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Efficient Reliability Estimation in Two-Parameter Exponential Distributions

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Abstract. This article concerns reliability estimation in two-parameter exponential distributions setup with known scale parameters, and unknown location parameters. Based on the uniformly minimum variance unbiased estimator, we propose a new estimator and study its theoretical properties. Simulation results reveal that the suggested estimator could be highly efficient.

1. Introduction

Consider a system with a random strength Y which is subjected to a random stress X. The system fails at any moment that the applied stress (or load) is greater than its strength or resistance. The reliability is therefore given by

$$\xi = P(X < Y).$$

This stress-strength model was first introduced in Birnbaum [2]. Since then, it has been widely used in many areas, including economics, quality control, psychology, medicine and clinical trials. Basu [1] and Johnson [7] contain reviews of many results primarily related to inference for ξ. Kotz et al. [9] present the theoretical and practical results on the theory and applications of the stress-strength relationships in industrial and economic systems.

When X and Y are independent exponential random variables, estimating ξ has been discussed by several authors. For example, see Enis and Geisser [6], Tong [20], Kelley et al. [8], Shah and Sathe [18], and Chao [3]. Kunda and Gupta [10] considered estimation of ξ when X and Y are independent random variables with generalized exponential distribution. Recently, Chaturvedi and Sharma [4] obtained the uniformly minimum variance unbiased estimator (UMVUE) for ξ when X and Y follow two-parameter exponential distribution with known scale parameter. In this article, we suggest a more efficient alternative estimator in the aforesaid setup. Toward this end, we utilize a sampling design which is delineated in the sequel.

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Ranked set sampling (RSS) is fitting for situations where quantification of sampling units is either costly or difficult, but ranking the units in a small set is easy and inexpensive. Ranking can be based on expert judgment, visual inspection or any means that does not involve actually quantifying the observations. The RSS was first proposed by Mclntyre [13] for use in estimating average yields in agriculture. Takahasi and Wakimoto [19] established theoretical foundation for statistical inference based on RSS. Since then, RSS has been applied in a variety of fields including forestry, environmental science and medicine. See Chen et al. [5] for an excellent treatment of RSS literature.

A ranked set sample of size m is drawn as follows. One first selects m^2 units at random from the population and partitions them into m sets, each of size m. The units in each set are ranked, without making actual measurements, with respect to the variable of interest. In the ith ($i = 1, \ldots, m$) set, the item judged to be ith smallest is taken for full quantification. Let $X_{[i]}$ ($i = 1, \ldots, m$) be the ith judgement order statistic from the ith set. Then, the resultant sample is denoted by $X_{[1]}, \ldots, X_{[m]}$. Here, the square bracket is used to indicate that the judgement ranks may not be correct. If our ranking is accurate (perfect ranking holds), then we replace the square brackets with the round ones, and $X_{(i)}$ becomes the ith true order statistic from the ith set. The RSS often leads to more efficient inference than is possible with simple random sampling (SRS) of comparable size. This can be justified by noting that a ranked set sample contains not only information carried by quantified observations but also information provided by the preparatory rankings.

The parameter m is called set size which should be kept small to reduce errors in the judgment ranking process. If a ranked set sample of larger size is required, then the above procedure can be repeated k times (cycles) to obtain a sample of size mk. Efficiency of RSS-based methods is generally increased by improving accuracy of the necessary rankings. To achieve this goal, the basic RSS scheme can be modified. For example, suppose in the RSS procedure, the smallest/largest unit is measured from each set of size m. We consider the situation that the sample comprises smallest judgement order statistics, which is denoted by $X^1_{[1]}, \ldots, X^m_{[1]}$. This configuration is closely related to a design called extreme ranked set sampling (ERSS) which was introduced by Samawi et al. [16]. So our sampling technique will be referred to as modified ERSS.

The stress-strength model under RSS has also drawn some attention. Sengupta and Mukhuti [17] studied nonparametric reliability estimation in RSS based on empirical distribution function. Mahdizadeh and Zamanzade [12] developed an alternative estimator based on kernel density estimation, and provided applications in the context of agriculture and medicine. Mahdizadeh and Zamanzade [11] dealt with nonparametric estimation of a dynamic reliability index in RSS. Muttlak et al. [14] derived some estimators of ξ using RSS in the case of exponential distribution. In this work, we deal with reliability estimation in two-parameter exponential distributions setup under modified ERSS.

In Section 2, our estimator is presented and some theoretical results are provided. Section 3 is given to a simulation study performed to compare the suggested estimator with its analog in SRS. Final conclusions appear in Section 4. Figures are collected in an appendix.

2. The proposed estimator

The random variable Z is said to follow a two-parameter exponential distribution with location parameter μ and scale parameter σ if its probability density function (pdf) is given by

$$f(z; \mu, \sigma) = \frac{1}{\sigma} \exp\left\{-\frac{(z-\mu)}{\sigma}\right\}, \quad z > \mu; \mu \in \mathbb{R}, \sigma > 0,$$

and it is denoted by $Z \sim E(\mu, \sigma)$. For independent random variables $X \sim E(\mu_x, \sigma_x)$ and $Y \sim E(\mu_y, \sigma_y)$, it can be shown that

$$\xi = \begin{cases} 1 - \frac{\rho}{\rho+1} e^{-\delta/\sigma_x} & \delta \geq 0 \\ \frac{1}{\rho+1} e^{\delta/\sigma_y} & \delta < 0 \end{cases}, \tag{1}$$

where $\rho = \sigma_x/\sigma_y$ and $\delta = \mu_y - \mu_x$.

Suppose $X_1, \ldots, X_m \overset{iid}{\sim} E(\mu_x, \sigma_x)$ and $Y_1, \ldots, Y_n \overset{iid}{\sim} E(\mu_y, \sigma_y)$ are two independent simple random samples. Also, $X_{(1)} = \min\{X_1, \ldots, X_m\}$ and $Y_{(1)} = \min\{Y_1, \ldots, Y_n\}$. Chaturvedi and Sharma [4] derived UMVUE for ξ which is given by

$$\hat{\xi}_{\text{SRS}} = \begin{cases} 1 - \frac{(m-1)(n\rho+1)}{mn(\rho+1)} e^{-D/\sigma_x} & D \geq 0 \\ \frac{(n-1)(\rho+m)}{mn(\rho+1)} e^{D/\sigma_y} & D < 0 \end{cases}, \tag{2}$$

where $D = Y_{(1)} - X_{(1)}$.

Let $X^1_{(1)}, \ldots, X^m_{(1)}$ and $Y^1_{(1)}, \ldots, Y^n_{(1)}$ be two independent ranked set samples drawn by modified ERSS design from $E(\mu_x, \sigma_x)$ and $E(\mu_y, \sigma_y)$, respectively. The corresponding set sizes are m and n. Here, the perfect ranking is assumed. This is not a strict assumption as one can simply identify the smallest judgment order statistic in a sample of size m or n. It is worth noting that modified ERSS scheme allows to draw m (n) copies of $X_{(1)}$ ($Y_{(1)}$) only based on m (n) measurements. In comparable SRS design, however, there is a single observation of $X_{(1)}$ and $Y_{(1)}$. An unbiased estimator of ξ can be constructed as

$$\hat{\xi}_{\text{ERSS}} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{\xi}_{i,j}, \tag{3}$$

where $\hat{\xi}_{i,j}$ is the same $\hat{\xi}_{\text{SRS}}$ evaluated at $D = Y^j_{(1)} - X^i_{(1)}$.

In the following, properties of the above estimator are studied.

Proposition 1 *The variance of $\hat{\xi}_{ERSS}$ is given by*

$$\begin{aligned} Var\left(\hat{\xi}_{ERSS}\right) &= \frac{1}{mn}\Bigg(m - \frac{2m\rho}{\rho+1} e^{-\delta/\sigma_x} + \frac{(m-1)(n\rho+1)^2}{n(\rho+1)^2(2\rho^{-1}+n)} \left[1 + \frac{m-1}{m(m-2)}\right] e^{-2\delta/\sigma_x} \\ &+ \frac{n}{m\rho^{-1}+n} \left[\frac{2(n\rho+1)}{n(\rho+1)} - \frac{(m-1)^2(n\rho+1)^2}{m(m-2)n^2(\rho+1)^2} + \frac{(n-1)^2(\rho+m)^2}{mn^2(\rho+1)^2(2\rho+m)} - 1\right] e^{-m\delta/\sigma_x} \\ &+ \frac{(n-1)(\rho+m)^2}{m(\rho+1)^2(2\rho+m)} e^{2\delta/\sigma_y} - (m+n-1)\xi^2\Bigg). \end{aligned} \tag{4}$$

Proof: From theory of U-statistics (see Rohatgi and Ehsanes Saleh [15], Chapter 13), one can verify that

$$Var\left(\hat{\xi}_{\text{ERSS}}\right) = \frac{1}{mn} \sum_{r=0}^{1} \sum_{s=0}^{1} \binom{m-1}{1-r} \binom{n-1}{1-s} \zeta_{r,s}, \tag{5}$$

where $\zeta_{r,s}$ is the covariance between $\hat{\xi}_{i,j}$ and $\hat{\xi}_{k,\ell}$ with exactly r $X_{(1)}$'s, and s $Y_{(1)}$'s in common. That is to say

$$\zeta_{0,0} = 0,$$

$$\zeta_{0,1} = E\left(\hat{\xi}_{i,j}\, \hat{\xi}_{k,j}\right) - \xi^2,$$

$$\zeta_{1,0} = E\left(\hat{\xi}_{i,j}\, \hat{\xi}_{i,\ell}\right) - \xi^2$$

and

$$\zeta_{1,1} = E\left(\hat{\xi}^2_{i,j}\right) - \xi^2.$$

It is well-known that $X^i_{(1)} \sim E(\mu_x, \sigma_x/m)$ and $Y^j_{(1)} \sim E(\mu_y, \sigma_y/n)$. These results are essential in obtaining the above quantities. After some computation, it can bee seen that

$$E\left(\hat{\xi}_{i,j} \middle| Y^j_{(1)} = y\right) = 1 - \frac{n\rho+1}{n(\rho+1)} e^{-(y-\mu_x)/\sigma_x}.$$

So we may write

$$\begin{aligned}\zeta_{0,1} &= E\left(E^2\left(\hat{\xi}_{i,j}\middle|Y^j_{(1)}=y\right)\right)-\xi^2\\ &= 1-\frac{2\rho}{\rho+1}\,e^{-\delta/\sigma_x}+\frac{(n\rho+1)^2}{n(\rho+1)^2(2\rho^{-1}+n)}\,e^{-2\delta/\sigma_x}-\xi^2. \qquad (6)\end{aligned}$$

Similarly, it is concluded that

$$E\left(\hat{\xi}_{i,j}\middle|X^i_{(1)}=x\right)=\frac{\rho+m}{m(\rho+1)}\,e^{-(x-\mu_y)/\sigma_y},$$

and therefore

$$\begin{aligned}\zeta_{1,0} &= E\left(E^2\left(\hat{\xi}_{i,j}\middle|X^i_{(1)}=x\right)\right)-\xi^2\\ &= \frac{(\rho+m)^2}{m(\rho+1)^2(2\rho+m)}\,e^{2\delta/\sigma_y}-\xi^2. \qquad (7)\end{aligned}$$

Proceeding in the same way,

$$\begin{aligned}E\left(\hat{\xi}^2_{i,j}\middle|Y^j_{(1)}=y\right) &= 1-\frac{2(n\rho+1)}{n(\rho+1)}\,e^{-(y-\mu_x)/\sigma_x}+\frac{(m-1)^2(n\rho+1)^2}{m(m-2)n^2(\rho+1)^2}\,e^{-2(y-\mu_x)/\sigma_x}\\ &+ \left[\frac{2(n\rho+1)}{n(\rho+1)}-\frac{(m-1)^2(n\rho+1)^2}{m(m-2)n^2(\rho+1)^2}+\frac{(n-1)^2(\rho+m)^2}{mn^2(\rho+1)^2(2\rho+m)}-1\right]e^{-m(y-\mu_x)/\sigma_x},\end{aligned}$$

which implies that

$$\begin{aligned}\zeta_{1,1} &= E\left(E\left(\hat{\xi}^2_{i,j}\middle|Y^j_{(1)}=y\right)\right)-\xi^2\\ &= 1-\frac{2\rho}{\rho+1}\,e^{-\delta/\sigma_x}+\frac{(m-1)^2(n\rho+1)^2}{m(m-2)n(\rho+1)^2(2\rho^{-1}+n)}\,e^{-2\delta/\sigma_x}\\ &+ \frac{n}{m\rho^{-1}+n}\left[\frac{2(n\rho+1)}{n(\rho+1)}-\frac{(m-1)^2(n\rho+1)^2}{m(m-2)n^2(\rho+1)^2}+\frac{(n-1)^2(\rho+m)^2}{mn^2(\rho+1)^2(2\rho+m)}-1\right]e^{-m\delta/\sigma_x}-\xi^2. \qquad (8)\end{aligned}$$

Putting (5)-(8) together, and some algebra complete the proof. □

The next result attends to asymptotic distribution of $\hat{\xi}_{\text{ERSS}}$.

Proposition 2 *Let $\hat{\xi}_{ERSS}$ be as in (3), and $N=m+n$. If $m,n\to\infty$ and $m/N\to\lambda\in(0,1)$, then*

$$\sqrt{N}(\hat{\xi}_{ERSS}-\xi)\xrightarrow{d}N(0,\sigma^2),$$

where $\sigma^2=\zeta_{1,0}/\lambda+\zeta_{0,1}/(1-\lambda)$ with

$$\zeta_{1,0}=\frac{(\rho+m)^2}{m\,(\rho+1)^2\,(2\rho+m)}\,e^{2\delta/\sigma_y}-\xi^2$$

and

$$\zeta_{0,1}=1-\frac{2\rho}{\rho+1}\,e^{-\delta/\sigma_x}+\frac{(n\rho+1)^2}{n\,(\rho+1)^2\,(2\rho^{-1}+n)}\,e^{-2\delta/\sigma_x}-\xi^2.$$

Proof. It is concluded from asymptotic theory of U-statistics (Rohatgi and Ehsanes Saleh [15], Chapter 13), and the previous proposition. □

It is difficult to compare variances of $\hat{\xi}_{\text{SRS}}$ and $\hat{\xi}_{\text{ERSS}}$ analytically. In the following section, the two estimators are compared by means of Monte Carlo experiment.

3. Simulation study

A simulation study was carried out to examine the performances of the two reliability estimators. For simplicity, we set $\mu_x = 0$ and $\sigma_x = 1$, and thus $\mu_y = \delta$ and $\sigma_y = \rho^{-1}$. Figure 1 depicts ξ as a function of δ, for $\rho = 0.5, 2$. It is seen that ξ is strictly increasing in δ which can also be verified by (1). In our analysis, set sizes $(m, n) \in \{(3, 3), (5, 5), (7, 7)\}$ were selected.

The efficiency of $\hat{\xi}_{\text{ERSS}}$ with respect to $\hat{\xi}_{\text{SRS}}$ is estimated as follows. For each combination of $\rho \in \{0.5, 2\}$ and $\delta \in (-2, 2)$, 10,000 pairs of samples were generated in SRS and ERSS designs. The two estimators were computed from each pair of samples in the corresponding designs, and their variances were determined. The relative efficiency (RE) is defined as

$$RE = \frac{\widehat{Var}\left(\hat{\xi}_{\text{SRS}}\right)}{\widehat{Var}\left(\hat{\xi}_{\text{ERSS}}\right)}.$$

The RE values larger than one indicate that $\hat{\xi}_{\text{ERSS}}$ is more efficient than $\hat{\xi}_{\text{SRS}}$. Figures 2-4 display the results.

Figure 2 is corresponding to $(m, n) = (3, 3)$ case. It is observed that RSS-based estimator is always superior to its SRS analog. Interestingly, efficiency gain is sizeable even with these small sample sizes. It turns out that the RE values have more fluctuations as δ deviates from zero, especially for $\rho = 2$. The results for $(m, n) = (5, 5), (7, 7)$ are presented in Figures 3 and 4, respectively. The general trends are similar to those in Figure 2. The only marked difference is that the REs are higher due to larger sample sizes, as expected.

4. Conclusion

The RSS is a cost-efficient alternative to the usual SRS scheme in situations where exact measurements of sample units are difficult or expensive to obtain but ranking of them according to the variable of interest is relatively easy and cheap.

As a variation of RSS, ERSS has a potential to reduce errors in the judgement ranking process. In this article, we build on a modification of ERSS to propose a reliability estimator in two-parameter exponential distributions setup. Monte Carlo evidence suggests that the estimator is highly efficient as compared with the UMVUE based on SRS.

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Appendix

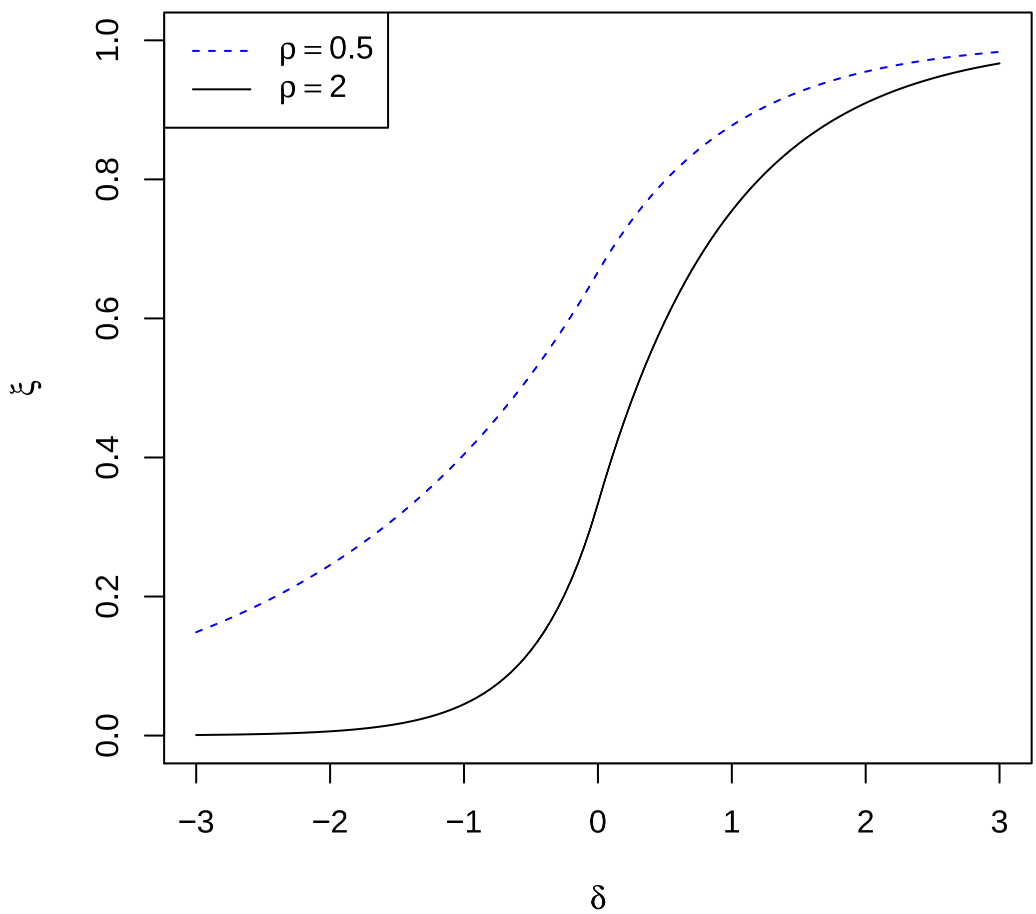

Figure 1: Values of ξ as a function of δ

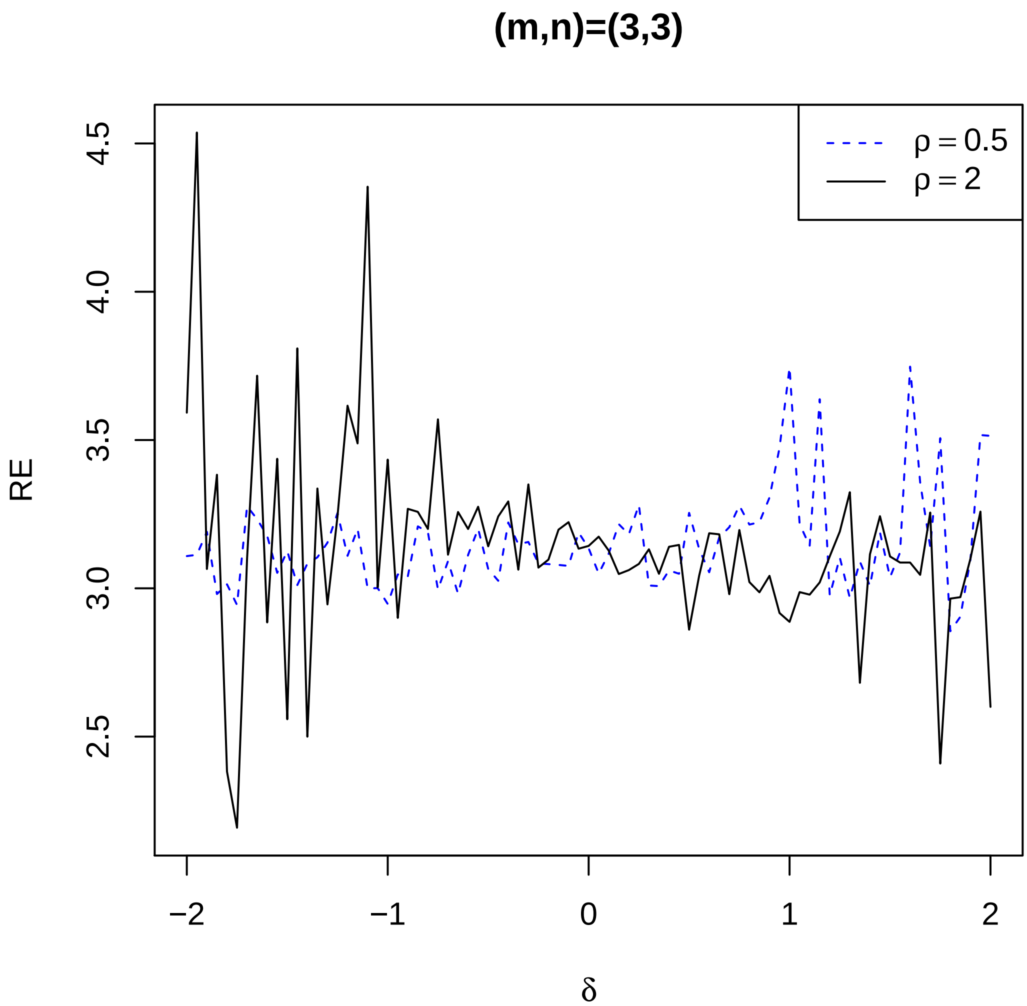

Figure 2: Estimated RE as a function of δ

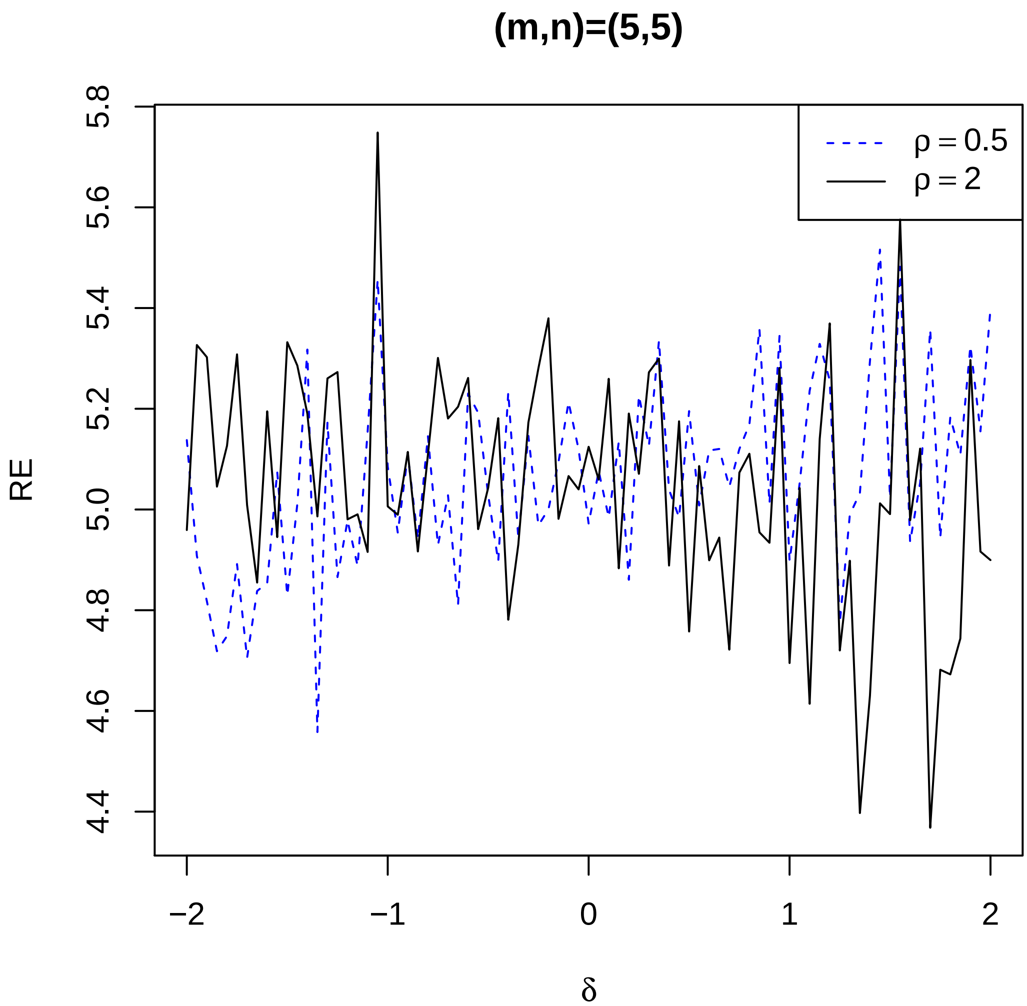

Figure 3: Estimated RE as a function of δ

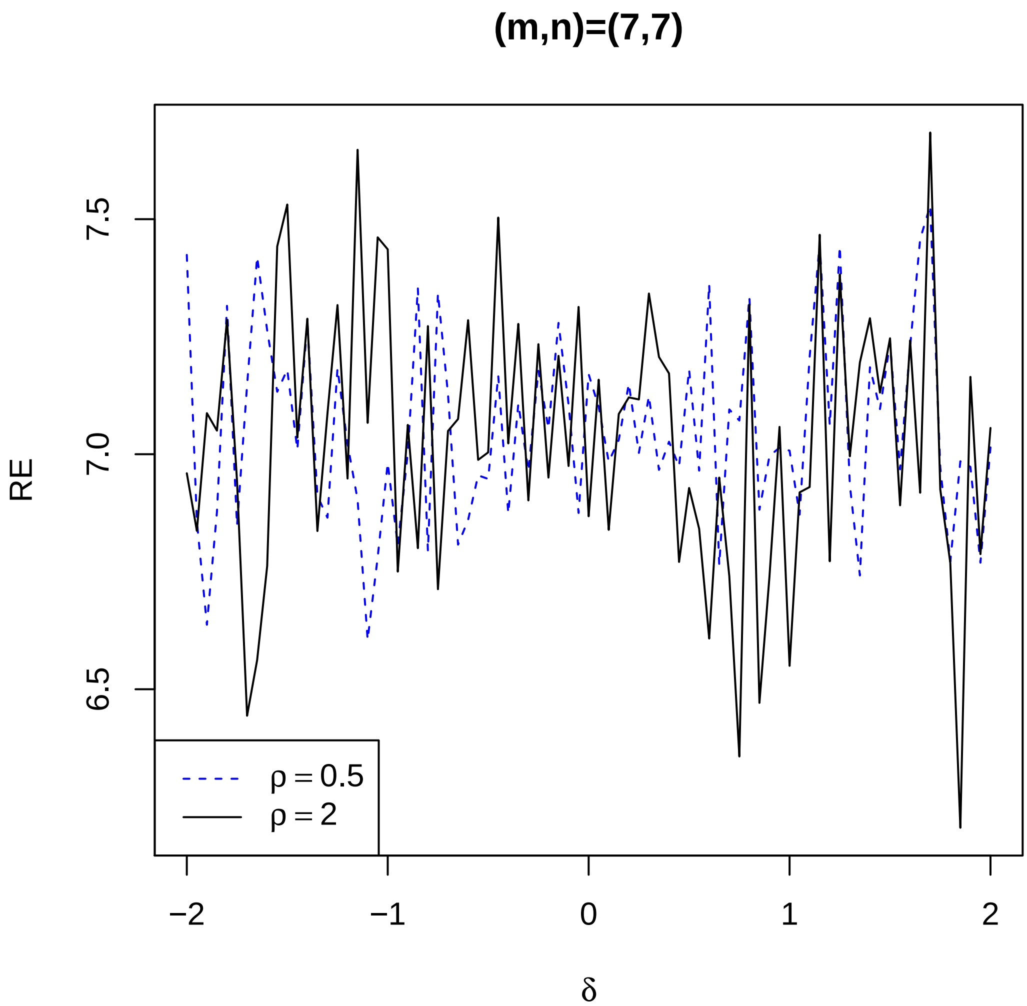

Figure 4: Estimated RE as a function of δ