Filomat 32:2 (2018), 423–438
https://doi.org/10.2298/FIL1802423A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Warped Product Pointwise Pseudo-Slant Submanifolds of Locally Product Riemannian Manifolds

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Abstract. In [3], it was shown that there are no warped product submanifolds of a locally product Riemannian manifold such that the spherical submanifold of a warped product is proper slant. In this paper, we introduce the notion of warped product submanifolds with a slant function and show that there exists a class of non-trivial warped product submanifolds of a locally product Riemannian manifold such that the spherical submanifold is pointwise slant by giving some examples. We present a characterization theorem and establish a sharp relationship between the squared norm of the second fundamental form and the warping function in terms of the slant function for such warped product submanifolds of a locally product Riemannian manifold. The equality case is also considered.

1. Introduction

Pseudo-slant submanifolds were defined and studied by A. Carriazo as a particular class of bi-slant submanifold under the name of anti-slant submanifolds in [10]. We note that a pseudo-slant submanifold is a special case of generic submanifold which was introduced by Ronsse [33]. We also note that the pseudo-slant submanifolds are also studied under the name of hemi-slant slant submanifolds (see [31], [34]).

On the other hand, F. Etayo [20] introduced the notion of ponitwise slant submanifolds of almost Hermitian manifolds under the name of quasi-slant submnifolds. Recently, B.-Y. Chen and O.J. Garay [17] studied these submanifolds in almost Hermitian manifolds and obtained several fundamental results. We note that every slant submanifold is a pointwise slant submanifold. Pointwise slant submanifolds of other structures are also studied in [26] and [21]. Recently, B. Sahin [32] introduced the idea of pointwise semi-slant submanifolds of Kaehler manifolds. Using this notion, he investigated warped product pointwise semi-slant submanifolds of Kaehler manifolds. In [31], Sahin studied warped product pseudo-slant submanifolds of Kaehler manifolds under the name of warped product hemi-slant submanifolds. He proved a non-existence result of the warped product of the form $M_\perp \times M_\theta$ of a Kaehler manifold $\widetilde{M}$, where $M_\perp$ and M_θ are totally real and proper slant submanifolds of $\widetilde{M}$, respectively. Then he introduced the notion of hemi-slant warped

2010 *Mathematics Subject Classification*. 53C15; 53C40; 53C42; 53B25
Keywords. slant submanifolds; pointwise slant; semi-slant; pseudo-slant; mixed totally geodesic; warped product; locally product Riemannian manifolds
Received: 01 October 2016; Revised: 21 November 2016; Accepted: 26 November 2016
Communicated by Mića S. Stanković
The research was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. G-687-247-37. The authors, therefore, acknowledge with thanks DSR for technical and financial support.
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products of the form $M_\theta \times_f M_\perp$ and obtained many important results, including a characterization and an inequality for such warped products.

In [3], M. Atceken proved the non-existence of warped product submanifolds of the form $M_\perp \times_f M_\theta$ and $M_T \times_f M_\theta$ of a locally product Riemannian manifold $\widetilde{M}$, where M_T, $M_\perp$ and M_θ are invariant, anti-invariant and proper slant submanifolds of $\widetilde{M}$, respectively. The warped product submanifolds of locally product Riemannian manifolds are also studied in (see [2], [5, 6], [29, 30], [35]).

In [39], we investigated the geometry of warped product pseudo-slant submanifolds $M_\theta \times_f M_\perp$ of a locally product Riemannian manifold $\widetilde{M}$. The warped product pseudo-slant submanifolds also have been studied for different structures in [31] and [37, 38, 40, 41]. For the survey on warped product submanifolds we refers to Chen's books [15, 18] and his survey article [16].

In this paper, we introduce the idea of pointwise pseudo-slant submanifolds of locally product Riemannian manifolds and using this notion we investigate the geometry of warped product pointwise pseudo-slant submanifolds of the form $M_\perp \times_f M_\theta$ of a locally product Riemannian manifold $\widetilde{M}$ where $M_\perp$ is an anti-invariant submanifold of $\widetilde{M}$ and M_θ is a proper pointwise slant submanifold of $\widetilde{M}$ with slant function θ. As we know that the warped product pointwise pseudo-slant submanifold $M_\theta \times_f M_\perp$, where M_θ is poinwise slant submanifold is a particular class of warped product pseudo-slant submanifold $M_\theta \times_f M_\perp$ studied in [39], therefore we are not interested to repeat this study for pointwise pseudo-slant warped products.

The paper is organised as follows: Section 2 is devoted to give preliminaries and basic definitions. In Section 3, we define and study pointwise pseudo-slant submanifolds of locally product Riemannian manifolds. In this section we investigate the geometry of the leaves of the involves distributions. In Section 4, we study warped product pointwise pseudo-slant submanifolds. In this section, we give some examples and prove a characterization theorem of such type of warped products. In Section 5, we establish Chen type inequality for the squared norm of the second fundamental form in terms of the warping function. The equality case of the inequality is also considered.

2. Preliminaries

Let $\widetilde{M}$ be a m-dimensional differentiable manifold with a tensor field F of type $(1, 1)$ such that $F^2 = I$ and $F \neq \pm I$, then we say that $\widetilde{M}$ is an almost product manifold with almost structure F. If an almost product manifold $\widetilde{M}$ has a Riemannian metric g such that

$$g(FX, FY) = g(X, Y) \tag{1}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, then $\widetilde{M}$ is called an *almost product Riemannian manifold* [43], where $\Gamma(T\widetilde{M})$ denotes the set all vector fields of $\widetilde{M}$. Let $\widetilde{\nabla}$ denotes the Levi-Civita connedtion on $\widetilde{M}$ with respect to the Riemannian metric g. If $(\widetilde{\nabla}_X F)Y = 0$, for any $X, Y \in \Gamma(T\widetilde{M})$, then $\widetilde{M}$ is called a *locally product Riemannian manifold* [23].

Let M be a Riemannian manifold isometrically immersed in $\widetilde{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$, the set of all vector fields normal to M. Let ∇ be the Levi-Civita connection on M, then the Gauss and Weingarten formulas are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^\perp_X N \tag{3}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ and A_N is the shape operator of M with respect to N. Moreover, $h : TM \times TM \to T^\perp M$ is the second fundamental form of M in $\widetilde{M}$. Furthermore, A_N and h are related by

$$g(h(X, Y), N) = g(A_N X, Y) \tag{4}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$.

For any X tanget to M, we write

$$FX = TX + \omega X, \tag{5}$$

where TX and ωX are the tangential and normal components of FX, respectively. Then T is an endomorphism of tangent bundle TM and ω is a normal bundle valued 1-form on TM. Similarly, for any vector field N normal to M, we put

$$FN = BN + CN, \tag{6}$$

where BN and CN are the tangential and normal components of FN, respectively. Moreover, from (1) and (5), we have $g(TX, Y) = g(X, TY)$, for any $X, Y \in \Gamma(TM)$.

A sumanifold M is said to be *F-invariant* if ω is identically zero, i.e., $FX \in \Gamma(TM)$, for any $X \in \Gamma(TM)$. On the other hand, M is said to be *F-anti-invariant* if T is identically zero i.e., $FX \in \Gamma(T^{\perp}M)$, for any $X \in \Gamma(TM)$.

A submanifold M of a locally product Riemnnian manifold $\widetilde{M}$ is said to be *totally umbilical submanifold* if $h(X, Y) = g(X, Y)H$, for any $X, Y \in \Gamma(TM)$, where $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$, the mean curvature vector of M. A submanifold M is said to be totally geodesic if $h(X, Y) = 0$. A totally umbilical submanifold of dimension greater than or equal to 2 with non-vanishing parallel mean curvature vector is called an *extrinsic sphere*.

Also, we set

$$h_{ij}^{r} = g(h(e_i, e_j), e_r), \quad i, j = 1, \cdots, n; \; r = n+1, \cdots, m \tag{7}$$

and

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) \tag{8}$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of the tangent space T_pM, for any $p \in M$.

For a differentiable function f on a m-dimensional manifold $\widetilde{M}$, the gradient $\vec{\nabla} f$ of f is defined as $g(\vec{\nabla} f, X) = Xf$, for any X tangent to $\widetilde{M}$. As a consequence, we have

$$||\vec{\nabla} f||^2 = \sum_{i=1}^{m} (e_i(f))^2 \tag{9}$$

for an orthonormal frame $\{e_1 \cdots, e_m\}$ on $\widetilde{M}$.

By the analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a locally product Riemannian manifold were considered.

(1) A submanifold M of a locally product Riemannian manifold $\widetilde{M}$ is called a *semi-invariant submanifold* [27, 29] of $\widetilde{M}$ if there exist a differentiable distribution $\mathcal{D} : p \to \mathcal{D}_p \subset T_pM$ such that $\mathcal{D}$ is invariant with respect to F and the complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant with respect to F.

(2) A submanifold M of a locally product Riemannian manifold $\widetilde{M}$ is said to be slant (see [11, 12], [28]), if for each non-zero vector X tangent to M, the angle $\theta(X)$ between FX and T_pM is a constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM$.

(3) A submanifold M of a locally product Riemannian manifold $\widetilde{M}$ is called semi-slant (see [25], [9] and [23]), if it is endowed with two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\theta}$, where $\mathcal{D}$ is invariant with respect to F and $\mathcal{D}^{\theta}$ is slant, i.e., $\theta(X)$ is the angle between FX and $\mathcal{D}_p^{\theta}$ is constant for any $X \in \mathcal{D}_p^{\theta}$ and $p \in M$.

(4) A submanifold M of a locally product Riemannian manifold $\widetilde{M}$ is said be pseudo-slant (or hemi–slant) (see [31] and [37]), if it is endowed with two orthogonal distributions $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$, where $\mathcal{D}^{\perp}$ is anti-invariant with respect to F and $\mathcal{D}^{\theta}$ is slant.

(5) A submanifold M of a locally product Riemannian manifold $\widetilde{M}$ is called pointwise slant [21], if at each point $p \in M$, the Wirtinger angle $\theta(X)$ between FX and T_pM is independent of the choice of the non-zero vector $X \in T_pM$. In this case, the Wirtinger angle gives rise a real-valued function $\theta : TM - \{0\} \to \mathbb{R}$ which is called the *Wirtinger function* or *slant function* of the pointwise slant submanifold.

We note that a pointwise slant submanifold of a locally product Riemannian manifold is called *slant*, in the sense of [28] and [4], if its Wirtinger function θ is globally constant. Moreover, F-invariant and F-anti-invariant submanifolds introduced in [43] and in [1] are pointwise slant submanifolds with slant function $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A pointwise slant submanifold of a locally product Riemannian manifold is called a *proper pointwise slant submanifold* if it is neither F-invariant nor F-anti-invariant.

On the similar line of Chen's result (Lemma 2.1) of [17], it is known that M is a pointwise slant submanifold of a locally product Riemannian manifold $\widetilde{M}$ if and only if

$$T^2 = (\cos^2\theta)I, \tag{10}$$

for some real-valued function θ defined on M, where I denotes the identity transformation of the tangent bundle TM of M. The following relations are the consequences of (10) as

$$g(TX, TY) = \cos^2\theta\, g(X, Y), \tag{11}$$

$$g(\omega X, \omega Y) = \sin^2\theta\, g(X, Y) \tag{12}$$

for any $X, Y \in \Gamma(TM)$. Another important relation for a poitwise slant submanifold of a locally product Riemannian manifold is obtained by using (5), (6) and (10) as

$$B\omega X = \sin^2\theta\, X, \quad C\omega X = -\omega TX \tag{13}$$

for any $X \in \Gamma(TM)$.

3. Pointwise pseudo-slant submanifolds

In this section, we define and study pointwise pseudo-slant submanifolds of a locally product Riemannian manifold. We give examples of pointwise pseudo-slant submanifolds and investigate the geometry of the leaves of distributions.

Definition 3.1. Let $\widetilde{M}$ be a locally product Riemannian manifold and M a real submanifold of $\widetilde{M}$. Then, we say that M is a pointwise pseudo-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ on M such that

(i) The tangent space TM admits the orthogonal direct decomposition $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta$.
(ii) The distribution $\mathcal{D}^\perp$ is F-anti-invariant, i.e. $F(\mathcal{D}^\perp) \subset T^\perp M$.
(iii) The distribution $\mathcal{D}^\theta$ is pointwise slant with slant function θ.

In the above definition, the angle θ is called the slant function of the pointwise slant distribution $\mathcal{D}^\theta$. The anti-invariant distribution $\mathcal{D}^\perp$ of a pointwise pseudo-slant submanifold is a pointwise slant distribution with slant function $\theta = \frac{\pi}{2}$. If we denote the dimensions of $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ by p and q, respectively, then we have the following possible cases:

(i) If $p = 0$, then M is an anti-invariant submanifold.
(ii) If $q = 0$, then M is a pointwise slant submanifold.
(iii) If $q = 0$ and $\theta = 0$, then M is an invariant submanifold.
(iv) If θ is constant on M, then M is a pseudo-slant submanifold with slant angle θ.
(v) If $\theta = 0$, then M is a semi-invariant submanifold.

We note that a pointwise pseudo-slant submanifold is proper if $q \neq 0$ and θ is not a constant.

Now, we construct the following examples of pointwise pseudo-slant submanifolds.

Example 3.2. Consider a submanifold M of $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ with cartesian coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ and the product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}, \; F\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial y_j}, \;\; 1 \leq i, j \leq 3. \tag{14}$$

For any $\theta, \varphi \in \left(0, \frac{\pi}{2}\right)$, consider a submanifold M of $\mathbb{R}^6$ defined as

$$\chi(\theta, \varphi) = (\cos\theta\, \cos\varphi,\; \sin\theta\, \cos\varphi,\; \sin\varphi,\; \sin\theta\, \sin\varphi,\; \cos\theta\, \sin\varphi,\; \cos\varphi)$$

such that φ is a real-valued function on M. Then, the tangent space TM of M is spanned by the following vector fields

$$Z_1 = -\sin\theta\, \cos\varphi \frac{\partial}{\partial x_1} + \cos\theta\, \cos\varphi \frac{\partial}{\partial x_2} + \cos\theta\, \sin\varphi \frac{\partial}{\partial y_1} - \sin\theta\, \sin\varphi \frac{\partial}{\partial y_2},$$

$$Z_2 = -\cos\theta\, \sin\varphi \frac{\partial}{\partial x_1} - \sin\theta\, \sin\varphi \frac{\partial}{\partial x_2} + \cos\varphi \frac{\partial}{\partial x_3} + \sin\theta\, \cos\varphi \frac{\partial}{\partial y_1} + \cos\theta\, \cos\varphi \frac{\partial}{\partial y_2} - \sin\varphi \frac{\partial}{\partial y_3}.$$

Thus, with respect to the product Riemannian structure F, we obtian

$$FZ_1 = \sin\theta\, \cos\varphi \frac{\partial}{\partial x_1} - \cos\theta\, \cos\varphi \frac{\partial}{\partial x_2} + \cos\theta\, \sin\varphi \frac{\partial}{\partial y_1} - \sin\theta\, \sin\varphi \frac{\partial}{\partial y_2},$$

$$FZ_2 = \cos\theta\, \sin\varphi \frac{\partial}{\partial x_1} + \sin\theta\, \sin\varphi \frac{\partial}{\partial x_2} - \cos\varphi \frac{\partial}{\partial x_3} + \sin\theta\, \cos\varphi \frac{\partial}{\partial y_1} + \cos\theta\, \cos\varphi \frac{\partial}{\partial y_2} - \sin\varphi \frac{\partial}{\partial y_3}.$$

It is easy to see that FZ_2 is orthogonal to TM, thus the anti-invariant distribution is $\mathcal{D}^\perp = \mathrm{Span}\{Z_2\}$ and $\mathcal{D}^{\theta_1} = \mathrm{Span}\{Z_1\}$ is a pointwise slant distribution with slant function $\theta_1 = \arccos\left(\frac{g(FZ_1, Z_1)}{\|FZ_1\|\|Z_1\|}\right) = 2\varphi$ and hence M is a proper pointwise pseudo-slant submanifold with slant function $\theta_1 = 2\varphi$.

Example 3.3. Consider a submanifold M of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ with cartesian coordinates (x_1, x_2, y_1, y_2) and the product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \; F\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial y_i}, \;\; i = 1, 2.$$

For a real valued function v and M, define an immersion

$$\phi(u, v) = (u + v,\; \sin v,\; -u - v,\; \cos v), \quad u, v \neq 0.$$

Its tangent space TM is spanned by the vectors

$$Z_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, \; Z_2 = \frac{\partial}{\partial x_1} + \cos v \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_1} + \sin v \frac{\partial}{\partial y_2}.$$

Then with respect to the product Riemannian structure F and the usual metric tensor of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, $F(TM)$ becomes

$$FZ_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}, \; FZ_2 = \frac{\partial}{\partial x_1} + \cos v \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1} - \sin v \frac{\partial}{\partial y_2}.$$

It is easy to see that FZ_1 is orthogonal to TM and hence the anti-invariant distribution is $\mathcal{D}^\perp = \mathrm{Span}\{Z_1\}$ and $\mathcal{D}^\theta = \mathrm{Span}\{Z_2\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{\cos 2v}{3}\right)$. Since v is a real-valued function on M, then the slant function θ is not a constant and hence M is a proper pointwise pseudo-slant submanifold.

Now, we give the following useful lemma.

Lemma 3.4. *Let M be a pointwise pseudo-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$. Then*

(i) For any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$\cos^2\theta\, g(\nabla_X Y, Z) = g(A_{FZ}TY, X) + g(A_{\omega TY}Z, X). \tag{15}$$

(ii) For any $Z, V \in \Gamma(\mathcal{D}^\perp)$ and $X \in \Gamma(\mathcal{D}^\theta)$, we have

$$\cos^2\theta\, g(\nabla_Z V, X) = -g(A_{FV}TX, Z) - g(A_{\omega TX}V, Z). \tag{16}$$

Proof. We prove (i) and (ii) in a similar way. For any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(F\widetilde{\nabla}_X Y, FZ).$$

Using the locally product structure and (5), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\widetilde{\nabla}_X TY, FZ) + g(\widetilde{\nabla}_X \omega Y, FZ) \\ &= g(h(X, TY), FZ) + g(\widetilde{\nabla}_X F\omega Y, Z). \end{aligned}$$

Then from (6), we get

$$g(\nabla_X Y, Z) = g(h(A_{FZ}TY, X) + g(\widetilde{\nabla}_X B\omega Y, Z) + g(\widetilde{\nabla}_X C\omega Y, Z).$$

Thus from (13), we derive

$$\begin{aligned} g(\nabla_X Y, Z) &= g(h(A_{FZ}TY, X) + g(\widetilde{\nabla}_X \sin^2\theta Y, Z) - g(\widetilde{\nabla}_X \omega TY, Z). \\ &= g(h(A_{FZ}TY, X) + \sin^2\theta\, g(\widetilde{\nabla}_X Y, Z) + \sin 2\theta\, X(\theta)\, g(Y, Z) + g(A_{\omega TY}X, Z). \end{aligned}$$

Then by the orthogonality of two distributions and the symmetry of the shape operator, we get (i). In a similar way we can prove (ii). □

Theorem 3.5. *Let M be a proper pointwise pseudo-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$. Then*

(i) The distribution $\mathcal{D}^\perp$ is integrable if and only if

$$g(h(X, V), FZ) = g(h(X, Z), FV),$$

for any $X \in \Gamma(\mathcal{D}^\theta)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$.

(ii) The distribution $\mathcal{D}^\theta$ is integrable if and only if

$$g(A_{FZ}TY, X) - g(A_{FZ}TX, Y) = g(A_{\omega TX}Z, Y) - g(A_{\omega TY}Z, X),$$

for any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. Using polarization identity in Lemma 3.4 (ii), we have

$$\cos^2\theta\, g(\nabla_V Z, X) = -g(A_{FZ}TX, V) - g(A_{\omega TX}Z, V), \tag{17}$$

for $Z, V \in \Gamma(\mathcal{D}^\perp)$ and $X \in \Gamma(\mathcal{D}^\theta)$. Then, relations (16), (17) and the symmetry of the shape operator imply that

$$\cos^2\theta\, g([Z, V], X) = g(h(TX, V), FZ) - g(h(TX, Z), FV),$$

which gives the assertion by interchanging X by TX and using (10). Similaly, by using the polarization identity in Lemma 3.4 (i) and the definition of Lie bracket, we obtain (ii). □

Theorem 3.6. *Let M be a proper pointwise pseudo-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$. Then*

(i) The anti-invariant distribution $\mathcal{D}^{\perp}$ defines a totally geodesic foliation if and only if

$$g(h(TX, Z), FV) = -g(h(Z, V), \omega TX),$$

for any $Z, V \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D}^{\theta})$.

(ii) The pointwise slant distribution $\mathcal{D}^{\theta}$ defines a totally geodesic foliation if and only if

$$g(h(X, TY), FV) = -g(h(X, V), \omega TY),$$

for any $X, Y \in \Gamma(\mathcal{D}^{\theta})$ and $V \in \Gamma(\mathcal{D}^{\perp})$.

Proof. The proof follows from Lemma 3.4. □

Thus, the following corollary is an immediate consequence of Theorem 3.6.

Corollary 3.7. *Let M be a proper pointwise pseudo-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$. Then, M is a locally Riemannian product manifold $M = M_{\perp} \times M_{\theta}$ if and only if*

$$A_{\omega TX} V = -A_{FV} TX,$$

for any $V \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D}^{\theta})$.

4. Warped products $M_{\perp} \times_f M_{\theta}$ in locally product Riemannian manifolds

In [8], R.L. Bishop and B. O'Neill in [8] introduced the notion of warped product manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1\star} X, \pi_{1\star} Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star} X, \pi_{2\star} Y)$$

for any vector field X, Y tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* or simply a *Riemannian product manifold* if the warping function f is constant. Let X be a vector field tangent to M_1 and Z be an another vector field on M_2, then from Lemma 7.3 of [8], we have

$$\nabla_X Z = \nabla_Z X = X(\ln f) Z \tag{18}$$

where ∇ is the Levi-Civita connection on M. If $M = M_1 \times_f M_2$ be a warped product manifold then M_1 is a totally geodesic submanifold of M and M_2 is a totally umbilical submanifold of M [8, 13].

A warped product submanifold $M = M_1 \times_f M_2$ of a locally product Riemannian manifold $\widetilde{M}$ is said to be *mixed totally geodesic* if $h(X, Z) = 0$, for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, where M_1 and M_2 are any Riemannian submanifolds of $\widetilde{M}$.

In this section, we investigate the geometry of warped product pointwise pseudo-slant submanifolds of a locally product Riemannian manifold. First, we give the following example of a warped product pseudo-slant submanifold $M_{\theta} \times_f M_{\perp}$.

Example 4.1. For any $u \neq 0,\ v \in \left(0, \frac{\pi}{2}\right)$, consider a submanifold M of $\mathbb{R}^5 = \mathbb{R}^3 \times \mathbb{R}^2$ with the cartesian coordinates $(x_1, x_2, x_3, y_1, y_2)$ and the product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i},\ \ F\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial y_j},\ \ i = 1, 2, 3 \ \&\ j = 1, 2.$$

The submanifold M is given by the equations

$$x_1 = u\cos v,\ x_2 = u\sin v,\ x_3 = 2u,\ y_1 = u\cos v,\ y_2 = u\sin v.$$

Then the tangent bundle TM is spanned by Z_1 and Z_2, where

$$Z_1 = \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} + \cos v\frac{\partial}{\partial y_1} + \sin v\frac{\partial}{\partial y_2},$$

$$Z_2 = -u\sin v\frac{\partial}{\partial x_1} + u\cos v\frac{\partial}{\partial x_2} - u\sin v\frac{\partial}{\partial y_1} + u\cos v\frac{\partial}{\partial y_2}.$$

From the product Riemannian structure, we find that

$$FZ_1 = \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} - \cos v\frac{\partial}{\partial y_1} - \sin v\frac{\partial}{\partial y_2},$$

$$FZ_2 = -u\sin v\frac{\partial}{\partial x_1} + u\cos v\frac{\partial}{\partial x_2} + u\sin v\frac{\partial}{\partial y_1} - u\cos v\frac{\partial}{\partial y_2}.$$

It is easy to see that FZ_2 is orthogonal to TM and hence the anti invariant distribution is $\mathcal{D}^\perp = \mathrm{Span}\{Z_2\}$. Also, the slant distribution is spanned by the vector Z_1, i.e., $\mathcal{D}^\theta = \mathrm{Span}\{Z_1\}$ with slant angle $\theta = \cos^{-1}\left(\frac{2}{3}\right)$. Thus M is a pseudo-slant submanifold of $\mathbb{R}^5$. It is easy to check that both the distributions are integrable. We denote the integral manifolds of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by $M_\perp$ and M_θ, respectively. Then, the metric tensor g of the product manifold M is given by

$$g = 6du^2 + 2u^2\,dv^2 = g_{M_\theta} + 2u^2 g_{M_\perp}.$$

Thus M is a non-trivial warped product pseudo-slant submanifold of $\mathbb{R}^5$ of the form $M_\theta \times_{\sqrt{2}u} M_\perp$.

In [3], M. Atceken proved that there is no warped product pseudo-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$ of the form $M_\perp \times_f M_\theta$, where $M_\perp$ and M_θ are anti-invariant and proper slant submanifolds of $\widetilde{M}$, respectively. In the following examples we can see that the warped product $M_\perp \times_f M_\theta$ exists only when the spherical manifold is a pointwise slant submanifold and we call such warped product, a *pointwise pseudo-slant warped product*.

Example 4.2. Consider a submanifold of $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ with the cartesian coordinates and the product structure given in Example 3.2. Then the immersed submanifold M of $\mathbb{R}^6$ is given by

$$\chi(u, v) = (u\cos v,\ u\sin v,\ v,\ u\cos v,\ u\sin v,\ -2v)$$

such that $u \in \mathbb{R} - \{0\}$ is a real-valued function on M and $v \in \left(0, \frac{\pi}{2}\right)$. The tangent space of M is spanned by the following vectors

$$Z_1 = \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial x_2} + \cos v\frac{\partial}{\partial y_1} + \sin v\frac{\partial}{\partial y_2},$$

$$Z_2 = -u\sin v\frac{\partial}{\partial x_1} + u\cos v\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} - u\sin v\frac{\partial}{\partial y_1} + u\cos v\frac{\partial}{\partial y_2} - 2\frac{\partial}{\partial y_3}.$$

Then from the considered product Remannian structure of $\mathbb{R}^6$ in Example 3.2, we obtain

$$FZ_1 = -\cos v\frac{\partial}{\partial x_1} - \sin v\frac{\partial}{\partial x_2} + \cos v\frac{\partial}{\partial y_1} + \sin v\frac{\partial}{\partial y_2},$$

$$FZ_2 = u\sin v\frac{\partial}{\partial x_1} - u\cos v\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - u\sin v\frac{\partial}{\partial y_1} + u\cos v\frac{\partial}{\partial y_2} - 2\frac{\partial}{\partial y_3}.$$

Thus it is clear that FZ_1 is orthogonal to TM and hence the anti-invariant distribution is $\mathcal{D}^{\perp} = \text{Span}\{Z_1\}$ and $\mathcal{D}^{\theta} = \text{Span}\{Z_2\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{3}{5+2u^2}\right)$. Thus M is a poitwise pseudo-slant submanifold of $\mathbb{R}^6$. Also, it is easy to see that both the distributions are integrable. If we denote the integral manifolds of $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ by $M_{\perp}$ and M_{θ}, respectively then, the metric g of the product manifold M is given by

$$g = 2du^2 + \left(5 + 2u^2\right) dv^2 = g_{M_{\perp}} + \left(\sqrt{5 + 2u^2}\right)^2 g_{M_{\theta}}.$$

Hence, we conclude that M is a warped product pointwise pseudo-slant submanifold of $\mathbb{R}^6$ of the form $M_{\perp} \times_f M_{\theta}$ with the warping function $f = \sqrt{5 + 2u^2}$.

Example 4.3. Let $\mathbb{R}^6$ be an Euclidean space with the cartesian coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ and the almost product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \quad F\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial y_j}, \quad 1 \leq i, j \leq 3.$$

Consider a submanifold M of $\mathbb{R}^6$ defined by

$$\chi(u, v, w) = (u\cos v,\ u\sin v,\ w,\ w\cos v,\ w\sin v,\ -u)$$

for non-vanishing real-valued functions u, w on M such that $u \neq w$. Then the tangent bundle TM is spanned by Z_1, Z_2 and Z_3, where

$$Z_1 = \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_3},$$

$$Z_2 = -u\sin v\frac{\partial}{\partial x_1} + u\cos v\frac{\partial}{\partial x_2} - w\sin v\frac{\partial}{\partial y_1} + w\cos v\frac{\partial}{\partial y_2}.$$

$$Z_3 = \frac{\partial}{\partial x_3} + \cos v\frac{\partial}{\partial y_1} + \sin v\frac{\partial}{\partial y_2},$$

thus, we find that

$$FZ_1 = \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_3},$$

$$FZ_2 = -u\sin v\frac{\partial}{\partial x_1} + u\cos v\frac{\partial}{\partial x_2} + w\sin v\frac{\partial}{\partial y_1} - w\cos v\frac{\partial}{\partial y_2}.$$

$$FZ_3 = \frac{\partial}{\partial x_3} - \cos v\frac{\partial}{\partial y_1} - \sin v\frac{\partial}{\partial y_2}.$$

It is easy to see that FZ_1 and FZ_3 are orthogonal to TM. Then $\mathcal{D}^{\perp} = \text{Span}\{Z_1, Z_3\}$ is an anti-invariant distribution and $\mathcal{D}^{\theta} = \text{Span}\{Z_2\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{u^2-w^2}{u^2+w^2}\right)$. Thus M is a pointwise pseudo-slant submanifold of $\mathbb{R}^6$. It is easy to check that both the distributions are integrable. If we denote the integral manifolds of $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ by $M_{\perp}$ and M_{θ}, respectively then, the metric tensor g of the product manifold M is given by

$$g = 2du^2 + 2dw^2 + \left(u^2 + w^2\right) dv^2 = g_{M_{\perp}} + \left(u^2 + w^2\right) g_{M_{\theta}}.$$

Hence, M is a non-trivial warped product pointwise pseudo-slant submanifold of $\mathbb{R}^6$ of the form $M_{\perp} \times_f M_{\theta}$ with the warping function $f = \sqrt{u^2 + w^2}$.

Example 4.4. Consider a submanifold of $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ with the cartesian coordinates and the product structure given in Example 3.2. Let M be a submanifold of $\mathbb{R}^6$ given by the equations

$$x_1 = u\cosh v,\ x_2 = u\sinh v,\ x_3 = -v,\ y_1 = u\sinh v,\ y_2 = u\cosh v,\ y_3 = \sqrt{2}\,v$$

such that $u, v \in \mathbb{R} - \{0\}$ are real-valued functions on M and $v \in \left(0, \frac{\pi}{2}\right)$. Then the tangent bundle of M is spanned by Z_1 and Z_2, where

$$Z_1 = \cosh v\frac{\partial}{\partial x_1} + \sinh v\frac{\partial}{\partial x_2} + \sinh v\frac{\partial}{\partial y_1} + \cosh v\frac{\partial}{\partial y_2},$$

$$Z_2 = u\sinh v\frac{\partial}{\partial x_1} + u\cosh v\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + u\cosh v\frac{\partial}{\partial y_1} + u\sinh v\frac{\partial}{\partial y_2} + \sqrt{2}\frac{\partial}{\partial y_3}.$$

Hence, we find

$$FZ_1 = -\cosh v\frac{\partial}{\partial x_1} - \sinh v\frac{\partial}{\partial x_2} + \sinh v\frac{\partial}{\partial y_1} + \cosh v\frac{\partial}{\partial y_2},$$

$$FZ_2 = -u\sinh v\frac{\partial}{\partial x_1} - u\cosh v\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + u\cosh v\frac{\partial}{\partial y_1} + u\sinh v\frac{\partial}{\partial y_2} + \sqrt{2}\frac{\partial}{\partial y_3}.$$

Since FZ_1 is orthogonal to TM and

$$\theta = \cos^{-1}\left(\frac{g(Z_2, FZ_2)}{\|Z_2\|.\|FZ_2\|}\right) = \cos^{-1}\left(\frac{1}{3 + 2u^2\cosh 2v}\right).$$

Then the anti-invariant distribution is $\mathcal{D}^\perp = \mathrm{Span}\{Z_1\}$, and $\mathcal{D}^\theta = \mathrm{Span}\{Z_2\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{1}{3+2u^2\cosh 2v}\right)$. Hence, M is a pointwise pseudo-slant submanifold of $\mathbb{R}^6$. Clearly, both the distributions are integrable. If the integral manifolds of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ are $M_\perp$ and M_θ, respectively then, the metric g of the product manifold M is given by

$$\begin{aligned} g &= 2\cosh 2v\, du^2 + \left(3 + 2u^2\cosh 2v\right) dv^2 \\ &= \left(\sqrt{2\cosh 2v}\right)^2 g_{M_\perp} + \left(\sqrt{3 + 2u^2\cosh 2v}\right)^2 g_{M_\theta}. \end{aligned}$$

Thus M is a warped product pointwise pseudo-slant submanifold of $\mathbb{R}^6$ of the form ${}_{f_2}M_\perp \times_{f_1} M_\theta$ with warping functions $f_1 = \sqrt{3 + 2u^2\cosh 2v}$ and $f_2 = \sqrt{2\cosh 2v}$. In fact, M is a doubly warped product submanifold of $\mathbb{R}^6$ with the warping functions f_1 and f_2.

Now, we investigate the geometry of the warped product pointwise pseudo-slant submanifolds of form $M_\perp \times_f M_\theta$. First, we prove the following useful lemma for later use.

Lemma 4.5. *Let $M = M_\perp \times {}_f M_\theta$ be a warped product pointwise pseudo-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$. Then*

(i) $g(h(Z, V), \omega X) = -g(h(X, Z), FV)$;
(ii) $g(h(X, Z), \omega Y) = -g(h(Y, Z), \omega X)$

for any $Z, V \in \Gamma(TM_\perp)$ and $X, Y \in \Gamma(TM_\theta)$.

Proof. For any $Z, V \in \Gamma(TM_\perp)$ and $X, Y \in \Gamma(TM_\theta)$, we have

$$\begin{aligned} g(h(Z, V), \omega X) &= g(\widetilde{\nabla}_Z V, \omega X) \\ &= g(\widetilde{\nabla}_Z V, FX) - g(\widetilde{\nabla}_Z V, TX) \\ &= g(\widetilde{\nabla}_Z FV, X) + g(\widetilde{\nabla}_Z TX, V). \end{aligned}$$

Then from (2), (3) and (18), we obtain

$$g(h(Z,V),\omega X) = -g(A_{FV}Z,X) + Z(\ln f)\,g(TX,V).$$

Second term in the right hand side of above relation vanishes identically by the orthogonality of the vector fields, thus we have

$$g(h(Z,V),\omega X) = -g(h(X,Z),FV)$$

which is (i). For the second part of the lemma, we have

$$g(\widetilde{\nabla}_Z X,TY) = Z(\ln f)\,g(X,TY) \tag{19}$$

for any $X,Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$. On the other hand, we also have

$$\begin{aligned} g(\widetilde{\nabla}_Z X,TY) &= g(\widetilde{\nabla}_Z X,FY) - g(\widetilde{\nabla}_Z X,\omega Y) \\ &= g(\widetilde{\nabla}_Z FX,Y) - g(h(X,Z),\omega Y) \\ &= g(\widetilde{\nabla}_Z TX,Y) + g(\widetilde{\nabla}_Z \omega X,Y) - g(h(X,Z),\omega Y) \\ &= Z(\ln f)\,g(TX,Y) - g(\widetilde{\nabla}_Z Y,\omega X) - g(h(X,Z),\omega Y). \end{aligned}$$

Then from (1) and (3), we find that

$$g(\widetilde{\nabla}_Z X,TY) = Z(\ln f)\,g(X,TY) - g(h(Y,Z),\omega X) - g(h(X,Z),\omega Y). \tag{20}$$

Thus, (ii) follows from (19) and (20), which proves the lemma completely. □

Lemma 4.6. *Let $M = M_\perp \times {}_f M_\theta$ be a warped product pointwise pseudo-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$. Then*

(i) $g(h(X,Y),FZ) = -Z(\ln f) g(X,TY)$,
(ii) $g(h(TX,Y),FZ) = -\cos^2\theta\, Z(\ln f)\,g(X,Y) - g(h(Y,Z),\omega TX)$

for any $X,Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. For any $X,Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we have

$$g(h(X,Y),FZ) = g(\widetilde{\nabla}_X Y,FZ) = g(\widetilde{\nabla}_X FY,Z).$$

Using (5), we obtain

$$g(h(X,Y),FZ) = g(\widetilde{\nabla}_X TY,Z) + g(\widetilde{\nabla}_X \omega Y,Z) = -g(\widetilde{\nabla}_X Z,TY) - g(A_{\omega Y}X,Z).$$

Then from (4) and (18), we derive

$$g(h(X,Y),FZ) = -Z(\ln f)\,g(X,TY) - g(h(X,Z),\omega Y). \tag{21}$$

By polarization identity, we derive

$$g(h(X,Y),FZ) = -Z(\ln f)\,g(Y,TX) - g(h(Y,Z),\omega X). \tag{22}$$

Using (1) and Lemma 4.5 (ii), we arrive at

$$g(h(X,Y),FZ) = -Z(\ln f)\,g(X,TY) + g(h(X,Z),\omega Y). \tag{23}$$

Thus from (21) and (23), we get (i). The second part of the lemma follows from (22) by interchanging X by TX and using (11). Hence, the proof is complete. □

We can easily find the following relations by interchanging X by TX and Y by TY, for any $X, Y \in \Gamma(TM_\theta)$ in Lemma 4.6 (ii) as follows.

$$g(h(TX, Y), FZ) = -Z(\ln f)\cos^2\theta\, g(X, Y), \tag{24}$$

$$g(h(X, TY), FZ) = -Z(\ln f)\cos^2\theta\, g(X, Y) \tag{25}$$

and

$$g(h(TX, TY), FZ) = -Z(\ln f)\cos^2\theta\, g(X, TY). \tag{26}$$

Then from (21) and (26), we get

$$g(h(TX, TY), FZ) = g(h(X, Y), FZ).$$

Also, from (24) and (25), we have

$$g(h(TX, Y), FZ) = g(h(X, TY), FZ).$$

In order to give a characterization we need the following well known result of S. Hiepko [22].

Hiepko's Theorem. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two orthogonal distribution on a Riemannian manifold M. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ both are involutive such that $\mathcal{D}_1$ is a totally geodesic foliation and $\mathcal{D}_2$ is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.*

The following result give a characterization of warped product pointwise pseudo-slant submanifolds.

Theorem 4.7. *Let M be a pointwise pseudo-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$. Then M is locally a warped product submanifold of the form $M_\perp \times_f M_\theta$ if and only if*

$$A_{\omega TX}V + A_{FV}TX = -V(\mu)\cos^2\theta\, X, \ \ \forall\, X \in \Gamma(\mathcal{D}^\theta),\ V \in \Gamma(\mathcal{D}^\perp) \tag{27}$$

for some smooth function μ on M satisfying $Y(\mu) = 0$, for any $Y \in \Gamma(\mathcal{D}^\theta)$.

Proof. Let $M = M_\perp \times_f M_\theta$ be a warped product pointwise pseudo-slant submanifold. Then from Lemma 4.5 (i), we have $g(A_{\omega X}V, Z) = -g(A_{FV}X, Z)$, for any $X \in \Gamma(TM_\theta)$ and $Z, V \in \Gamma(TM_\perp)$. Interchanging X by TX, we get $g(A_{\omega TX}V + A_{FV}TX, Z) = 0$, which means that $A_{\omega TX}V + A_{FV}TX$ has no component in $TM_\perp$, i.e., $A_{\omega TX}V + A_{FV}TX$ lies in TM_θ. Using this fact with Lemma 4.6 (ii), we get (27).

Conversely, if M is a pointwise pseudo-slant submanifold such that (27) holds, then from Lemma 3.4 (ii), we have

$$g(\nabla_Z V, X) = -\sec^2\theta\, g(A_{\omega TX}V + A_{FV}TX, Z).$$

Using the hypothesis of the theorem, i.e., the relation (27) and the orthogonality of two distributions, we arrive at

$$g(\nabla_Z V, X) = 0$$

for any $Z, V \in \Gamma(\mathcal{D}^\perp)$ and $X \in \Gamma(\mathcal{D}^\theta)$, which means that the leaves of the distribution $\mathcal{D}^\perp$ are totally geodesic in M. Let $M_\perp$ be a leaf of $\mathcal{D}^\perp$, thus $M_\perp$ is totally geodesic in M. Also, from Lemma 3.4 (i), we have

$$\cos^2\theta\, g(\nabla_X Y, V) = g(A_{FV}TY + A_{\omega TY}V, X).$$

Using (27), we derive

$$g(\nabla_X Y, V) = -V(\mu)\, g(X, Y). \tag{28}$$

By polarization identity, we obtain

$$g(\nabla_Y X, V) = -V(\mu)\, g(X, Y). \tag{29}$$

Subtracting (29) from (28) and using the definition of Lie bracket, we find that

$$g([X, Y], V) = 0,$$

which implies that the pointwise slant distribution $\mathcal{D}^\theta$ is integrable. Let M_θ be the integral manifold of $\mathcal{D}^\theta$ and h^θ be the second fundamental form of M_θ in M. Then, for any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $V \in \Gamma(\mathcal{D}^\perp)$, from (28) we have

$$g(h^\theta(X, Y), V) = g(\nabla_X Y, V) = -V(\mu)\, g(X, Y)$$

or equivalently, we have

$$h^\theta(X, Y) = -\vec{\nabla}\mu\, g(X, Y) \tag{30}$$

where $\vec{\nabla}\mu$ is the gradient vector of the function μ which means that M_θ is totally umbilical in M with mean curvature vector $H^\theta = -\vec{\nabla}\mu$. Furthermore, $Y(\mu) = 0$, $Y \in \Gamma(\mathcal{D}^\theta)$ implies that H^θ is parallel with respect to the normal connection D^n of M_θ in M. Thus M_θ is a totally umbilical submanifold with non-vanishing parallel mean curvature vector H^θ. Hence the spherical condition is also fulfilled, that is M_θ is an extrinsic sphere in M. Then, from Hiepko Theorem, M is a non-trivial warped product of the form $M = M_\perp \times_\mu M_\theta$, which proves the theorem completely. □

Remark 4.8. If we assume $\theta = 0$ in Theorem 4.7, then the warped product pointwise pseudo-slant submanifolds reduce to warped product semi-invariant submanifolds of the form $M_\perp \times_f M_T$ which have been discussed in [29], thus Theorem 4.7 is a generalization of Theorem 4.1 of [29].

5. An optimal inequality for warped products $M_\perp \times_f M_\theta$

In this section, we establish a sharp inequality for the squared norm of the second fundamental form in terms of the warping function. First, we construct the following frame fields for an $n = (p + q)$-dimensional warped product pointwise pseudo-slant submanifold $M = M_\perp \times_f M_\theta$ of a m-dimensional locally product Riemannian manifold $\widetilde{M}$, where $M_\perp$ and M_θ are anti-invariant and proper pointwise slant submanifolds of $\widetilde{M}$, respectively. Let us denote by $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ the tangent bundles of $M_\perp$ and M_θ, respectively. Also, if we consider the $\dim(M_\perp) = q$ and $\dim(M_\theta) = p$, then the orthonormal frames of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively are given by

$$\{e_1, \cdots, e_q\}$$

and

$$\{e_{q+1} = e_1^* = \sec\theta\, Te_1^*, \cdots, e_n = e_{q+p} = e_p^* = \sec\theta\, Te_p^*\}.$$

Then the orthonormal frame fields of the normal subbundles of $F\mathcal{D}^\perp$, $\omega\mathcal{D}^\theta$ and ν, respectively are

$$\{e_{n+1} = Fe_1, \cdots, e_{n+q} = Fe_q\},$$

$$\{e_{n+q+1} = \widetilde{e}_1 = \csc\theta\, \omega e_1^*, \cdots, e_{2n} = \widetilde{e}_p = \csc\theta\, \omega e_p^*\}$$

and

$$\{e_{n+q+p+1} = \widetilde{e}_{p+1}, \cdots, e_m = \widetilde{e}_{m-2n}\}.$$

Now, we are able to establish the following inequality with the help of the above constructed frame fields and some previous formulas which we have obtained for warped product semi-slant submanifolds of a locally product Riemannian manifold.

Theorem 5.1. *Let $M = M_\perp \times_f M_\theta$ be a proper warped product pointwise pseudo-slant submanifold of a locally product Riemannian manifold $\widetilde{M}$, where $M_\perp$ and M_θ are anti-invariant and proper pointwise slant submanifolds of $\widetilde{M}$, respectively. Then*

(i) The squared norm of the second fundamental form of M satisfies

$$\|h\|^2 \geq p\cos^2\theta\, \|\vec{\nabla} \ln f\|^2 \tag{31}$$

where $p = \dim M_\theta$ and $\vec{\nabla}\ln f$ is gradient of $\ln f$.

(ii) If equality sign in (i) holds identically, then $M_\perp$ and M_θ are totally geodesic and totally umbilical submanifolds of $\widetilde{M}$, respectively. Furthermore, $M_\perp \times_f M_\theta$ is a mixed totally geodesic submanifold of $\widetilde{M}$

Proof. From the definition of h, we have

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=n+1}^{m}\sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2.$$

Thus from the frame fields of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, we find

$$\|h\|^2 = \sum_{r=n+1}^{m}\sum_{i,j=1}^{q} g(h(e_i, e_j), e_r)^2 + 2\sum_{r=n+1}^{m}\sum_{i=1}^{q}\sum_{j=q+1}^{n} g(h(e_i, e_j), e_r)^2 + \sum_{r=n+1}^{m}\sum_{i,j=q+1}^{n} g(h(e_i, e_j), e_r)^2. \tag{32}$$

Leaving the second positive term in the right hand side of above relation. Then, we have

$$\|h\|^2 \geq \sum_{r=n+1}^{m}\sum_{i,j=1}^{q} g(h(e_i, e_j), e_r)^2 + \sum_{r=n+1}^{m}\sum_{i,j=1}^{p} g(h(e_i^*, e_j^*), e_r)^2.$$

Using the frame fields of $F\mathcal{D}^\perp$, $\omega\mathcal{D}^\theta$ and ν, the above equation takes the form

$$\begin{aligned}\|h\|^2 &= \sum_{r=1}^{q}\sum_{i,j=1}^{q} g(h(e_i, e_j), Fe_r)^2 + \sum_{r=1}^{p}\sum_{i,j=1}^{q} g(h(e_i, e_j), \widetilde{e}_r)^2 + \sum_{r=p+1}^{m-2n}\sum_{i,j=1}^{q} g(h(e_i, e_j), \widetilde{e}_r)^2 \\ &+ \sum_{r=1}^{q}\sum_{i,j=1}^{p} g(h(e_i^*, e_j^*), Fe_r)^2 + \sum_{r=1}^{p}\sum_{i,j=1}^{p} g(h(e_i^*, e_j^*), \widetilde{e}_r)^2 \sum_{r=p+1}^{m-2n}\sum_{i,j=1}^{p} g(h(e_i^*, e_j^*), \widetilde{e}_r)^2.\end{aligned} \tag{33}$$

The third and sixth term have ν-components and we have not found any relation for these components, therefore we can leave these two positive terms. Also, we could not find any relation for $g(h(e_i, e_j), Fe_r)$, for any $i, j, r = 1, \cdots, q$ and $g(h(e_i^*, e_j^*), \widetilde{e}_r)$, for any $i, j, r = 1, \cdots, p$, therefore we shall leave these positive terms also. After leaving these terms in the right hand side of (33) and using the constructed frame fields, we find

$$\|h\|^2 \geq \sum_{r=1}^{p}\sum_{i,j=1}^{q} g(h(e_i, e_j), \csc\theta\, \omega e_r^*)^2 + \sum_{r=1}^{q}\sum_{i,j=1}^{p} g(h(e_i^*, e_j^*), Fe_r)^2. \tag{34}$$

Again leaving the first positive term in the right hand side of above equation. Thus, from Lemma 4.6 (i), we derive

$$\|h\|^2 \geq \sum_{r=1}^{q}\sum_{i,j=1}^{p} \Big((e_r \ln f)\, g(e_i^*, Te_j^*)\Big)^2.$$

Then from the adopted frame fields of $\mathcal{D}^\theta$, we know that $Te_j^* = \cos\theta\, e_j^*$, using this fact in the above relation, then we have

$$||h||^2 \geq p\cos^2\theta \sum_{r=1}^{q} (e_r \ln f)^2.$$

Thus by using (9), we get

$$||h||^2 \geq p\cos^2\theta ||\vec{\nabla} \ln f||^2,$$

which is inequality (i). From the leaving second term in the right hand side of (32), we have

$$h(\mathcal{D}^\perp, \mathcal{D}^\theta) = 0. \tag{35}$$

Also, from the remaining first and third terms of (33), we obtain

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) \subset \omega\mathcal{D}^\theta. \tag{36}$$

On the other hand, from Lemma 4.5 (i) and (35), we find that

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp \omega\mathcal{D}^\theta. \tag{37}$$

Then from (36) and (37), we conclude that

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0. \tag{38}$$

Also, from the remaining fifth and sixth terms in the right hand side of (33), we find that

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) \subset F\mathcal{D}^\perp. \tag{39}$$

Since $M_\perp$ is totally geodesic in M [8, 13], using this fact with (37) we get $M_\perp$ is totally geodesic in $\widetilde{M}$. On the other hand, (38) implies that M_θ is totally umbilical in $\widetilde{M}$ due to M_θ being totally umbilical in M [8, 13]. Moreover, (35), (37) and (38) imply that M is a mixed totally geodesic submanifold of $\widetilde{M}$. Hence, the proof is complete $\square$

From the above theorem, we have the following remark.

Remark 5.2. In Theorem 5.1, if we put $\theta = 0$, then the warped product becomes $M = M_\perp \times_f M_T$ in a locally product Riemannian manifold $\widetilde{M}$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\widetilde{M}$, respectively, which is a case of warped product semi-nivariant submanifolds which have been discussed in ([5], [29]). Thus, Theorem 4.2 of [29] and Theorem 4.1 of [5] are the special cases of Theorem 5.1.

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