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On Pseudo-Slant Submanifolds of a Sasakian Space Form

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Abstract. In this paper, we study the geometry of the pseudo-slant submanifolds of a Sasakian space form. Necessary and sufficient conditions are given for a submanifold to be pseudo-slant submanifolds, pseudo-slant product, mixed geodesic and totally geodesic in Sasakian manifolds. Finally, we give some results for totally umbilical pseudo-slant submanifolds of Sasakian manifolds and Sasakian space forms.

1. Introduction

The differential geometry of slant submanifolds has shown an increasing development since B.-Y. Chen defined slant submanifolds in complex manifolds as a natural generalization of both holomorphic and totally real submanifolds [5, 6]. Many research articles have been appeared on the existence of these submanifolds in different knows spaces. The slant submanifols of an almost contact metric manifolds were defined and studied by Lotta [10]. After, such submanifolds were studied by Cabrerizo et al. of Sasakian manifolds [3]. Recently, in [2, 7, 8], Atçeken et al. studied slant and pseudo-slant submanifold in various manifolds. The notion of semi-slant submanifolds of an almost Hermitian manifold was introduced by Papagiuc [11]. Cabrerizo [4] defined and studied bi-slant immersions in almost contact metric manifolds and simultaneously gave the notion of pseudo-slant submanifolds. Pseudo-slant submanifolds also have been studied by Khan et al. in [9]. The present paper is organized as follows.

In this paper, we study pseudo-slant submanifolds of a Sasakian manifold. In Section 2, we review basic formulas and definitions for a Sasakian manifold and their submanifolds. In Section 3, we have recalled the definition and some basic results of a pseudo-slant submanifold of an almost contact metric manifold. In Section 4, we give some new results for totally umbilical pseudo-slant submanifolds in a Sasakian manifold and a Sasakian space form $\widetilde{M}(c)$.

2. Preliminaries

In this section, we give some notations used throughout this paper. We recall some necessary fact and formulas from the theory of Sasakian manifolds and their submanifolds.

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Given an odd-dimensional Riemannian manifold $(\widetilde{M}, g)$, let φ be a $(1,1)$-type tensor field, ξ is a unit vector field and η is a 1-form on $\widetilde{M}$. If we have

$$\varphi^2 X = -X + \eta(X)\xi, \quad g(X,\xi) = \eta(X), \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2}$$

for any vector fields on $\widetilde{M}$, then $\widetilde{M}$ is said to be have an almost contact metric structure (φ, ξ, η, g) and it is called an almost contact metric manifold.

Let Φ denotes the fundamental 2-form in $\widetilde{M}$, given by $\Phi(X,Y) = g(X, \varphi Y)$, for any vector fields X, Y on $\widetilde{M}$. If $\Phi = d\eta$, then $\widetilde{M}$ is said to be a contact metric manifold. Furthermore, a contact metric structure is called a K-contact structure if ξ is a Killing vector field, that is, $\widetilde{\nabla}_X \xi = -\varphi X$, for any vector field X on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Levi-Civita connection on $\widetilde{M}$.

The structure (φ, ξ, η, g) is said to be normal if $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A normal contact metric manifold is called Sasakian manifold. So every Sasakian manifold is a K-contact manifold. It is well-know that an almost contact metric manifold is a Sasakian if and only if

$$(\widetilde{\nabla}_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{3}$$

for any vector fields X, Y on $\widetilde{M}$.

Let $\widetilde{M}(c)$ be a Sasakian space form with constant φ-holomorphic sectional curvature c. Then the curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is given by

$$\begin{aligned} \widetilde{R}(X,Y)Z \;=\; & (\frac{c+3}{4})\{g(Y,Z)X - g(X,Z)Y\} + (\frac{c-1}{4})\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ & -g(Y,Z)\eta(X)\xi + \Phi(Y,Z)\varphi X - \Phi(X,Z)\varphi Y + 2\Phi(X,Y)\varphi Z\} \end{aligned} \tag{4}$$

for any vector fields X, Y, Z on $\widetilde{M}(c)$.

Now, let M be a submanifold of a contact metric manifold $\widetilde{M}$ with the induced metric g. Also, let ∇ and $\nabla^\perp$ be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively. Then the Gauss and Weingarten formulas are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{5}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{6}$$

for any vector fields X, Y on M, where h and A_V are the second fundamental form and the shape operator (corresponding to the normal vector field V), respectively, for the immersion of M into $\widetilde{M}$. The second fundamental form h and shape operator A_V are related by

$$g(A_V X, Y) = g(h(X,Y), V), \tag{7}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

The mean curvature vector H of M is given by $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$, where m is the dimension of M and $\{e_1, e_2, ..., e_m\}$ is a local orthonormal frame of M. A submanifold M of an contact metric manifold $\widetilde{M}$ is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H, \tag{8}$$

for any $X, Y \in \Gamma(TM)$. A submanifold M is said to be totally geodesic if $h = 0$ and M is said to be minimal if $H = 0$.

For any submanifold M of a Riemannian manifold $\widetilde{M}$, the equation of Gauss is given by

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\widetilde{\nabla}_X h)(Y,Z) - (\widetilde{\nabla}_Y h)(X,Z), \tag{9}$$

for any $X, Y, Z \in \Gamma(TM)$, where $\widetilde{R}$ and R denote the Riemannian curvature tensor of $\widetilde{M}$ and M, respectively. The covariant derivative $\widetilde{\nabla} h$ of h is defined by

$$(\widetilde{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y). \tag{10}$$

The normal component of (9) is said to be the Codazzi equation and it is given by

$$(\widetilde{R}(X,Y)Z)^{\perp} = (\widetilde{\nabla}_X h)(Y,Z) - (\widetilde{\nabla}_Y h)(X,Z), \tag{11}$$

where $(\widetilde{R}(X,Y)Z)^{\perp}$ denotes the normal part of $\widetilde{R}(X,Y)Z$. If $(\widetilde{R}(X,Y)Z)^{\perp} = 0$, then M is said to be curvature-invariant submanifold of $\widetilde{M}$. The Ricci equation is given by

$$g(\widetilde{R}(X,Y)V,U) = g(\widetilde{R}^{\perp}(X,Y)V,U) + g([A_U, A_V]X, Y), \tag{12}$$

for any $X, Y \in \Gamma(TM)$ and $V, U \in \Gamma(T^{\perp}M)$, where $\widetilde{R}^{\perp}$ denotes the Riemannian curvature tensor of the normal $T^{\perp}M$. If $\widetilde{R}^{\perp} = 0$, then the normal connection of the submanifold M is called flat.

A Sasakian manifold $\widetilde{M}$ is said to be η-Einstein if its Ricci tensor S of type $(0,2)$ is of the from $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where a, b are smooth functions on $\widetilde{M}$.

Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$. Then for any $X \in \Gamma(TM)$, we can write

$$\varphi X = PX + FX, \tag{13}$$

where PX is the tangential component and FX is the normal component of φX. Similarly for $V \in \Gamma(T^{\perp}M)$, we can write

$$\varphi V = BV + CV, \tag{14}$$

where BV is the tangential component and CV is also the normal component of φV. A submanifold M is said to be invariant if F is identically zero, that is, $\varphi X \in \Gamma(TM)$, for all $X \in \Gamma(TM)$. On the other hand, M is said to be anti-invariant if P is identically zero, that is, $\varphi X \in \Gamma(T^{\perp}M)$, for all $X \in \Gamma(TM)$.

Taking into account (4) and (12), we have

$$\begin{aligned} g(\widetilde{R}^{\perp}(X,Y)V,U) = {} & g([A_V, A_U]X, Y) + (\frac{c-1}{4})\{g(X,\varphi V)g(U,\varphi Y) - g(Y,\varphi V)g(\varphi X, U) \\ & +2g(X,\varphi Y)g(\varphi V, U)\}, \end{aligned} \tag{15}$$

for any $X, Y \in \Gamma(TM)$ and $V, U \in \Gamma(T^{\perp}M)$. By using (4) and (9), the Riemannian curvature tensor R of an immersed submanifold M of a Sasakian space form $\widetilde{M}(c)$ is given by

$$\begin{aligned} R(X,Y)Z = {} & (\frac{c+3}{4})\{g(Y,Z)X - g(X,Z)Y\} + (\frac{c-1}{4})\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi \\ & -\eta(X)g(Y,Z)\xi + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\ & +A_{h(Y,Z)}X - A_{h(X,Z)}Y + (\widetilde{\nabla}_Y h)(X,Z) - (\widetilde{\nabla}_X h)(Y,Z). \end{aligned} \tag{16}$$

The normal part of (16), we have

$$(\widetilde{\nabla}_X h)(Y,Z) - (\widetilde{\nabla}_Y h)(X,Z) = (\frac{c-1}{4})\{g(X,PZ)FY - g(Y,PZ)FX + 2g(X,PY)FZ\} \tag{17}$$

Thus by using (1), (13) and (14), we obtain

$$P^2 = -I + \eta \otimes \xi - BF \ \ and \ \ FP + CF = 0, \tag{18}$$

$$PB + BC = 0 \ \ and \ \ FB + C^2 = -I. \tag{19}$$

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(FX, Y) = -g(X, FY)$ and $V, U \in \Gamma(T^{\perp}M)$, we get $g(U, CV) = -g(CU, V)$. These relations show that P and C are also skew-symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have $g(FX, V) = -g(X, BV)$,which gives the relation between F and B.

On the other hand, the covariant derivatives of the tensor fields P, F, B and C, respectively, defined by

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{20}$$

$$(\nabla_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \tag{21}$$

$$(\nabla_X B)V = \nabla_X BV - B\nabla^{\perp}_X V \tag{22}$$

and

$$(\nabla_X C)V = \nabla^{\perp}_X CV - C\nabla^{\perp}_X V \tag{23}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X, Y \in \Gamma(TM)$.

By an easy computation, we obtain the following formulas

$$(\nabla_X P)Y = A_{FY}X + Bh(X, Y) + g(X, Y)\xi - \eta(Y)X, \tag{24}$$

$$(\nabla_X F)Y = Ch(X, Y) - h(X, PY), \tag{25}$$

$$(\nabla_X B)V = A_{CV}X - PA_V X \tag{26}$$

and

$$(\nabla_X C)V = -h(BV, X) - FA_V X \tag{27}$$

for any $V \in \Gamma(T^{\perp}M)$ and $X, Y \in \Gamma(TM)$.

Since ξ is tangent to M, making use of (3), (5), (7) and (13), we infer that

$$\nabla_X \xi = -PX, \ \ h(X, \xi) = -FX, \ \ A_V \xi = BV, \tag{28}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$.

3. Pseudo-Slant Submanifolds of a Sasakian Manifold

In this section, we study pseudo-slant submanifolds in a Sasakian manifold and we give some characterization results.

A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be slant if for any $x \in M$ and $X \in T_xM - \xi$, the angle between T_xM and φX is constant. The constant angle $[0, \frac{\pi}{2}]$ is then called slant angle of M. If $\theta = 0$, than M is invariant and if $\theta = \frac{\pi}{2}$ then, it is anti-invariant. On the other hand, if $\theta \in (0, \frac{\pi}{2})$ then M is a proper slant submanifold [10]. The tangent bundle TM of M is decomposed as $TM = D \oplus \xi$, where the orthogonal complementary distribution D of ξ is know as the slant distribution on M. We have the following result in the setting of almost contact manifolds given by Cabrerizo et.al.

Theorem 3.1. *Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in \Gamma(TM)$. Then M is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi). \tag{29}$$

In this case, if θ is the slant angle of M, then it satisfies $\lambda = \cos^2 \theta$[4].

Thus, one has the following consequences of above formulae

$$g(PX, PY) = \cos^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\} \tag{30}$$

and

$$g(FX, FY) = \sin^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\} \tag{31}$$

for any $X, Y \in \Gamma(TM)$.

Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$. M is said to be *a pseudo-slant* of $\widetilde{M}$ if there exist two orthogonal distributions $D^\perp$ and D_θ on M such that:
i) $TM = D^\perp \oplus D_\theta$, $\xi \in \Gamma(D_\theta)$.
ii) The distribution $D^\perp$ is anti-invariant, that is, $\varphi D^\perp \subset T^\perp M$.
iii) The distribution D_θ is slant, that is, the slant angle θ between D_θ and $\varphi(X)$ is a constant, for any $X \in \Gamma(D_\theta)$[9].

Let m_1 =dim($D^\perp$) and m_2=dim(D_θ). We distinguish the following six cases.
i) If $m_2 = 0$, then M is an anti-invariant submanifold.
ii) If $m_1 = 0$ and $\theta = 0$, then M is an invariant submanifold.
iii) If $m_1 = 0$ and $\theta \neq \{0, \frac{\pi}{2}\}$, then M is a proper slant submanifold.
iv)If $\theta = \frac{\pi}{2}$ then, M is an anti-invariant submanifold.
v) If $m_2 m_1 \neq 0$ and $\theta = 0$, then M is a semi-invariant submanifold.
vi) If $m_2 m_1 \neq 0$ and $\theta \neq \{0, \frac{\pi}{2}\}$, then M is a pseudo-slant submanifold.

If μ is the invariant subspace of the bundle $T^\perp M$ then in the case of pseudo-slant submanifold $T^\perp M$ can be decomposed as follows $T^\perp M = F(D^\perp) \oplus F(D_\theta) \oplus \mu$.

Now we construct an example of a pseudo-slant submanifold in an almost contact metric manifold.

Example 3.2. *Let M be a submanifold of $\mathbb{R}^9$ defined by the following equation*

$$M = \chi(u, v, s, t, z) = (3u \sin\alpha, -v\cos\alpha, -4u\sin\alpha, v\cos\alpha, s\cos t, -\cos t, s\sin t, -\sin t, z).$$

We can easily to see that the tangent bundle of M is spanned by the tangent vectors

$$\begin{aligned} e_1 &= 3\sin\alpha\frac{\partial}{\partial x_1} - 4\sin\alpha\frac{\partial}{\partial x_2},\ e_2 = -\cos\alpha\frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial y_2}, \\ e_3 &= \cos t\frac{\partial}{\partial x_3} + \sin t\frac{\partial}{\partial x_4},\ e_4 = -s\sin t\frac{\partial}{\partial x_3} + \sin t\frac{\partial}{\partial y_3} + s\cos t\frac{\partial}{\partial x_4} - \cos t\frac{\partial}{\partial y_4} \end{aligned}$$

and

$$e_5 = \xi = \frac{\partial}{\partial z}.$$

For the almost contact metric structure φ of $\mathbb{R}^9$, whose coordinate systems (x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, z), choosing

$$\varphi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i},\ \varphi(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j},\ \varphi(\frac{\partial}{\partial z}) = 0,\ 1 \leq i, j \leq 4$$

then we have

$$\varphi e_1 = 3\sin\alpha\frac{\partial}{\partial y_1} - 4\sin\alpha\frac{\partial}{\partial y_2}\,,\ \varphi e_2 = \cos\alpha\frac{\partial}{\partial x_1} - \cos\alpha\frac{\partial}{\partial x_2},\ \varphi e_3 = \cos t\frac{\partial}{\partial y_3} + \sin t\frac{\partial}{\partial y_4},$$

and

$$\varphi e_4 = -s\sin t\frac{\partial}{\partial y_3} - \sin t\frac{\partial}{\partial x_3} + s\cos t\frac{\partial}{\partial y_4} + \cos t\frac{\partial}{\partial x_4}, \varphi e_5 = 0.$$

By direct calculations, we infer that $D_\theta = span\{e_1, e_2\}$ is a slant distribution with slant angle $\cos\theta = \frac{g(e_1,\varphi e_2)}{\|e_1\|\|\varphi e_2\|} = \frac{7\sqrt{2}}{10}$, $\theta = cos^{-1}(\frac{7\sqrt{2}}{10})$. Since $g(\varphi e_3, e_i) = 0$, $i = 1,2,4,5$ and $g(\varphi e_4, e_j) = 0$, $j = 1,2,3,5, e_3, e_4$ are orthogonal to M, $D^\perp = span\{e_3, e_4\}$ is an anti-invariant distribution. Thus M is a 5-dimensional proper pseudo-slant submanifold of $\mathbb{R}^9$ with it's usual almost contact metric structure.

A pseudo-slant submanifold M of a Sasakian manifold $\widetilde{M}$ is said to be D_θ-totally geodesic (resp. $D^\perp$-totally geodesic) if $h(X,Y) = 0$ for any $X,Y \in \Gamma(D_\theta)$ (resp. $h(Z,W) = 0$ for any $Z,W \in \Gamma(D^\perp)$. If for any $X \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^\perp)$, $h(X,Z) = 0$, then M is called a mixed totally geodesic.

Theorem 3.3. *Let M be a proper pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. Then, either M is a mixed-totally geodesic or an anti-invariant submanifold.*

Proof. By using (2), (3), (5), (6), (13) and (14), we have

$$\begin{aligned} g(A_V X, Y) &= g(\widetilde{\nabla}_X Y, V) = -g(\widetilde{\nabla}_X V, Y) \\ &= -g(\varphi\widetilde{\nabla}_X V, \varphi Y) = g((\widetilde{\nabla}_X\varphi)V - \widetilde{\nabla}_X\varphi V, \varphi Y) \\ &= -g(\widetilde{\nabla}_X BV + \widetilde{\nabla}_X CV, FY) \\ &= -g(h(X,BV), FY) - g(\nabla^\perp_X CV, FY), \end{aligned}$$

for any $X \in \Gamma(D_\theta)$, $Y \in \Gamma(D^\perp)$ and $V \in \Gamma(T^\perp M)$. Taking into account (23), (27) and (30), we get

$$\begin{aligned} g(A_V X, Y) &= -g(h(X,BV), FY) - g((\nabla_X C)V + C\nabla^\perp_X V, FY) \\ &= -g(h(X,BV), FY) - g(-h(X,BV) - FA_V X, FY) \\ &= g(FA_V X, FY) = -g(BFA_V X, Y). \end{aligned}$$

By using (18), we obtain

$$\begin{aligned} g(A_V X, Y) &= -g(-A_V X + \eta(A_V X)\xi - P^2 A_V X, Y) \\ &= g(A_V X, Y) - \eta(A_V X)\eta(Y) + g(P^2 A_V X, Y), \end{aligned}$$

that is,

$$-\cos^2\theta g(A_V X - \eta(A_V X)\xi, Y) = -\cos^2\theta g(A_V X, Y) = 0.$$

This tells us that either M is mixed-totally geodesic or it is an anti-invariant submanifold. □

Theorem 3.4. *Let M be a proper pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. Then, either M is $D^\perp$-totally geodesic or an anti-invariant submanifold of $\widetilde{M}$.*

Proof. By using (2), (3), (5), (6), (13) and (14), we obtain

$$\begin{aligned} g(h(Z,W), V) &= -g(\widetilde{\nabla}_W V, Z) = -g(\varphi\widetilde{\nabla}_W V, \varphi Z) \\ &= g((\widetilde{\nabla}_W\varphi)V - \widetilde{\nabla}_W\varphi V, \varphi Z) \\ &= g(g(W,V)\xi - \eta(V)W, FZ) \\ &\quad -g(\widetilde{\nabla}_W BV, FZ) - g(\widetilde{\nabla}_W CV, FZ) \\ &= -g(h(W,BV), FZ) - g(\nabla^\perp_W CV, FZ), \end{aligned}$$

for any $Z, W \in \Gamma(D^{\perp})$ and $V \in \Gamma(T^{\perp}M)$. Hence, by using (23), (27) and (30), we reach

$$\begin{aligned} g(h(Z,W),V) &= -g(h(W,BV),FZ) - g((\nabla_W C)V,FZ) \\ &= -g(h(W,BV),FZ) + g(h(BV,W) + FA_V W,FZ) \\ &= g(FA_V W,FZ) = -g(BFA_V W,Z) = -g(-A_V W + \eta(A_V W)\xi - P^2 A_V W,Z) \\ &= g(A_V W,Z) + g(P^2 A_V W,Z), \end{aligned}$$

or

$$-\cos^2\theta g(A_V W - \eta(A_V W)\xi, Z) = -\cos^2\theta g(A_V W,Z) = 0.$$

The last relation yields $\cos^2\theta g(h(Z,W),V) = 0$, which means that either M is $D^{\perp}$-totaly geodesic or it is an anti-invariant submanifold. □

Given a proper pseudo-slant submanifold M of a Sasakian manifold $\widetilde{M}$, if the distributions D_θ and $D^{\perp}$ are totally geodesic in M, then M is said to be contact pseudo-slant product.

For any $X, Y \in \Gamma(D_\theta < \xi >)$ and $Z \in \Gamma(D^{\perp})$, by using (3), (5), (6), (21), (25) and (30), we have

$$\begin{aligned} g(\nabla_X Y,Z) &= g(\varphi\widetilde{\nabla}_X Y,\varphi Z) = g(\widetilde{\nabla}_X\varphi Y - (\widetilde{\nabla}_X\varphi)Y,\varphi Z) \\ &= g(h(X,PY),FZ) + g(\nabla^{\perp}_X FY,FZ) \\ &= g(h(X,PY),FZ) + g((\nabla_X F)Y + F\nabla_X Y,FZ) \\ &= g(h(X,PY),FZ) + g(Ch(X,Y),FZ) \\ &- g(h(X,PY),FZ) + g(F\nabla_X Y,FZ) = g(F\nabla_X Y,FZ) = -g(BF\nabla_X Y,Z) \\ &= -g(-\nabla_X Y + \eta(\nabla_X Y)\xi - P^2\nabla_X Y,Z), \end{aligned}$$

which implies that

$$g(P^2\nabla_X Y,Z) = -\cos^2\theta g(\nabla_X Y - \eta(\nabla_X Y)\xi,Z) = -\cos^2\theta g(\nabla_X Y,Z) = 0. \tag{32}$$

and

$$\begin{aligned} g(\nabla_W Z,X) &= -g(\widetilde{\nabla}_W X,Z) = -g(\varphi\widetilde{\nabla}_W X,\varphi Z) \\ &= g((\widetilde{\nabla}_W\varphi)X,\phi Z) - g(\widetilde{\nabla}_W\varphi X,\varphi Z) \\ &= -g(h(PX,W),FZ) - g(\nabla^{\perp}_W FX,FZ) \\ &= -g(h(PX,W),FZ) - g((\nabla_W F)X + F\nabla_W X,FZ) \\ &= -g(h(PX,W),FZ) - g(Ch(X,W),FZ) \\ &+ g(h(W,PX),FZ) - g(F\nabla_W X,FZ) = g(BF\nabla_W X,Z) \\ &= -g(-\nabla_W X + \eta(\nabla_W X)\xi - P^2\nabla_W X,Z) \\ &= g(\nabla_W X,X) + g(P^2\nabla_W X,Z), \end{aligned}$$

that is,

$$\cos^2\theta g(\nabla_W X - \eta(\nabla_W X)\xi,Z) = \cos^2\theta g(\nabla_W X,Z) = -\cos^2\theta g(\nabla_W Z,X) = 0. \tag{33}$$

for any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D_\theta)$. Thus from (32) and (33), we have the following Theorem.

Theorem 3.5. *Every proper pseudo-slant submanifold M of a Sasakian manifold $\widetilde{M}$ is a contact pseudo-slant product.*

4. Pseudo-Slant Submanifolds of a Sasakian Space Form

In this section, we study pseudo-slant submanifolds in a Sasakian space form $\widetilde{M}(c)$ with constant φ-sectional curvature c. We obtain some results for such submanifolds in terms of curvature tensor.

Theorem 4.1. *Let M be a pseudo-slant submanifold of a Sasakian space form $\widetilde{M}(c)$ such that $c \neq 1$. If M is a pseudo-slant curvature-invariant submanifold, then*
(i) either M is invariant,
(ii) or M anti-invariant
(iii) or $\dim(M) = 1$.

Proof. Suppose that M is a pseudo-slant curvature-invariant submanifold of a Sasakian space form $\widetilde{M}(c)$ such that $c \neq 1$. Then from (11) and (17), we have

$$g(X, PZ)FY - g(Y, PZ)FX + 2g(X, PY)FZ = 0, \tag{34}$$

for any $X, Y, Z \in \Gamma(TM)$. If we put, than $X = Z$ and $Y = PZ$ we have, $g(PZ, PZ)FZ = 0$. Here, by using (30), we obtain

$$\cos^2\theta \left\{g(Z, Z) - \eta^2(Z)\right\}^2 FZ = 0,$$

which implies that, either M is invariant or anti-invariant submanifold or $\dim(M) = 1$. □

Theorem 4.2. *Let M be a pseudo-slant submanifold of a Sasakian space form $\widetilde{M}(c)$ with flat normal connection such that $c \neq 1$. If $PA_V = A_VP$ for any vector V normal to M, then M is either anti- invariant or it is a generic submanifold of $\widetilde{M}(c)$.*

Proof. If the normal connection of M is flat, then from (15), we have

$$\begin{aligned} g([A_U, A_V]X, Y) &= (\frac{c-1}{4})\{g(X, \varphi V)g(U, \varphi Y) - g(Y, \varphi V)g(\varphi X, U) \\ &\quad +2g(X, \varphi Y)g(\varphi V, U)\} \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$. Here, choosing $U = CV$ and $Y = PX$, by direct calculations, we can state

$$g([A_V, A_{CV}]X, PX) = -(\frac{c-1}{2})\{g(PX, PX)g(CV, CV)\},$$

that is,

$$g(A_{CV}A_VPX - A_VA_{CV}PX, X) = -(\frac{c-1}{2})\{g(PX, PX)g(CV, CV)\},$$

from which

$$tr(A_{CV}A_VP) - tr(A_VA_{CV}P) = (\frac{c-1}{2})tr(P^2)g(CV, CV).$$

If $PA_V = A_VP$, then we conclude that $tr(A_{CV}A_VP) = tr(A_VA_{CV}P)$ and thus

$$(\frac{c-1}{2})tr(P^2)g(CV, CV) = 0,$$

from here $dim(M) = 2p + q + 1$, then we can easily to see that $(2p + q + 1)cos^2\theta g(CV, CV) = 0$. Thus θ is either $\frac{\pi}{2}$ or $C = 0$. This implies that M is either anti-invariant or it is a generic submanifold. □

Theorem 4.3. *Let M be a pseudo-slant submanifold of a Sasakian space form $\widetilde{M}(c)$. Then the Ricci tensor S of M is given by*

$$\begin{aligned} S(X,W) &= \left\{(\frac{c+3}{4})(2p+q)+(\frac{c-1}{4})(3\cos^2\theta-1)\right\}g(X,W) \\ &+(\frac{c-1}{4})(1-q-2p-3\cos^2\theta)\eta(X)\eta(W) \\ &+(2p+q+1)g(h(X,W),H)-\sum_{m=1}^{2p+q+1}g(h(e_m,W),h(X,e_m)) \end{aligned} \tag{35}$$

for any $X, W \in \Gamma(TM)$.

Proof. For any $X, Y, Z, W \in \Gamma(TM)$, by using (16), we have

$$\begin{aligned} g(R(X,Y)Z,W) = \; & (\frac{c+3}{4})\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}+(\frac{c-1}{4})\{\eta(X)\eta(Z)g(Y,W) \\ & -\eta(Y)\eta(Z)g(X,W)+\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z)+g(X,\varphi Z)g(\varphi Y,W) \\ & -g(Y,\varphi Z)g(\varphi X,W)+2g(X,\varphi Y)g(\varphi Z,W)\}+g(h(X,W),h(Y,Z))-g(h(Y,W),h(X,Z)). \end{aligned}$$

Now, let $e_1, e_2, ..., e_p, e_{p+1} = \sec\theta Pe_1, e_{p+2} = \sec\theta Pe_2, ..., e_{2p} = \sec\theta Pe_p, e_{2p+1} = \xi, e_{2p+2}, e_{2p+3}, ..., e_{2p+q+1}$ be an orthonormal basis of $\Gamma(TM)$ such that $e_1, e_2, ..., e_p, e_{p+1} = \sec\theta Pe_1, e_{p+2} = \sec\theta Pe_2, ..., e_{2p} = \sec\theta Pe_p, e_{2p+1} = \xi$ are tangent to $\Gamma(D_\theta)$ and $e_{2p+2}, e_{2p+3}, ..., e_{2p+q+1}$ are tangent to $\Gamma(D^\perp)$. Hence, taking $Y = Z = e_i, e_j, e_k$ and $1 \le i \le p, 1 \le j \le p, \xi, 2p+2 \le k \le 2p+q+1$ then we obtain

$$\begin{aligned} S(X,W) &= \sum_{i=1}^{p} g(R(X,e_i)e_i,W)+\sum_{j=p+1}^{2p} g(R(X,\sec\theta Pe_j)\sec\theta Pe_j,W) \\ &+g(R(X,\xi)\xi,W)+\sum_{k=2p+2}^{2p+q+1} g(R(X,e_k)e_k,W). \end{aligned}$$

It follows that

$$\begin{aligned} S(X,W) &= (\frac{c+3}{4})\{(2p+q)g(X,W)\}+(\frac{c-1}{4})\{-(2p+q-1)\eta(X)\eta(W) \\ &+3\cos^2\theta\{g(X,W)-\eta(X)\eta(W)\}-g(X,W)\}+(2p+q+1)g(h(X,W),H) \\ &-\sum_{i=1}^{p} g(h(e_i,W),h(X,e_i))-\sum_{j=p+1}^{2p} g(h(\sec\theta Pe_j,W),h(X,\sec\theta Pe_j)) \\ &-g(h(\xi,W),h(X,\xi))-\sum_{k=2p+2}^{2p+q+1} g(h(e_k,W),h(X,e_k)). \end{aligned}$$

Hence, the proof follows from the above relation. □

Theorem 4.4. *Let M be a pseudo-slant submanifold of a Sasakian space form $\widetilde{M}(c)$. Then the scalar curvature ρ of M is given by*

$$\begin{aligned} \rho &= \{(\frac{c+3}{4})(2p+q)+(\frac{c-1}{4})(3\cos^2\theta-1)\}(2p+q+1) \\ &+(\frac{c-1}{4})(3\cos^2\theta+2p+q-1)+(2p+q+1)^2\,\|H)\|^2-\|h\|^2 \end{aligned} \tag{36}$$

.

Proof. From (35) by using $X = W = e_m$, we have $\rho = \sum_{m=1}^{2p+q+1} S(e_m, e_m)$ which gives (36). Thus, the proof is complete. □

Theorem 4.5. *Let M be a totally umbilical proper pseudo-slant submanifold of a Sasakian space form $\widetilde{M}(c)$ such that $c \neq 1$. Then,*
(i) either M is semi-invariant,
(ii) or M anti-invariant
(iii) or $\dim(D_\theta) = 1$.

Proof. Suppose that M is a totally umbilical pseudo-slant submanifold in Sasakian space form $\widetilde{M}$. From (28), we have $h(\xi, \xi) = 0$. If M is a totally umbilical submanifold of a Sasakian manifold $\widetilde{M}$, $h(X, Y) = g(X, Y)H$, for any $X, Y \in \Gamma(TM)$. For $X = Y = \xi$, we get $H = 0$. This tells us that every totally umbilical submanifold in Sasakian manifold is totally geodesic. So we have

$$g(\widetilde{R}(X, Y)Z, \varphi Z) = g((\widetilde{\nabla}_X h)(Y, Z) - (\widetilde{\nabla}_Y h)(X, Z), \varphi Z) = 0 \tag{37}$$

for any $X, Y \in \Gamma(D_\theta < \xi >)$ and $Z \in \Gamma(D^\perp)$. Since the ambient space is a Sasakian space form, then from (4) we infer

$$g(\widetilde{R}(X, Y)Z, \varphi Z) \;=\; (\frac{c-1}{2})g(X, \varphi Y)g(FZ, FZ) = 0. \tag{38}$$

Taking $Y = PX$ in equation (38), we have

$$(\frac{c-1}{2})g(X, \varphi PX)g(FZ, FZ) = 0.$$

Here, by using, (30) and (31), we obtain

$$\cos^2\theta \sin^2\theta g(Z, Z)g(X, X) - \eta^2(X) = 0,$$

thus, $\sin^2 2\theta g(Z, Z)g(X, X) - \eta^2(X) = 0$, which implies that, either M is semi-invariant or anti-invariant submanifold or $\dim(D_\theta) = 1$.
□

Theorem 4.6. *Let M be a totally umbilical pseudo-slant submanifold of a Sasakian space form $\widetilde{M}(c)$. Then the Ricci tensor S of M is given by*

$$\begin{aligned} S(X, W) &= \left\{(\frac{c+3}{4})(2p+q) + (\frac{c-1}{4})(3\cos^2\theta - 1)\right\} g(X, W) \\ &\quad +(\frac{c-1}{4})(1 - q - 2p - 3\cos^2\theta)\eta(X)\eta(W) \end{aligned} \tag{39}$$

for any $X, W \in \Gamma(TM)$.

Proof. From (35) by using (8), we obtain

$$\begin{aligned} S(X, W) &= \left\{(\frac{c+3}{4})(2p+q) + (\frac{c-1}{4})(3\cos^2\theta - 1)\right\} g(X, W) \\ &\quad +(\frac{c-1}{4})(1 - q - 2p - 3\cos^2\theta)\eta(X)\eta(W) \\ &\quad +(2p + q + 1)g(X, W)\,\|H\|^2 - \sum_{m=1}^{2p+q+1} g(g(e_m, W)H, g(X, e_m)H). \end{aligned}$$

Thus, the proof follows from the above relations, which proves the theorem completely. □

Thus we have the following corollary.

Corollary 4.7. *Every totally umbilical pseudo-slant submanifold M of a Sasakian space form $\widetilde{M}(c)$ is an η-Einstein submanifold.*

Theorem 4.8. *Let M be a totally umbilical pseudo-slant submanifold of a Sasakian space form $\widetilde{M}(c)$. Then the scalar curvature ρ of M is given by*

$$\begin{aligned} \rho &= \left\{(\frac{c+3}{4})(2p+q) + (\frac{c-1}{4})(3\cos^2\theta - 1)\right\}(2p+q+1) \\ &\quad +(\frac{c-1}{4})(1-q-2p-3\cos^2\theta). \end{aligned} \tag{40}$$

Proof. From (39), by using $X = W = e_m$, we have $\rho = \sum_{m=1}^{2p+q+1} S(e_m, e_m)$ which gives (40). Thus the proof is complete. □

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