S_{n,q_n}(t^2)(x) = x^2.$$

We prove the uniform convergence of the moment of order 1.

$$\left| \frac{x}{1 + \frac{1-q_n^n}{1+q_n^{n-1}}x} - x \right| = \left| \frac{\frac{1-q_n^n}{1+q_n^{n-1}}x^2}{1 + \frac{1-q_n^n}{1+q_n^{n-1}}x} \right| \leq |1 - q_n^n| a \to 0,\ n \to \infty.$$

The uniform convergence of the moment of order 2 results in the following way

$$\left| \frac{x^2 q_n + \frac{x}{[n]_{q_n}}}{\left(1 + \frac{1-q_n^n}{1+q_n^{n-1}}x\right)\left(1 + q_n\frac{1-q_n^n}{1+q_n^{n-1}}x\right)} - x^2 \right| \leq$$

$$\leq \left| \frac{x}{[n]_{q_n}} - x^2(1 - q_n) - \frac{1 - q_n^n}{1 + q_n^{n-1}}(q_n + 1)x^3 - q_n\frac{(1 - q_n^n)^2}{(1 + q_n^{n-1})^2}x^4 \right| \leq$$

$$\leq \frac{a}{[n]_{q_n}} + |1 - q_n|a^2 + |1 - q_n^n|\frac{1 + q_n}{1 + q_n^{n-1}}a^3 + |1 - q_n^n|^2\frac{q_n}{(1 + q_n^{n-1})^2}a^4 \to 0,\ n \to \infty.$$

The following convergences hold for $x \in [0, a]$

$$S_{n,q_n}(1)(x) \rightrightarrows 1,\ n \to \infty,$$

$$S_{n,q_n}(t)(x) \rightrightarrows x,\ n \to \infty,$$

$$S_{n,q_n}(t)(x) \rightrightarrows x^2,\ n \to \infty,$$

where $\rightrightarrows$ means uniform convergence. We further have

$$\lim_{n\to\infty} S_{n,q_n}(t^2 - 2xt + x^2 1)(x) = \lim_{n\to\infty}[S_{n,q_n}(t^2)(x) - 2xS_{n,q_n}(t)(x) + x^2 S_{n,q_n}(1)(x)] = 0$$

The conditions in the previous theorem are therefore satisfied. ■

Finally, we consider the case when q is constant.

Remark 3.3. *Let $q \in (0,1)$, q fixed. The following formulae hold for $x \in \left[0, \frac{1+q^{n-1}}{1-q^n}\right)$*

$$S_{n,q}(1)(x) \rightrightarrows 1,\ n \to \infty,$$

$$S_{n,q}(t)(x) \rightrightarrows \frac{x}{1+x},\ n \to \infty,$$

where $\rightrightarrows$ means uniform convergence.

The proof for $S_{n,q}(1)(x)$ is obvious. Since $q \in (0,1)$, we have $\lim_{n\to\infty} q^n = 0$. From the second formula in Lemma 2.3 it follows that

$$\lim_{n\to\infty} S_{n,q}(t)(x) = \frac{x}{1+x}.$$

Moreover,

$$\left|\frac{x}{1+\frac{1-q^n}{1+q^{n-1}}x} - \frac{x}{1+x}\right| = |x^2|\left|\frac{1-\frac{1-q^n}{1+q^{n-1}}}{(1+x)(1+\frac{1-q^n}{1+q^{n-1}}x)}\right| \le |x^2|\left|\frac{q^{n-1}(1+q)}{1+q^{n-1}}\right| <$$

$$< \frac{q^{n-1}(1+q)(1+q^{n-1})}{(1-q^n)^2} \to 0, n \to \infty,$$

which implies uniform convergence.

In the following we will consider only the case when q depends on n, i.e. $q = q_n$ and it verifies the condition $q_n^n \to a$ as $n \to \infty$, $0 < a \le 1$.

4. Voronovskaja Theorem

We will first prove the following result.

Lemma 4.1. *Let $(q_n)_{n\in\mathbb{N}^*}$ be a sequence such that $q_n \in (0,1)$, $\forall n \in \mathbb{N}^*$, $q_n \to 1$ and $q_n^n \to 1$ as $n \to \infty$. Then the following formulae hold true*

$$\lim_{n\to\infty} [n]_{q_n} S_{n,q_n}(t-x)(x) = 0, \tag{28}$$

$$\lim_{n\to\infty} [n]_{q_n} S_{n,q_n}\left((t-x)^2\right)(x) = x, \tag{29}$$

$$\lim_{n\to\infty} [n]_{q_n}^2 S_{n,q_n}((t-x)^4)(x) = 3x^2. \tag{30}$$

Moreover, the convergences are uniform on any compact interval $[0, a]$, $a > 0$.

Proof First we compute the following limits

$$\lim_{n\to\infty}\frac{(1-q_n^n)^2}{1-q_n}=\lim_{n\to\infty}(1-q_n^n)(1+q_n+\cdots+q_n^{n-1})=\lim_{n\to\infty}(1+q_n+\cdots+q_n^{n-1}-q_n^n-$$

$$-q_n^{n+1}-\cdots-q_n^{2n-1})=0 \text{ because } q_n\to 1 \text{ and } q_n^n\to 1, n\to\infty. \tag{31}$$

$$\lim_{n\to\infty}\frac{(1-q_n^n)^3}{1-q_n}=\lim_{n\to\infty}(1-q_n^n)(1+q_n+\cdots+q_n^{n-1}-q_n^n-q_n^{n+1}-\cdots-q_n^{2n-1})=$$

$$=\lim_{n\to\infty}[1+q_n+\cdots+q_n^{n-1}-$$

$$-2(q_n^n+\cdots+q_n^{2n-1})+q_n^{2n}+\cdots+q_n^{3n-1}]=0 \text{ because } q_n\to 1 \text{ and } q_n^n\to 1, n\to\infty. \tag{32}$$

The calculations hold true for $x\in\left[0,\frac{2}{(1-q_n)[n]_{q_n}}\right)$. Since the expression $\frac{2}{(1-q_n)[n]_{q_n}}\to\infty$ as $n\to\infty$, there exists a rank N_0 such that if $n\geq N_0$ then $[0,a]\subset\left[0,\frac{2}{(1-q_n)[n]_{q_n}}\right)$, where $a\in\mathbb{R}_+$. The first formula can be proved in the following way

$$\left|[n]_{q_n}S_{n,q_n}(t-x)(x)\right|=\left|[n]_{q_n}(S_{n,q_n}(t)(x)-xS_{n,q_n}(1)(x))\right|=\left|[n]_{q_n}\left(\frac{x}{1+(1-q_n)\frac{[n]_{q_n}x}{2}}-\right.\right.$$

$$\left.-x)\right|=\left|-\frac{(1-q_n)[n]_{q_n}^2}{2}\cdot\frac{x^2}{1+(1-q_n)\frac{[n]_{q_n}x}{2}}\right|=\left|\frac{1-q_n}{2}\left(\frac{1-q_n^n}{1-q_n}\cdot\frac{2}{1+q_n^{n-1}}\right)^2\cdot\right.$$

$$\left.\cdot\frac{x^2}{1+\frac{1-q_n^n}{1+q_n^{n-1}}x}\right|=\left|\frac{(1-q_n^n)^2}{1-q_n}\cdot\frac{2}{(1+q_n^{n-1})^2}\cdot\frac{x^2}{1+\frac{1-q_n^n}{1+q_n^{n-1}}x}\right|\leq\frac{(1-q_n^n)^2}{1-q_n}\cdot 2a^2\to 0,$$

as $n\to\infty$ according to (31). The second formula can be proved in the following way

$$[n]_{q_n}S_{n,q_n}\left((t-x)^2\right)(x)=[n]_{q_n}\left[S_{n,q_n}(t^2)(x)-2xS_{n,q_n}(t)(x)+x^2S_{n,q_n}(1)(x)\right]=$$

$$=[n]_{q_n}\left\{\frac{x^2q_n+\frac{x}{[n]_{q_n}}}{\left[1+(1-q_n)\frac{[n]_{q_n}x}{2}\right]\cdot\left[1+q_n(1-q_n)\frac{[n]_{q_n}x}{2}\right]}-2x\cdot\frac{x}{1+(1-q_n)\frac{[n]_{q_n}x}{2}}+x^2\right\}=$$

$$=\frac{[n]_{q_n}}{Q_{n,0}Q_{n,1}}\left[x^2q_n+\frac{x}{[n]_{q_n}}-2x^2\left(1+q_n(1-q_n)\frac{[n]_{q_n}x}{2}\right)+x^2\left(1+(1-q_n)\frac{[n]_{q_n}x}{2}+\right.\right.$$

$$\left.\left.+q_n(1-q_n)\frac{[n]_{q_n}x}{2}+q_n(1-q_n)^2\frac{[n]_{q_n}^2x^2}{4}\right)\right]=\frac{[n]_{q_n}}{Q_{n,0}Q_{n,1}}\left[x^4q_n(1-q_n)^2\frac{[n]_{q_n}^2}{4}+\right.$$

$$\left.+x^3(1-q_n)^2\frac{[n]_{q_n}}{2}+x^2(q_n-1)+\frac{x}{[n]_{q_n}}\right]=\frac{1}{Q_{n,0}Q_{n,1}}\left[x^4\frac{q_n(1-q_n)^2}{4}\left(\frac{1-q_n^n}{1-q_n}\cdot\frac{2}{1+q_n^{n-1}}\right)^3+\right.$$

$$\left.+x^3\frac{(1-q_n)^2}{2}\left(\frac{1-q_n^n}{1-q_n}\cdot\frac{2}{1+q_n^{n-1}}\right)^2+x^2(q_n-1)\cdot\frac{1-q_n^n}{1-q_n}\cdot\frac{2}{1+q_n^{n-1}}+x\right]=$$

$$=\frac{1}{Q_{n,0}Q_{n,1}}\left[x^4q_n\frac{(1-q_n^n)^3}{1-q_n}\cdot\frac{2}{(1+q_n^{n-1})^3}+x^3(1-q_n^n)^2\frac{2}{(1+q_n^{n-1})^2}-x^2(1-q_n^n)\frac{2}{1+q_n^{n-1}}+x\right].$$

As $n \to \infty$, by means of the formula (32) we obtain the limit (29).

$$\left|[n]_{q_n} S_{n,q_n}((t-x)^2)(x) - x\right| = \left|\frac{1}{Q_{n,0}Q_{n,1}}\left[x^4 q_n \frac{(1-q_n^n)^3}{1-q_n} \cdot \frac{2}{(1+q_n^{n-1})^3} + x^3(1-q_n^n)^2 \cdot\right.\right.$$

$$\left.\left.\cdot \frac{2-q_n}{(1+q_n^{n-1})^2} - -x^2(1-q_n^n)\frac{1-q_n}{1+q_n^{n-1}}\right]\right| \leq 2a^4\frac{(1-q_n^n)^3}{1-q_n} + 2a^3(1-q_n^n)^2 + a^2(1-q_n^n),$$

which tends to 0 by (32) as $n \to \infty$, which proves the uniform convergence. The third formula can be proven analogously. ■

We will use the following result.

Remark 4.2. *If we consider the conditions in the previous lemma, then formula (29) implies that $\|S_{n,q_n}((t-\cdot)^2)\| \to 0$ as $n \to \infty$, uniformly on compact intervals.*

Recall that if we consider $I \subset \mathbb{R}$ to be an interval, $f : I \to \mathbb{R}$ a bounded function and $\delta \geq 0$, then the modulus of continuity of first order is defined as [12]

$$\omega(f,\delta) = \sup\{|f(u) - f(v)| : u, v \in I, |u - v| \leq \delta\}. \tag{33}$$

Moreover, the following inequality holds for f bounded: $\omega(f,\delta) \leq \left[1 + \left(\frac{\delta}{\eta}\right)^2\right]\omega(f,\eta)$, $\forall\ \delta, \eta > 0$.

The main result of this section is the following theorem.

Theorem 4.3. *Let $(q_n)_{n\in\mathbb{N}^*}$ be a sequence such that $q_n \in (0,1)$, $\forall\ n \in \mathbb{N}^*$, $q_n \to 1$ and $q_n^n \to 1$ as $n \to \infty$. Then for any function f that is continuous and bounded on $[0,\infty)$ such that f' and f'' are continuous and bounded on $[0,\infty)$ the following uniform convergence holds on any compact interval $[0,a]$, $a > 0$*

$$\lim_{n\to\infty}[n]_{q_n}\left[S_{n,q_n}(f)(x) - f(x)\right] = \frac{1}{2}f''(x) \cdot x. \tag{34}$$

Proof Let the functions f, f' and f'' be continuous and bounded on $[0,\infty)$. Let $x \in [0,\infty)$ be fixed. We use Taylor's formula with the remainder in its integral form

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du.$$

We use the following notation

$$R_2(t,x) = \int_x^t (t-u)f''(u)du = \int_x^t (t-u)f''(x)du + \int_x^t (t-u)\left(f''(u) - f''(x)\right)du =$$

$$= f''(x)\frac{(t-x)^2}{2} + \int_x^t (t-u)\left(f''(u) - f''(x)\right)du.$$

It follows that

$$f(t) = f(x) + f'(x)(t-x) + f''(x)\frac{(t-x)^2}{2} + \int_x^t (t-u)\left(f''(u) - f''(x)\right)du.$$

We consider the function $r(t;x) = \frac{1}{(t-x)^2}\int_x^t (t-u)\left[f''(u) - f''(x)\right]du$. It follows that

$$f(t) = f(x) + f'(x)(t-x) + f''(x)\frac{(t-x)^2}{2} + r(t;x)(t-x)^2. \tag{35}$$

Then we prove that $\lim_{t\to x} r(t;x) = 0$. We prove that $r(t;x)$ is bounded for all $t \in [0,a]$ and that $r(t;x)$ is continuous.

According to (35) it results that $r(t;x)$ can be rewritten in the following way

$$r(t;x) = \frac{f(t) - f(x) - f'(x)(t-x) - f''(x)\frac{(t-x)^2}{2}}{(t-x)^2}.$$

Then

$$\lim_{t\to x} r(t;x) = \lim_{t\to x} \frac{f(t) - f(x) - f'(x)(t-x) - f''(x)\frac{(t-x)^2}{2}}{(t-x)^2} \overset{l'H}{=}$$

$$= \lim_{t\to x} \frac{f'(t) - f'(x) - f''(x)(t-x)}{2(t-x)} \overset{l'H}{=} \lim_{t\to x} \frac{f''(t) - f''(x)}{2} = 0 \text{ because } f'' \text{ is continuous.}$$

Boundedness results from the following formula

$$|r(t;x)| = \frac{1}{(t-x)^2}\left|\int_x^t (t-u)\left[f''(u) - f''(x)\right]du\right| \leq \frac{1}{(t-x)^2}\int_x^t |t-u|\cdot|f''(u) - f''(x)|du \leq$$

$$\leq \frac{1}{(t-x)^2}\int_x^t (t-u)[|f''(u)| + |f''(x)|]du \leq \frac{1}{(t-x)^2}\cdot 2\|f''\|\int_x^t (t-u)du = \|f''\|.$$

We prove continuity. If $t = x$, then we consider by convention that $r(x;x) = 0$. If $t \neq x$, then

$$|r(t;x)| \leq \frac{1}{(t-x)^2}\int_x^t (t-u)|f''(u) - f''(x)|du.$$

By means of the modulus of continuity of first order it results that

$$|f''(u) - f''(x)| \leq \omega(f'', |u-x|) \leq \omega(f'', |t-x|).$$

It follows that $|r(t;x)| \leq \frac{1}{2}\omega(f'', |t-x|)$ for $t \neq x$. By applying S_{n,q_n} to (35) we obtain

$$S_{n,q_n}(f)(x) = f(x)S_{n,q_n}(1)(x) + f'(x)S_{n,q_n}(t-x)(x) + \frac{f''(x)}{2}S_{n,q_n}((t-x)^2)(x) +$$

$$+ S_{n,q_n}\left(r(t;x)(t-x)^2\right)(x).$$

Then

$$[n]_{q_n}\left[S_{n,q_n}(f)(x) - f(x)\right] = f'(x)[n]_{q_n}S_{n,q_n}(t-x)(x) + \frac{f''(x)[n]_{q_n}}{2}S_{n,q_n}((t-x)^2)(x) +$$

$$+ [n]_{q_n}S_{n,q_n}\left(r(t;x)(t-x)^2\right)(x).$$

By applying the Cauchy-Schwarz inequality we obtain

$$S_{n,q_n}\left(r(t;x)(t-x)^2\right)(x) \leq \sqrt{S_{n,q_n}\left(r^2(t;x)\right)(x)}\cdot\sqrt{S_{n,q_n}\left((t-x)^4\right)(x)}. \tag{36}$$

By multiplying (36) by $[n]_{q_n}$ it results that

$$[n]_{q_n}S_{n,q_n}\left(r(t;x)(t-x)^2\right)(x) \leq \sqrt{S_{n,q_n}\left(r^2(t;x)\right)(x)}\cdot\sqrt{[n]_{q_n}^2 S_{n,q_n}\left((t-x)^4\right)(x)}. \tag{37}$$

According to (30) it results that

$$\lim_{n\to\infty}\sqrt{[n]_{q_n}^2 S_{n,q_n}\left((t-x)^4\right)(x)} = \sqrt{3x^2}.$$

Then

$$|S_{n,q_n}(r(t;x))(x)| \leq S_{n,q_n}(|r(t;x)|)(x) \leq S_{n,q_n}\left[\omega(f'', |t-x|)\right](x) \leq$$

$$\leq S_{n,q_n}\left[\left(1+\frac{(t-x)^2}{\eta_n^2}\right)\omega(f,\eta_n)\right](x)=\omega(f,\eta_n)S_{n,q_n}(1)(x)+\frac{\omega(f,\eta_n)}{\eta_n^2}S_{n,q_n}((t-x)^2)(x)$$

We consider $\eta_n = \| \sqrt{S_{n,q_n}((t-x)^2)(x)}\|$, which implies that

$$|S_{n,q_n}(r(t;x))(x)| \leq 2\omega\left(f, \left\| \sqrt{S_{n,q_n}((t-x)^2)(x)}\right\|\right).$$

We apply Theorem 3.2 and the formulas (28) and (29) to complete the proof. ■

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