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Atomic Decompositions of Martingale Hardy-Lorentz Spaces and Interpolation

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Abstract. In this paper, we establish atomic decompositions for the martingale Hardy-Lorentz spaces. As an application, with the help of atomic decomposition, some interpolation theorems with a function parameter for these spaces are proved.

1. introduction and preliminaries

The main result of this paper is the atomic decompositions of martingale Hardy-Lorentz spaces which is a martingale function spaces built on the "classical" Lorentz spaces. The martingale Hardy type spaces is a main topic for theory of martingale function spaces. There are several generalizations obtained recently such as the martingale Hardy-Orlicz spaces [15], martingale Hardy-Morrey spaces [9] and martingale Hardy spaces with variable exponents [12]. Therefore, the martingale function spaces introduced in this article gives further generalizations on this topic.

Atomic decomposition plays a fundamental role in the classical martingale theory and harmonic analysis. For instance, atomic decomposition is a powerful tool for dealing with duality theorems, interpolation theorems and some fundamental inequalities both in martingale theory and harmonic analysis. In [3] Coifman used the Fefferman-Stein theory of H^p spaces [5] to decompose the functions of these spaces into basic building blocks (atoms). Coifman and Weiss have provided a comprehensive treatment of these ideas and many applications to harmonic analysis in [4]. In [11], Jiao et al. proved that the Lorentz martingales spaces also have an atomic decomposition. Hou and Ren [10] considered weak atomic decomposition of weak martingale Hardy spaces. Recently, Ho introduced the martingale Hardy-Lorentz-Karamata spaces and proved atomic decomposition of these martingale function spaces [7]. In this article, the atomic decomposition for the martingale Hardy-Lorentz spaces is established in section 2 which is the main result of this paper. By using these decompositions, we obtain the interpolation of the the martingale Hardy-Lorentz spaces by using the interpolation functor with function parameter. Notice that the interpolation functor used in this paper is a special case of a general family of interpolation functors appeared in [8]. To achieve our goal we first fix our notations and terminology. Let us denote the set of integers and the set of non–negative integers, by **Z** and **N**, respectively.

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Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration $(\mathcal{F}_n)_{n\in\mathbf{N}}$ is a non-decreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\cup_{n\in\mathbf{N}}\mathcal{F}_n)$. We denote by E and E_n the expectation and the conditional expectation operators with respect to $(\mathcal{F}_n)_{n\in\mathbf{N}}$. For simplicity, we assume that $E_n f = 0$ if $n = 0$.

For a martingale $f = (f_n, n \in \mathbf{N})$ relative to $(\Omega, \mathcal{F}, P)$, denote the martingale differences by $d_n f := f_n - f_{n-1}$ with convention $d_0 f = 0$. For an arbitrary stopping time ν and a martingale f, $f^\nu = (f_n^\nu, n \in \mathbf{N})$ is defined by

$$f_n^\nu := \sum_{m=0}^{n} \chi(\nu \geq m) d_m f.$$

The conditional square function of f is defined by

$$s_m(f) := \left(\sum_{n\leq m} E_{n-1} \mid d_n f \mid^2 \right)^{1/2} \quad , \quad s(f) := \left(\sum_{n\in\mathbf{N}} E_{n-1} \mid d_n f \mid^2 \right)^{1/2} .$$

Let us recall briefly the construction of Lorentz spaces and the real interpolation method. For measurable function f, we define a distribution function $m(s, f)$ by setting $m(s, f) = P(\{w \in \Omega : |f(w)| > s\})$. The function

$$f^*(t) = \inf\{s > 0 : m(s, f) \leq t\}, \quad (t \geq 0),$$

is called the decreasing rearrangement of f.

we say that a nonnegative function is a weight, if it is locally integrable. Let φ be a weight. The classical Lorentz spaces $\Lambda_q(\varphi)$ is defined to be the collection of all measurable functions f for which the quantity

$$\|f\|_{\Lambda_q(\varphi)} := \begin{cases} \left(\int_0^\infty (f^*(t)\varphi(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} & (0 < q < \infty), \\ \sup_t f^*(t)\varphi(t) & (q = \infty), \end{cases}$$

is finite. Recall that for $0 < q \leq \infty$, $\|.\|_{\Lambda_q(\varphi)}$ is only a quasi-norm. Also $\Lambda_q(\varphi)$ is a quasi–Banach space with the quasi-norm

$$\|f\|^q_{\Lambda_q(\varphi)} = q \int_0^\infty y^{q-1} w^q (m(y, f))\, dy, \qquad (0 < q < \infty),$$

where $w(t) = \left(\int_0^t \varphi^q(s) \frac{ds}{s} \right)^{\frac{1}{q}}$ is a non-decreasing weight and satisfies the Δ_2-condition, $w(2t) \leq Cw(t)$, for some $C > 0$ (see [2]).

For $q = \infty$ we have

$$\|f\|_{\Lambda_\infty(\varphi)} = \sup_s sw(m(s, f)).$$

For $0 < q \leq \infty$, martingale Hardy-Lorentz spaces $\Lambda_q^s(\varphi)$ is defined by:

$$\Lambda_q^s(\varphi) = \left\{ f = (f_n)_{n\in\mathbf{N}} : \|f\|_{\Lambda_q^s(\varphi)} := \|s(f)\|_{\Lambda_q(\varphi)} < \infty \right\}.$$

Note that if $\varphi(t) = t^{\frac{1}{p}}$, then $\Lambda_q(\varphi) = L_{p,q}$ and $\Lambda_q^s(\varphi) = H_{p,q}^s$. In particular, if $\varphi(t) = t^{\frac{1}{q}}$, then $\Lambda_q(\varphi) = L_q$ and $\Lambda_q^s(\varphi) = H_q^s$.

Let (A_0, A_1) be a quasi–Banach couple, that is, two quasi-Banach spaces A_0, A_1 which are continuously embedded in a Hausdorff topological vector space A. The K–functional is defined by

$$K(t, f, A_0, A_1) = K(t, f) := \inf_{f_0+f_1=f} \{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\}$$

for $t > 0$ and $f \in A_0 + A_1$, where $f_i \in A_i$, $i = 0, 1$.

For $0 < q \leq \infty$ and each measurable function ϱ, the real interpolation space $(A_0, A_1)_{\varrho,q}$ consists of all elements of $f \in A_0 + A_1$ such that the quantity

$$\|f\|_{(A_0,A_1)_{\varrho,q}} := \begin{cases} \left(\int_0^\infty \left(\frac{K(t,f)}{\varrho(t)} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} & (0 < q < \infty), \\ \sup_{t>0} \frac{K(t,f)}{\varrho(t)} & (q = \infty), \end{cases}$$

is finite. Let a and b be real numbers such that $a < b$. Following Persson's convention [16], we adopt the following notations. The notation $\varphi(t) \in Q[a,b]$ means that $\varphi(t)t^{-a}$ is non–decreasing and $\varphi(t)t^{-b}$ is non-increasing for all $t > 0$. Moreover, we say that $\varphi(t) \in Q(a,b)$, wherever $\varphi(t) \in Q[a+\epsilon, b-\epsilon]$ for some $\epsilon > 0$. The notation $\varphi(t) \in Q(a,-)$ (or $\varphi(t) \in Q(-,b)$) means that $\varphi(t) \in Q(a,c)$ (or $\varphi(t) \in Q(c,b)$) for some real number c and by $\varphi(t) \in Q(-,-)$, we mean that $\varphi(t) \in Q(c,c')$ for some real numbers c, c' such that $c < c'$. In this paper we shall consider the interpolation spaces $(A_0, A_1)_{\varrho,q}$ with a parameter function $\varrho = \varrho(t) \in Q(0,1)$ where A_0 and A_1 are the martingale spaces.

It is easy to see that $\varrho(t) = t^{\theta} (0 < \theta < 1)$ belongs to $Q(0,1)$, so by replacing measurable function $\varrho = \varrho(t)$ with t^{θ} we obtain $(A_0, A_1)_{\theta,q}$.

Let $0 < p < \infty, 0 < q \leq \infty$ and $\varrho \in Q(0,1)$. It was proved by Persson [16, Lemma 6.1] that

$$(L_p, L_\infty)_{\varrho,q} = \Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}})). \tag{1}$$

We shall need the following well-known result due to T. Aoki and S. Rolewicz, which states that every quasi-normed vector space can be equipped with an equivalent r–norm [13].

Theorem 1.1. *(Aoki-Rolewicz). Let X be a quasi-normed vector space. Then there is a* $C > 0$ *and* $0 < r \leq 1$ *such that for any* $x_1, ..., x_n \in X$,

$$\left\| \sum_{i=1}^{n} x_i \right\|_X \leq C \left(\sum_{i=1}^{n} \|x_i\|_X^r \right)^{\frac{1}{r}}.$$

In what follows, $a \lesssim b$ means that $a \leq Cb$ for some positive constant C independent of the quantities a and b. If both $a \lesssim b$ and $b \lesssim a$ are satisfied (with possibly different constants), we write $a \approx b$. We use C to denote a constant, which may be different in different places. Throughout this article, by w we mean

$$w(t) = \left(\int_0^t \varphi^q(s) \frac{ds}{s} \right)^{\frac{1}{q}}, \quad (q < \infty),$$

for a given weight φ in $\Lambda_q^s(\varphi)$, and $w \in \Delta_2$.

2. Atomic Decomposition

In this section, we provide an atomic decomposition for the martingale Hardy-Lorentz spaces $\Lambda_p^s(\varphi)$, which is an extension of the atomic decomposition of the martingale Hardy spaces H_p^s that was proved by Weisz [18].

Definition 2.1. *A measurable function a is called a* (p, ∞) *atom if there exists a stopping time* ν *such that*

1. $a_n := E_n a = 0$ *if* $\nu \geq n$.
2. $\| s(a) \|_\infty \leq P(\nu \neq \infty)^{-1/p}$.

Theorem 2.2. *If* $f = (f_n, n \in \mathbf{N}) \in \Lambda_q^s(\varphi)$ $(0 < q \leq \infty)$, *then there exists a sequence* $\{(a^k, \nu_k)\}_{k \in \mathbf{Z}}$ *of* (p, ∞) *atoms* $(0 < p < \infty)$ *such that*

$$\sum_{k=-\infty}^{\infty} \mu_k E_n a^k = f_n$$

where $\mu_k = 2^k 3P(\nu_k \neq \infty)^{1/p}$ *and*

$$\|\{2^k w(P(\nu_k \neq \infty))\}_{k \in \mathbf{Z}}\|_{l_q} \lesssim \|f\|_{\Lambda_q^s(\varphi)}.$$

Moreover, if $0 < q \leq 1$, *then*

$$\|f\|_{\Lambda_q^s(\varphi)} \approx \inf \|\{2^k w(P(\nu_k \neq \infty))\}_{k \in \mathbf{Z}}\|_{l_q}$$

where the infimum is taken over all the preceding decompositions of f.

Proof. Let $f = (f_n, n \in \mathbf{N}) \in \Lambda_q^s(\varphi)$. For any $k \in \mathbf{Z}$, define

$$\nu_k := \inf\left\{n \in \mathbf{N} : s_{n+1}(f) > 2^k\right\}.$$

Then ν_k is a stopping time and non–decreasing with respect to k and $\nu_k \to \infty$ when $k \to \infty$. It is easy to see that

$$\begin{aligned}\sum_{k\in\mathbf{Z}} (f_n^{\nu_{k+1}} - f_n^{\nu_k}) &= \sum_{k\in\mathbf{Z}} \left(\sum_{m=0}^{n} (\chi(\nu_{k+1} \geq m)d_m f - \chi(\nu_k \geq m)d_m f)\right) \\ &= \sum_{m=0}^{n} \left(\sum_{k\in\mathbf{Z}} \chi(\nu_k < m \leq \nu_{k+1})d_m f\right) = f_n.\end{aligned}$$

Now let

$$a_n^k = \frac{f_n^{\nu_{k+1}} - f_n^{\nu_k}}{\mu_k}.$$

We assume that $a_n^k = 0$ if $\mu_k = 0$. It is clear that for a fixed $k \in \mathbf{Z}$, $(a_n^k, n \in \mathbf{N})$ is a martingale. Since $s(f_n^{\nu_k}) = s_{\nu_k}(f_n) \leq 2^k$, then

$$s(a_n^k) \leq \frac{s(f_n^{\nu_{k+1}}) + s(f_n^{\nu_k})}{\mu_k} \leq P(\nu_k \neq \infty)^{-1/p}, \quad (n \in \mathbf{N}).$$

Consequently, (a_n^k) is L_2–bounded and so there exists $a^k \in L_2$ such that $E_n a^k = a_n^k$. If $n \leq \nu_k$, then $a_n^k = 0$ and $\| s(a) \|_\infty \leq P(\nu \neq \infty)^{-1/p}$. Therefore, a^k is a (p, ∞) atom and

$$f_n = \sum_{k\in\mathbf{Z}} (f_n^{\nu_{k+1}} - f_n^{\nu_k}) = \sum_{k\in\mathbf{Z}} \mu_k a_n^k = \sum_{k\in\mathbf{Z}} \mu_k E_n a^k.$$

Let $0 < q < \infty$. It follows from $\{\nu_k \neq \infty\} = \{s(f) > 2^k\}$ for any $k \in \mathbf{Z}$, that

$$\begin{aligned}\sum_{k\in\mathbf{Z}} 2^{kq} w^q(P(\nu_k \neq \infty)) &= \sum_{k\in\mathbf{Z}} 2^{kq} w^q(P(s(f) > 2^k)) \\ &\lesssim \sum_{k\in\mathbf{Z}} \int_{2^{k-1}}^{2^k} y^{q-1} dy w^q(P(s(f) > 2^k)) \\ &\lesssim \sum_{k\in\mathbf{Z}} \int_{2^{k-1}}^{2^k} y^{q-1} w^q(P(s(f) > y))dy \\ &\lesssim \int_0^\infty y^{q-1} w^q(P(s(f) > y))dy \\ &= \frac{1}{q} \|f\|^q_{\Lambda_q^s(\varphi)}.\end{aligned}$$

For $q = \infty$ we have

$$2^k w(P(\nu_k \neq \infty)) = 2^k w(P(s(f) > 2^k)) \lesssim \| s(f) \|_{\Lambda_\infty(\varphi)} =: \|f\|_{\Lambda_\infty^s(\varphi)}$$

which implies $\sup_{k\in\mathbf{Z}} 2^k w(P(\nu_k \neq \infty)) \lesssim \|f\|_{\Lambda_\infty^s(\varphi)}$.

Now we prove the last part of the theorem. Since $a_n^k = E_n a^k = 0$ on the set $\{\nu_k \geq n\}$,

$$\chi(\nu_k \geq n)E_{n-1} \mid d_n a \mid^2 = E_{n-1}\chi(\nu_k \geq n) \mid d_n a \mid^2 = 0.$$

Hence, $s(a^k) = 0$ on the set $\{\nu_k = \infty\}$.
So, we have

$$P(s(a^k) > y) \leq P(s(a^k) \neq 0) \leq P(\nu_k \neq \infty). \tag{2}$$

It follows from $\|s(a^k)\|_\infty < P(\nu_k \neq \infty)^{-1/p}$ and (2) that

$$\begin{aligned}\|a^k\|^q_{\Lambda^s_q(\varphi)} &= q\int_0^\infty y^{q-1}w^q(P(s(a^k)>y))dy\\ &= q\int_0^{P(\nu_k\neq\infty)^{-1/p}} y^{q-1}w^q(P(s(a^k)>y))dy\\ &\leq qw^q(P(\nu_k\neq\infty))\int_0^{P(\nu_k\neq\infty)^{-1/p}} y^{q-1}dy\\ &\leq w^q(P(\nu_k\neq\infty))P(\nu_k\neq\infty)^{-q/p}.\end{aligned}$$

Finally, since for $0 < q \leq 1$ by Theorem 1.1, the quasi-normed $\|.\|_{\Lambda^s_p(\varphi)}$ is equivalent to a q–norm,

$$\begin{aligned}\|f\|^q_{\Lambda^s_q(\varphi)} &\leq \left\|\sum_{k\in\mathbf{Z}}\mu_k s(a^k)\right\|^q_{\Lambda_q(\varphi)} \lesssim \sum_{k\in\mathbf{Z}}\mu_k^q\left\|s(a^k)\right\|^q_{\Lambda_q(\varphi)}\\ &\leq \sum_{k\in\mathbf{Z}}\mu_k^q w^q(P(\nu_k\neq\infty))P(\nu_k\neq\infty)^{-q/p} \lesssim \sum_{k\in\mathbf{Z}} 2^{kq}w^q(P(\nu_k\neq\infty)).\end{aligned}$$

The proof is complete. □

3. Interpolation

As an application of atomic decomposition, the interpolation spaces with a function parameter between the martingale Hardy-Lorentz spaces are identified.

Theorem 3.1. *Let $0 < p \leq 1$, $0 < q \leq \infty$ and $\varrho \in Q(0,1)$ be a parameter function. Then*

$$(H^s_p, H^s_\infty)_{\varrho,q} = \Lambda^s_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}})).$$

In order to prove the theorem 3.1, the authors of paper [17] used a standard method: their method needs a decreasing rearrangement function inequality. Our approach differs from their method: we will use atomic decomposition method. To prove Theorem 3.1, we need the following lemmata.

Lemma 3.2. *Let $f \in \Lambda^s_q(\varphi)$, $0 < q \leq \infty$, $y > 0$ and fix $0 < p \leq 1$. Then f can be decomposed into the sum of two martingales g and h such that*

$$\|g\|_{H^s_\infty} \leq 6y$$

and

$$\|h\|_{H^s_p} \lesssim \left(\int_{\{s(f)>y\}} s(f)^p dP\right)^{\frac{1}{p}}.$$

Proof. Let $f \in \Lambda^s_q(\varphi)$. For any fixed $y > 0$, choose $j \in \mathbf{Z}$ such that $2^{j-1} \leq y < 2^j$ and let

$$f = \sum_{k\in\mathbf{Z}}\mu_k a^k = \sum_{k=-\infty}^{j-1}\mu_k a^k + \sum_{k=j}^{\infty}\mu_k a^k = g + h,$$

where stopping times ν_k, atoms a^k and numbers $\mu_k (k \in \mathbf{Z})$ are as in Theorem 2.2. Now we have

$$\begin{aligned}\|g\|_{H^s_\infty} \leq \left\|\sum_{k=-\infty}^{j-1}\mu_k s(a^k)\right\|_\infty &\leq \sum_{k=-\infty}^{j-1}\mu_k\|s(a^k)\|_\infty\\ &\leq \sum_{k=-\infty}^{j-1}\mu_k P(\nu_k\neq\infty)^{-1/p} \leq \sum_{k=-\infty}^{j-1}2^k 3 \leq 2^j 3 \leq 6y.\end{aligned}$$

Since $s(a^k) = 0$ on the set $\{\nu_k = \infty\}$ and $\|s(a^k)\|_\infty < P(\nu_k \neq \infty)^{-1/p}$, then

$$\begin{aligned}
\|h\|^p_{H^s_p} &\leq \int_\Omega \left(\sum_{k=j}^{\infty} \mu_k s(a^k) \right)^p dP \\
&\lesssim \sum_{k=j}^{\infty} \mu_k^p \int_\Omega \left(s(a^k) \right)^p dP \\
&\leq \sum_{k=j}^{\infty} \mu_k^p \int_{\{\nu_k \neq \infty\}} \|s(a^k)\|^p_\infty dP \\
&\leq \sum_{k=j}^{\infty} \mu_k^p P(\nu_k \neq \infty)^{-1} P(\nu_k \neq \infty) \\
&= 3^p \sum_{k=j}^{\infty} 2^{kp}.P(\nu_k \neq \infty) \\
&= 3^p \sum_{k=j}^{\infty} 2^{kp}.P(s(f) > 2^k) \\
&\lesssim \int_{\{s(f)>2^j\}} s(f)^p dP, \quad \text{(by Abel rearrangement)} \\
&\lesssim \int_{\{s(f)>y\}} s(f)^p dP.
\end{aligned}$$

□

Lemma 3.3. *[16] Let $0 < q \leq \infty, 0 < p < \infty$ and $\psi(t) \in Q(-,-)$. Let $h(t)$ be a positive and non-increasing function on $(0, \infty)$.*

1. *If $\varphi(t) \in Q(-, 0)$, then*

$$\left(\int_0^\infty (\varphi(t))^q \left(\int_0^t (h(u)\psi(u))^p \frac{du}{u} \right)^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty (\varphi(t)h(t)\psi(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

2. *If $\varphi(t) \in Q(0, -)$, then*

$$\left(\int_0^\infty (\varphi(t))^q \left(\int_t^\infty (h(u)\psi(u))^p \frac{du}{u} \right)^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty (\varphi(t)h(t)\psi(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

(C depends only on q and the constants involved in the definition of φ and ψ.)

Proof of Theorem 3.1. Let f be a function in $\Lambda^s_q(\varphi)$ and s^* be the non-increasing rearrangement of $s(f)$ and choose y in Lemma 3.2 such that $y = s^*(t^p)$. First we prove that

$$K(t, f, H^s_p, H^s_\infty) \leq C \left(\int_0^{t^p} s^*(x)^p dx \right)^{\frac{1}{p}}, \qquad (t > 0). \tag{3}$$

For a fixed $t > 0$ set $E = \{s(f) > s^*(t^p)\}$. Using the inequality $m(f^*(s), f) \leq s$ we obtain $P(E) = m(s^*(t^p), s(f)) \leq t^p$ and since s^* is constant on $[P(E), t^p]$, henceforth

$$\int_E s(f)^p dP = \int_0^{P(E)} s^*(x)^p dx \leq \int_0^{t^p} s^*(x)^p dx. \tag{4}$$

Using inequality (4) and Lemma 3.2, we get

$$\begin{aligned} K(t,f,H_p^s,H_\infty^s) &\leq \|h\|_{H_p^s}+t\|g\|_{H_\infty^s} \\ &\leq C\left(\left(\int_{\{s(f)>y\}} s(f)^p dP\right)^{\frac{1}{p}} + ts^*(t^p)\right) \\ &\leq C\left(\left(\int_{\{s(f)>s^*(t^p)\}} s(f)^p dP\right)^{\frac{1}{p}} + \left(\int_0^{t^p} s^*(x)^p dx\right)^{\frac{1}{p}}\right), \quad \text{(by (4))} \\ &\leq C\left(\int_0^{t^p} s^*(x)^p dx\right)^{\frac{1}{p}}. \end{aligned}$$

Let $0 < q < \infty$. It is easy to see that $1/\varrho(t^{\frac{1}{p}}) \in Q(-\frac{1}{p},0)$ [16, Lemma 1.1]. So we have

$$\begin{aligned} \|f\|^q_{(H_p^s,H_\infty^s)_{\varrho,q}} &= \int_0^\infty \left(\frac{K(t,f,H_p^s,H_\infty^s)}{\varrho(t)}\right)^q \frac{dt}{t} \\ &\leq C\int_0^\infty \left(\frac{1}{\varrho(t)}\right)^q \left(\int_0^{t^p} s^*(x)^p dx\right)^{\frac{q}{p}} \frac{dt}{t}, \quad \text{(by (3))} \\ &\leq C\int_0^\infty \left(\frac{1}{\varrho(t^{\frac{1}{p}})}\right)^q \left(\int_0^{t} s^*(x)^p dx\right)^{\frac{q}{p}} \frac{dt}{t} \\ &\leq C\int_0^\infty \left(\frac{1}{\varrho(t^{\frac{1}{p}})}\right)^q t^{\frac{q}{p}} s^*(t)^p \frac{dt}{t}, \quad \text{(by Lemma 3.3)} \\ &= C\|s(f)\|^q_{\Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))} =: C\|f\|^q_{\Lambda_q^s(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))}. \end{aligned}$$

To prove the converse, we consider the operator $T : f \mapsto s(f)$. The sublinear operators $T : H_\infty^s \to L_\infty$ and $T : H_p^s \to L_p$ are bounded. By [16, Theprem 2.2] , the operator

$$T : \left(H_p^s, H_\infty^s\right)_{\varrho,q} \to \left(L_p, L_\infty\right)_{\varrho,q} = \Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))$$

is bounded in which the equality follows from (1). So we have

$$\|f\|_{\Lambda_q^s(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))} := \|s(f)\|_{\Lambda_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))} \leq C\|f\|_{(H_p^s,H_\infty^s)_{\varrho,q}}.$$

The proof is complete for $0 < q < \infty$. Let $q = \infty$. Since $\varrho \in Q(0,1)$, then $\varrho(t)t^{-\epsilon}$ is non–decreasing for some $\epsilon > 0$. So we have

$$\begin{aligned} \|f\|_{(H_p^s,H_\infty^s)_{\varrho,\infty}} &= \sup_{t>0} \frac{K(t,f,H_p^s,H_\infty^s)}{\varrho(t)} \\ &\leq C\sup_{t>0} \frac{\left(\int_0^{t^p} s^*(x)^p dx\right)^{\frac{1}{p}}}{\varrho(t)}, \quad \text{(by (3))} \\ &\leq C\sup_{t>0} \frac{\left(\int_0^{t} (s^*(x^p))^p x^{p-1} dx\right)^{\frac{1}{p}}}{\varrho(t)} \\ &\leq C\sup_{x>0} \frac{xs^*(x^p)}{\varrho(x)} . \sup_{t>0} \frac{\varrho(t)t^{-\epsilon}(\int_0^t x^{p\epsilon-1}dx)^{\frac{1}{p}}}{\varrho(t)} \\ &\leq C\|f\|_{\Lambda_\infty^s(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))}. \end{aligned}$$

To prove the converse, we consider the operator $T : f \mapsto s(f)$. The sublinear operators $T : H^s_\infty \to L_\infty$ and $T : H^s_p \to L_p$ are bounded. By [16, Theprem 2.2] , the operator

$$T : \left(H^s_p, H^s_\infty\right)_{\varrho,\infty} \to \left(L_p, L_\infty\right)_{\varrho,\infty} = \Lambda_\infty(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))$$

is bounded in which the equality follows from (1). Hence

$$\|f\|_{\Lambda^s_\infty(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))} := \|s(f)\|_{\Lambda_\infty(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}}))} \le C\|f\|_{(H^s_p,H^s_\infty)_{\varrho,\infty}}.$$

The proof is complete.

If we take $\varrho(t) = t^\theta$ in Theorem 3.1, then we get the following result, which has proved by Weisz [18].

Corollary 3.4. *If* $0 < \theta < 1, 0 < p_0 \le 1$ *and* $0 < q \le \infty$, *then*

$$(H^s_{p_0}, H^s_\infty)_{\theta,q} = H^s_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}.$$

Applying the Theorem 3.1 we get the next theorem.

Theorem 3.5. *Let* $\varphi_i(t) \in Q(0,-), i = 0,1,\ 0 < q_0, q_1, q \le \infty$ *and* $\varrho \in Q(0,1)$. *If* $\varphi_0(t)/\varphi_1(t) \in Q(0,-)$ *or* $\varphi_0(t)/\varphi_1(t) \in Q(-,0)$, *then*

$$\left(\Lambda^s_{q_0}(\varphi_0), \Lambda^s_{q_1}(\varphi_1)\right)_{\varrho,q} = \Lambda^s_q(\varphi),$$

where $\varphi(t) = \varphi_0(t)/\rho(\varphi_0(t)/\varphi_1(t))$.

Proof. Put $\varrho_i(t) = t/\varphi_i(t^p)$ and choose p so small that $\varrho_i(t) \in Q(0,1), i = 0,1$. According to [16, Corollary 4.4] and Theorem 3.1 we get

$$\begin{aligned}\left(\Lambda^s_{q_0}(\varphi_0), \Lambda^s_{q_1}(\varphi_1)\right)_{\varrho.q} &= \left((H^s_p, H^s_\infty)_{\varrho_0,q_0}, (H^s_p, H^s_\infty)_{\varrho_1,q_1}\right)_{\varrho,q} \\ &= \left(H^s_p, H^s_\infty\right)_{\varrho_0\varrho(\varrho_1/\varrho_0),q} \\ &= \Lambda^s_q(\varphi),\end{aligned}$$

where $\varphi(t) = \varphi_0(t)/\rho(\varphi_0(t)/\varphi_1(t))$. □

The following result is a simple application of Theorem 3.5, if we take $\varphi_i(t) = t^{\frac{1}{p_i}}, i = 0,1$.

Corollary 3.6. *Let* $0 < p_i < \infty, 0 < q_i, q \le \infty, i = 0,1$ *and* $\varrho \in Q(0,1)$. *If* $p_0 \ne p_1$, *then*

$$\left(H^s_{p_0,q_0}, H^s_{p_1,q_1}\right)_{\varrho,q} = \Lambda^s_q(t^{\frac{1}{p_0}}/\varrho(t^{\frac{1}{p_0}-\frac{1}{p_1}}))$$

and

$$\left(H^s_{p_0}, H^s_{p_1}\right)_{\varrho,q} = \Lambda^s_q(t^{\frac{1}{p_0}}/\varrho(t^{\frac{1}{p_0}-\frac{1}{p_1}})).$$

In particular, if $\varrho(t) = t^\theta$, *then*

$$\left(H^s_{p_0}, H^s_{p_1}\right)_{\theta,q} = H^s_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

According to Theorem 3.5 we have the following corollary.

Corollary 3.7. *Under the hypothesis of Theorem 3.5, we have*

$$\left(\Lambda^s_{q_0}(\varphi_0), \Lambda^s_{q_1}(\varphi_1)\right)_{\theta,q} = \Lambda^s_q(\varphi_0^{1-\theta}\varphi_1^\theta).$$

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