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Malliavin Calculus for Generalized and Test Stochastic Processes

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Abstract. We extend the Malliavin calculus from the classical finite variance setting to generalized processes with infinite variance and their test processes. The domain and the range of the basic Malliavin operators is characterized in terms of test processes and generalized processes. Various properties are proved such as the duality of the integral and the derivative in strong and in weak sense, the product rule with respect to ordinary and Wick multiplication and the chain rule in classical and in Wick sense.

1. Introduction

Stochastic processes with infinite variance (e.g. the white noise process) appear in many cases as solutions to stochastic differential equations. The Hida spaces and the Kondratiev spaces (see e.g. [3, 4]) have been introduced as the stochastic analogues of the Schwartz spaces of tempered distributions in order to provide a strict theoretical meaning for these kind of processes. The spaces of the test processes contain highly regular processes which are needed as windows through which one can detect the action of generalized processes.

The Malliavin derivative, the Skorokhod integral and the Ornstein-Uhlenbeck operator are fundamental for the stochastic calculus of variations. Each of them has a meaning also in quantum theory: they represent the annihilation, the creation and the number operator respectively. In stochastic analysis, the Malliavin derivative charachterizes densities of distributions, the Skorokhod integral is an extension of the Itô integral to non-adapted processes, and the Ornstein-Uhlenbeck operator plays the role of the stochastic Laplacian.

In the classical setting followed by [2, 13, 15], the domain of these operators is a strict subset of the set of processes with finite second moments $(L)^2$, leading to Sobolev type normed spaces. A more general characterization of the domain of these operators in Kondratiev generalized function spaces has been derived in [5, 6, 9, 10]. The range of the operators for generalized processes for $\rho = 1$ has been studied in [8]. As a conclusion to this series of papers, in the current paper we provide a setting for the domains of these operators for $\rho \in [0, 1]$ and a similar setting for test processes: first we construct a subset of the Kondratiev space which will be the domain of the operators, then we prove that the operators are linear, bounded, non-injective within the corresponding spaces and develop a representation of their range. In

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the second part of the paper we fully develop the calculus including the integration by parts, Leibnitz rule and chain rule etc. using the interplay of generalized processes with their test processes and different types of dual pairings.

The Malliavin derivative of generalized stochastic processes has first been considered in [1] using the S-transform of stochastic exponentials and chaos expansions with n-fold Itô integrals with some vague notion of the Itô integral of a generalized function. Our approach is different, it relies on chaos expansions via Hermite polynomials and it provides more precise results: a fine gradation of generalized and test functions is followed where each level has a Hilbert structure and consequently each level of singularity has its own domain, range, set of multipliers etc.

The organisation of the paper is the following: After a short preview of the basic setting and notions of chaos expansions (Subsection 2.1), spaces of generalized stochastic processes and test stochastic processes (Subsection 2.2-2.3), we turn to the question of their multiplication in Subsection 2.4. In Section 3 we provide the characterisation of the domains of the basic operators of Malliavin calculus and prove their linearity and boundedness. In Section 4 we provide explicit solutions to the equations $\mathcal{R}u = g$, $\mathbb{D}u = h$, $\delta u = f$. In Section 5 we prove some rules of the Malliavin calculus for generalized and test processes, such as the duality between the derivative $\mathbb{D}$ and the integral operator δ (integration by parts formula), the product rule for $\mathbb{D}$ and $\mathcal{R}$ both for ordinary multiplication and Wick multiplication, and eventually we prove the chain rule. Some accompanying examples, applications and supplementary material to our results are provided in [11].

2. Preliminaries

Consider the Gaussian white noise probability space $(S'(\mathbb{R}), \mathcal{B}, \mu)$, where $S'(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel $\sigma-$algebra generated by the weak topology on $S'(\mathbb{R})$ and μ the Gaussian white noise measure corresponding to the characteristic function $\int_{S'(\mathbb{R})} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}}$, $\phi \in S(\mathbb{R})$, given by the Bochner-Minlos theorem.

Denote by $h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}})$, $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the family of Hermite polynomials and $\xi_n(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{(n-1)!}} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x)$, $n \in \mathbb{N}$, the family of Hermite functions. The family of Hermite functions forms a complete orthonormal system in $L^2(\mathbb{R})$. We follow the characterization of the Schwartz spaces in terms of the Hermite basis: The space of rapidly decreasing functions as a projective limit space $S(\mathbb{R}) = \bigcap_{l\in\mathbb{N}_0} S_l(\mathbb{R})$ and the space of tempered distributions as an inductive limit space $S'(\mathbb{R}) = \bigcup_{l\in\mathbb{N}_0} S_{-l}(\mathbb{R})$ where

$$S_l(\mathbb{R}) = \{f = \sum_{k=1}^{\infty} a_k\,\xi_k : \ \|f\|_l^2 = \sum_{k=1}^{\infty} a_k^2 (2k)^l < \infty\},\ l \in \mathbb{Z},\ \mathbb{Z} = -\mathbb{N} \cup \mathbb{N}_0.$$

Note that $S_l(\mathbb{R})$ is a Hilbert space endowed with the scalar product $\langle\cdot,\cdot\rangle_l$ given by

$$\langle \xi_k, \xi_i\rangle_l = \begin{cases} 0, & k \neq i \\ \|\xi_k\|_l^2 = (2k)^l, & k = i. \end{cases}, \quad l \in \mathbb{Z}.$$

2.1. The Wiener chaos spaces

Let $\mathcal{I} = (\mathbb{N}_0^{\mathbb{N}})_c$ denote the set of sequences of nonnegative integers which have only finitely many nonzero components $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m, 0, 0\ldots)$, $\alpha_i \in \mathbb{N}_0$, $i = 1, 2, ..., m$, $m \in \mathbb{N}$. The kth unit vector $\varepsilon^{(k)} = (0, \cdots, 0, 1, 0, \cdots)$, $k \in \mathbb{N}$ is the sequence of zeros with the only entry 1 as its kth component. The multi-index $\mathbf{0} = (0, 0, 0, 0, \ldots)$ has all zero entries. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha| = \sum_{k=1}^{\infty} \alpha_k$.

Operations with multi-indices are carried out componentwise e.g. $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$, $\alpha! = \prod_{k=1}^{\infty} \alpha_k!$ and $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$. Note that $\alpha > \mathbf{0}$ (equivalently $|\alpha| > 0$) if there is at least one component $\alpha_k > 0$. We adopt the convention that $\alpha - \beta$ exists only if $\alpha - \beta > \mathbf{0}$ and otherwise it is not defined.

Let $(2\mathbb{N})^{\alpha} = \prod_{k=1}^{\infty} (2k)^{\alpha_k}$. Note that $\sum_{\alpha\in\mathcal{I}} (2\mathbb{N})^{-p\alpha} < \infty$ for $p > 1$ (see e.g. [4]).

Lemma 2.1. *The following estimates hold:*

1° $\binom{\alpha}{\beta} \le 2^{|\alpha|} \le (2\mathbb{N})^{\alpha}, \qquad \alpha \in \mathcal{I},$

2° $(\theta+\beta)! \le \theta!\beta!\,(2\mathbb{N})^{\theta+\beta}, \qquad \theta, \beta \in \mathcal{I}.$

Proof. 1° Since $\binom{n}{k} \le 2^n$, for all $n \in \mathbb{N}_0$ and $0 \le k \le n$, it follows that

$$\binom{\alpha}{\beta} = \prod_{i\in\mathbb{N}} \binom{\alpha_i}{\beta_i} \le \prod_{i\in\mathbb{N}} 2^{\alpha_i} = 2^{|\alpha|} \le \prod_{i\in\mathbb{N}} (2i)^{\alpha_i} = (2\mathbb{N})^{\alpha},$$

for all $\alpha \in \mathcal{I}$ and $\mathbf{0} \le \beta \le \alpha$.

2° From $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha-\beta)!}$ and (i) it follows that $\alpha! \le \beta!\,(\alpha-\beta)!\,(2\mathbb{N})^{\alpha}$. By substituting $\theta = \alpha - \beta$, we obtain $(\theta+\beta) \le \theta!\,\beta!\,(2\mathbb{N})^{\theta+\beta}$, for all $\theta, \beta \in \mathcal{I}$. □

Let $(L)^2 = L^2(S'(\mathbb{R}), \mathcal{B}, \mu)$ be the Hilbert space of random variables with finite second moments. Then

$$H_\alpha(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(\langle \omega, \xi_k \rangle), \quad \alpha \in \mathcal{I}, \tag{1}$$

forms the Fourier-Hermite orthogonal basis of $(L)^2$ such that $\|H_\alpha\|^2_{(L)^2} = \alpha!$. In particular, $H_{\mathbf{0}} = 1$ and for the kth unit vector $H_{\varepsilon^{(k)}}(\omega) = \langle \omega, \xi_k \rangle$, $k \in \mathbb{N}$. The prominent *Wiener-Itô chaos expansion theorem* states that each element $F \in (L)^2$ has a unique representation of the form

$$F(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha\, H_\alpha(\omega), \quad \omega \in S'(\mathbb{R}),\ c_\alpha \in \mathbb{R},\ \alpha \in \mathcal{I},$$

such that $\|F\|^2_{(L)^2} = \sum_{\alpha\in\mathcal{I}} c_\alpha^2\, \alpha! < \infty$.

2.2. *Kondratiev spaces and Hida spaces*

The stochastic analogue of Schwartz spaces as generalized function spaces are the Kondratiev spaces of generalized random variables. Let $\rho \in [0, 1]$.

Definition 2.2. *The space of the* Kondratiev test random variables $(S)_\rho$ *consists of elements* $f = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha \in (L)^2$, $c_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, *such that*

$$\|f\|^2_{\rho,p} = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{p\alpha} < \infty, \quad \textit{for all } p \in \mathbb{N}_0.$$

The space of the Kondratiev generalized random variables $(S)_{-\rho}$ *consists of formal expansions of the form* $F = \sum_{\alpha\in\mathcal{I}} b_\alpha H_\alpha$, $b_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, *such that*

$$\|F\|^2_{-\rho,-p} = \sum_{\alpha\in\mathcal{I}} b_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-p\alpha} < \infty, \quad \textit{for some } p \in \mathbb{N}_0.$$

This provides a sequence of spaces $(S)_{\rho,p} = \{f \in (L)^2 : \|f\|_{\rho,p} < \infty\}$, $\rho \in [0, 1]$, such that

$$(S)_{1,p} \subseteq (S)_{\rho,p} \subseteq (S)_{0,p} \subseteq (L)^2 \subseteq (S)_{0,-p} \subseteq (S)_{-\rho,-p} \subseteq (S)_{-1,-p},$$

$$(S)_{\rho,p} \subseteq (S)_{\rho,q} \subseteq (L)^2 \subseteq (S)_{-\rho,-q} \subseteq (S)_{-\rho,-p},$$

for all $p \geq q \geq 0$, the inclusions denote continuous embeddings and $(S)_{0,0} = (L)^2$. Thus, $(S)_\rho = \bigcap_{p\in\mathbb{N}_0} (S)_{\rho,p}$, can be equipped with the projective topology, while $(S)_{-\rho} = \bigcup_{p\in\mathbb{N}_0} (S)_{-\rho,-p}$ as its dual with the inductive topology. Note that $(S)_\rho$ is nuclear and the following Gel'fand triple

$$(S)_\rho \subseteq (L)^2 \subseteq (S)_{-\rho}$$

is obtained. Especially, the case $\rho = 0$ corresponds to the Hida spaces.

We will denote by $\ll \cdot,\cdot \gg_\rho$ the dual pairing between $(S)_{-\rho}$ and $(S)_\rho$. Its action is given by $\ll A, B \gg_\rho = \ll \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha, \sum_{\alpha\in\mathcal{I}} b_\alpha H_\alpha \gg_\rho = \sum_{\alpha\in\mathcal{I}} \alpha! a_\alpha b_\alpha$. In case of random variables with finite variance it reduces to the scalar product $\ll A, B \gg_{(L)^2} = E(AB)$. Especially, the Hida case will be of importance, thus note that for any fixed $p \in \mathbb{Z}$, $(S)_{0,p}$, $p \in \mathbb{Z}$, is a Hilbert space (we identify the case $p = 0$ with $(L)^2$) endowed with the scalar product

$$\ll H_\alpha, H_\beta \gg_{0,p} = \begin{cases} 0, & \alpha \neq \beta, \\ \alpha!(2\mathbb{N})^{p\alpha}, & \alpha = \beta, \end{cases}, \quad \text{for } p \in \mathbb{Z},$$

extended by linearity and continuity to

$$\ll A, B \gg_{0,p} = \sum_{\alpha\in\mathcal{I}} \alpha! a_\alpha b_\alpha (2\mathbb{N})^{p\alpha}, \quad p \in \mathbb{Z}.$$

In the framework of white noise analysis, the problem of pointwise multiplication of generalized functions is overcome by introducing the Wick product. It is well defined in the Kondratiev spaces of test and generalized stochastic functions $(S)_\rho$ and $(S)_{-\rho}$; see for example [3, 4].

Definition 2.3. *Let $F, G \in (S)_{-\rho}$ be given by their chaos expansions $F(\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha(\omega)$ and $G(\omega) = \sum_{\beta\in\mathcal{I}} g_\beta H_\beta(\omega)$, for unique $f_\alpha, g_\beta \in \mathbb{R}$. The* Wick product *of F and G is the element denoted by $F\lozenge G$ and defined by*

$$F\lozenge G(\omega) = \sum_{\alpha\in\mathcal{I}} \sum_{\beta\in\mathcal{I}} f_\alpha \; g_\beta \; H_{\alpha+\beta}(\omega) = \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) H_\gamma(\omega).$$

The same definition is provided for the Wick product of test random variables belonging to $(S)_\rho$.

For the Fourier-Hermite polynomials (1), for all $\alpha, \beta \in \mathcal{I}$ it holds $H_\alpha \lozenge H_\beta = H_{\alpha+\beta}$.

The nth Wick power is defined by $F^{\lozenge n} = F^{\lozenge(n-1)} \lozenge F$, $F^{\lozenge 0} = 1$. Note that $H_{n\varepsilon^{(k)}} = H_{\varepsilon^{(k)}}^{\lozenge n}$ for $n \in \mathbb{N}_0$, $k \in \mathbb{N}$.

Note that the Kondratiev spaces $(S)_\rho$ and $(S)_{-\rho}$ are closed under the Wick multiplication [4], while the space $(L)^2$ is not closed under it. The most important property of the Wick multiplication is its relation to the Itô-Skorokhod integration [3, 4], since it reproduces the fundamental theorem of calculus. It also represents a renormalization of the ordinary product and the highest order stochastic approximation of the ordinary product [14].

In the sequel we will need the notion of *Wick-versions of analytic functions*.

Definition 2.4. *If $\varphi : \mathbb{R} \to \mathbb{R}$ is a real analytic function at the origin represented by the power series*

$$\varphi(x) = \sum_{n=0}^{\infty} a_n \, x^n, \quad x \in \mathbb{R},$$

then its Wick version $\varphi^\lozenge : (S)_{-\rho} \to (S)_{-\rho}$, for $\rho \in [0, 1]$, is given by

$$\varphi^\lozenge(F) = \sum_{n=0}^{\infty} a_n \, F^{\lozenge n}, \quad F \in (S)_{-\rho}.$$

2.3. Generalized stochastic processes

Let $\tilde{X}$ be a Banach space endowed with the norm $\|\cdot\|_{\tilde{X}}$ and let $\tilde{X}'$ denote its dual space. In this section we describe $\tilde{X}$−valued random variables. Most notably, if $\tilde{X}$ is a space of functions on $\mathbb{R}$, e.g. $\tilde{X} = C^k([a, b])$, $-\infty < a < b < \infty$ or $\tilde{X} = L^2(\mathbb{R})$, we obtain the notion of a stochastic process. We will also define processes where $\tilde{X}$ is not a normed space, but a nuclear space topologized by a family of seminorms, e.g. $\tilde{X} = S(\mathbb{R})$ (see e.g. [16]).

Definition 2.5. *Let f have the formal expansion*

$$f = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha\,, \quad \text{where } \ f_\alpha \in X,\ \alpha \in \mathcal{I}. \tag{2}$$

Let $\rho \in [0, 1]$. Define the following spaces:

$$\begin{aligned} X \otimes (S)_{\rho,p} &= \{f:\ \|f\|^2_{X\otimes(S)_{\rho,p}} = \sum_{\alpha\in\mathcal{I}} \alpha!^{1+\rho}\, \|f_\alpha\|^2_X (2\mathbb{N})^{p\alpha} < \infty\},\\ X \otimes (S)_{-\rho,-p} &= \{f:\ \|f\|^2_{X\otimes(S)_{-\rho,-p}} = \sum_{\alpha\in\mathcal{I}} \alpha!^{1-\rho}\, \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty\}, \end{aligned}$$

where X denotes an arbitrary Banach space (allowing both possibilities $X = \tilde{X}$, $X = \tilde{X}'$). Especially, for $\rho = 0$ and $p = 0$, $X \otimes (S)_{0,0}$ will be denoted by

$$X \otimes (L)^2 = \{f:\ \|f\|^2_{X\otimes(L)^2} = \sum_{\alpha\in\mathcal{I}} \alpha! \|f_\alpha\|^2_X < \infty\}.$$

We will denote by $E(F) = f_{\mathbf{0}}$ the generalized expectation of the process F.

Definition 2.6. Generalized stochastic processes *and* test stochastic processes *in Kondratiev sense are elements of the spaces*

$$X \otimes (S)_{-\rho} = \bigcup_{p\in\mathbb{N}_0} X \otimes (S)_{-\rho,-p}, \quad X \otimes (S)_\rho = \bigcap_{p\in\mathbb{N}_0} X \otimes (S)_{\rho,p}, \quad \rho \in [0, 1]$$

respectively.

Remark 2.7. *The symbol $\otimes$ denotes the projective tensor product of two spaces, i.e. $\tilde{X}' \otimes (S)_{-\rho}$ is the completion of the tensor product with respect to the π-topology.*

The Kondratiev space $(S)_\rho$ is nuclear and thus $(\tilde{X} \otimes (S)_\rho)' \cong \tilde{X}' \otimes (S)_{-\rho}$. Note that $\tilde{X}' \otimes (S)_{-\rho}$ is isomorphic to the space of linear bounded mappings $\tilde{X} \to (S)_{-\rho}$, and it is also isomporphic to the space of linear bounded mappings $(S)_\rho \to \tilde{X}'$.

In [19] and [20] a general setting of S'-valued generalized stochastic process is provided: $S'(\mathbb{R})$-valued generalized stochastic processes are elements of $X \otimes S'(\mathbb{R}) \otimes (S)_{-\rho}$ and they are given by chaos expansions of the form

$$f = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \otimes H_\alpha = \sum_{\alpha\in\mathcal{I}} b_\alpha \otimes H_\alpha = \sum_{k\in\mathbb{N}} c_k \otimes \xi_k, \tag{3}$$

where $b_\alpha = \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \in X \otimes S'(\mathbb{R})$, $c_k = \sum_{\alpha\in\mathcal{I}} a_{\alpha,k} \otimes H_\alpha \in X \otimes (S)_{-\rho}$ and $a_{\alpha,k} \in X$. Thus,

$$X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-\rho,-p} = \left\{ f = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \otimes H_\alpha :\ \|f\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-\rho,-p}} = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha!^{1-\rho} \|a_{\alpha,k}\|^2_X\, (2k)^{-l} (2\mathbb{N})^{-p\alpha} < \infty \right\}$$

and

$$X \otimes S'(\mathbb{R}) \otimes (S)_{-\rho} = \bigcup_{p,l \in \mathbb{N}_0} X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-\rho,-p}.$$

The generalized expectation of an S'-valued stochastic process f is given by $E(f) = \sum_{k\in\mathbb{N}} a_{(0,0,\dots),k} \otimes \xi_k = b_{\mathbf{0}}$.

In an analogue way, we define S-valued test processes as elements of $X \otimes S(\mathbb{R}) \otimes (S)_\rho$, which are given by chaos expansions of the form (3), where $b_\alpha = \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \in X \otimes S(\mathbb{R})$, $c_k = \sum_{\alpha\in\mathcal{I}} a_{\alpha,k} \otimes H_\alpha \in X \otimes (S)_\rho$ and $a_{\alpha,k} \in X$. Thus,

$$X \otimes S_l(\mathbb{R}) \otimes (S)_{\rho,p} = \left\{ f = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \otimes H_\alpha : \|f\|^2_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}} = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha!^{1+\rho} \|a_{\alpha,k}\|^2_X (2k)^l (2\mathbb{N})^{p\alpha} < \infty \right\}$$

and

$$X \otimes S(\mathbb{R}) \otimes (S)_\rho = \bigcap_{p,l\in\mathbb{N}_0} X \otimes S_l(\mathbb{R}) \otimes (S)_{\rho,p}.$$

The Hida spaces are obtained for $\rho = 0$. Especially, for $p = l = 0$, one obtains the space of processes with finite second moments and square integrable trajectories $X \otimes L^2(\mathbb{R}) \otimes (L)^2$. It is isomporphic to $X \otimes L^2(\mathbb{R}\times\Omega)$ and if X is a separable Hilbert space, then it is also isomorphic to $L^2(\mathbb{R}\times\Omega; X)$.

2.4. Multiplication of stochastic processes

We generalize the definition of the Wick product of random variables to the set of generalized stochastic processes in the way as it is done in [7, 17] and [18]. For this purpose we will assume that X is closed under multiplication, i.e. that $x \cdot y \in X$, for all $x, y \in X$.

Definition 2.8. *Let $F, G \in X \otimes (S)_{\pm\rho}$, $\rho \in [0,1]$, be generalized (resp. test) stochastic processes given in chaos expansions of the form* (2). *Then the* Wick product *$F\lozenge G$ is defined by*

$$F\lozenge G = \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) \otimes H_\gamma. \tag{4}$$

Theorem 2.9. *Let $\rho \in [0,1]$ and let the stochastic processes F and G be given in their chaos expansion forms $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha$ and $G = \sum_{\alpha\in\mathcal{I}} g_\alpha \otimes H_\alpha$.*

1° *If $F \in X \otimes (S)_{-\rho,-p_1}$ and $G \in X \otimes (S)_{-\rho,-p_2}$ for some $p_1, p_2 \in \mathbb{N}_0$, then $F\lozenge G$ is a well defined element in $X \otimes (S)_{-\rho,-q}$, for $q \geq p_1 + p_2 + 4$.*

2° *If $F \in X \otimes (S)_{\rho,p_1}$ and $G \in X \otimes (S)_{\rho,p_2}$ for $p_1, p_2 \in \mathbb{N}_0$, then $F\lozenge G$ is a well defined element in $X \otimes (S)_{\rho,q}$, for $q \leq \min\{p_1, p_2\} - 4$.*

Proof. 1° By the Cauchy-Schwartz inequality, the following holds

$$\begin{aligned}
\|F\lozenge G\|^2_{X\otimes(S)_{-\rho,-q}} &= \sum_{\gamma\in\mathcal{I}} \| \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \|^2_X (\gamma!)^{1-\rho} (2\mathbb{N})^{-q\gamma} \leq \sum_{\gamma\in\mathcal{I}} \| \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \|^2_X (\gamma!)^{1-\rho} (2\mathbb{N})^{-(p_1+p_2+4)\gamma} \\
&= \sum_{\gamma\in\mathcal{I}} \| \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta (\alpha+\beta)!^{\frac{1-\rho}{2}} (2\mathbb{N})^{-\frac{p_1+1}{2}\gamma} (2\mathbb{N})^{-\frac{p_2+1}{2}\gamma} \|^2_X (2\mathbb{N})^{-2\gamma} \\
&\leq \sum_{\gamma\in\mathcal{I}} \| \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta (\alpha!\beta!(2\mathbb{N})^{\alpha+\beta})^{\frac{1-\rho}{2}} (2\mathbb{N})^{-\frac{p_1+1}{2}\alpha} (2\mathbb{N})^{-\frac{p_2+1}{2}\beta} \|^2_X (2\mathbb{N})^{-2\gamma} \\
&\leq \sum_{\gamma\in\mathcal{I}} \| \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \alpha!^{\frac{1-\rho}{2}} \beta!^{\frac{1-\rho}{2}} (2\mathbb{N})^{-\frac{p_1+\rho}{2}\alpha} (2\mathbb{N})^{-\frac{p_2+\rho}{2}\beta} \|^2_X (2\mathbb{N})^{-2\gamma} \\
&\leq \sum_{\gamma\in\mathcal{I}} (2\mathbb{N})^{-2\gamma} \| \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \alpha!^{\frac{1-\rho}{2}} \beta!^{\frac{1-\rho}{2}} (2\mathbb{N})^{-\frac{p_1\alpha}{2}} (2\mathbb{N})^{-\frac{p_2\beta}{2}} \|^2_X
\end{aligned}$$

$$\leq \sum_{\gamma\in\mathcal{I}}(2\mathbb{N})^{-2\gamma}\left(\sum_{\alpha+\beta=\gamma}\|f_\alpha\|_X^2(\alpha!)^{1-\rho}(2\mathbb{N})^{-p_1\alpha}\right)\left(\sum_{\alpha+\beta=\gamma}\|g_\beta\|_X^2(\beta!)^{1-\rho}(2\mathbb{N})^{-p_2\beta}\right)$$
$$\leq \sum_{\gamma\in\mathcal{I}}(2\mathbb{N})^{-2\gamma}\left(\sum_{\alpha\in\mathcal{I}}\|f_\alpha\|_X^2(\alpha!)^{1-\rho}(2\mathbb{N})^{-p_1\alpha}\right)\left(\sum_{\beta\in\mathcal{I}}\|g_\beta\|_X^2(\beta!)^{1-\rho}(2\mathbb{N})^{-p_2\beta}\right)$$
$$= M\cdot\|F\|^2_{X\otimes(S)_{-\rho,-p_1}}\cdot\|G\|^2_{X\otimes(S)_{-\rho,-p_2}}<\infty,$$

since $M=\sum_{\gamma\in\mathcal{I}}(2\mathbb{N})^{-2\gamma}<\infty$. We also applied Lemma 2.1 part 1°, inequalities $(2\mathbb{N})^{-\frac{p_1+1}{2}\gamma}\leq(2\mathbb{N})^{-\frac{p_1+1}{2}\alpha}$ and $(2\mathbb{N})^{-\frac{p_2+1}{2}\gamma}\leq(2\mathbb{N})^{-\frac{p_2+1}{2}\beta}$ since $\gamma\geq\alpha,\gamma\geq\beta$, as well as $(2\mathbb{N})^{-\frac{p_1+\rho}{2}\alpha}\leq(2\mathbb{N})^{-\frac{p_1\alpha}{2}}$ because $\rho\in[0,1]$.

2° Let now $F\in X\otimes(S)_{\rho,p_1}$ and $G\in X\otimes(S)_{\rho,p_2}$ for all $p_1,p_2\in\mathbb{N}_0$. Then the chaos expansion form of $F\lozenge G$ is given by (4) and

$$\begin{aligned}\|F\lozenge G\|^2_{X\otimes(S)_{\rho,q}} &= \sum_{\gamma\in\mathcal{I}}\gamma!^{1+\rho}\|\sum_{\alpha+\beta=\gamma}f_\alpha g_\beta\|_X^2(2\mathbb{N})^{q\gamma}=\sum_{\gamma\in\mathcal{I}}(2\mathbb{N})^{-2\gamma}\|\sum_{\alpha+\beta=\gamma}\gamma!^{\frac{1+\rho}{2}}f_\alpha g_\beta(2\mathbb{N})^{\frac{q+2}{2}\gamma}\|_X^2\\ &\leq \sum_{\gamma\in\mathcal{I}}(2\mathbb{N})^{-2\gamma}\|\sum_{\alpha+\beta=\gamma}\alpha!^{\frac{1+\rho}{2}}\beta!^{\frac{1+\rho}{2}}(2\mathbb{N})^{\frac{1+\rho}{2}(\alpha+\beta)}f_\alpha g_\beta(2\mathbb{N})^{\frac{q+2}{2}(\alpha+\beta)}\|_X^2\\ &\leq M\left(\sum_{\alpha+\beta=\gamma}\alpha!^{1+\rho}\|f_\alpha\|_X^2(2\mathbb{N})^{p_1\alpha}\right)\left(\sum_{\alpha+\beta=\gamma}\beta!^{1+\rho}\|g_\beta\|_X^2(2\mathbb{N})^{p_2\beta}\right)\\ &\leq M\left(\sum_{\alpha\in\mathcal{I}}\alpha!^{1+\rho}\|f_\alpha\|_X^2(2\mathbb{N})^{(q+4))\alpha}\right)\left(\sum_{\beta\in\mathcal{I}}\beta!^{1+\rho}\|g_\beta\|_X^2(2\mathbb{N})^{(q+4)\beta}\right)\\ &= M\cdot\|F\|^2_{X\otimes(S)_{\rho,p_1}}\cdot\|G\|^2_{X\otimes(S)_{\rho,p_2}}<\infty,\end{aligned}$$

if $q\leq p_1-4$ and $q\leq p_2-4$. We used the Cauchy-Schwartz inequality along with the estimate $(\alpha+\beta)!\leq\alpha!\,\beta!\,(2\mathbb{N})^{\alpha+\beta}$, from Lemma 2.1. □

Remark 2.10. *A test stochastic process $u\in X\otimes(S)_{\rho,p}$, $p\geq 0$ can be considered as a generalized stochastic process from $X\otimes(S)_{-\rho,-q}$, $q\geq 0$ since $\|u\|^2_{X\otimes(S)_{-\rho,-q}}\leq\|u\|^2_{X\otimes(S)_{\rho,p}}$. Therefore, if $F\in X\otimes(S)_{\rho,p_1}$ and $G\in X\otimes(S)_{-\rho,-p_2}$ for some $p_1,p_2\in\mathbb{N}_0$, then $F\lozenge G$ is a well defined element in $X\otimes(S)_{-\rho,-q}$, for $q\geq p_2+4$. This follows from Theorem 2.9 part 1° by letting $p_1=0$.*

Applying the well-known formula for the Fourier-Hermite polynomials (see [4])

$$H_\alpha\cdot H_\beta=\sum_{\gamma\leq\min\{\alpha,\beta\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}H_{\alpha+\beta-2\gamma} \tag{5}$$

one can define the ordinary product $F\cdot G$ of two stochastic processes F and G. Thus, by applying formally (5) we obtain

$$\begin{aligned}F\cdot G &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}f_\alpha\,g_\beta\otimes H_\alpha\cdot H_\beta=\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}f_\alpha\,g_\beta\otimes\sum_{\mathbf{0}\leq\gamma\leq\min\{\alpha,\beta\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}H_{\alpha+\beta-2\gamma}\\ &= F\lozenge G+\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}f_\alpha\,g_\beta\otimes\sum_{\mathbf{0}<\gamma\leq\min\{\alpha,\beta\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}H_{\alpha+\beta-2\gamma}.\end{aligned}$$

After a change of variables $\delta=\alpha-\gamma$, $\theta=\beta-\gamma$, we obtain $H_\alpha\cdot H_\beta=\sum_{\substack{\gamma,\delta,\theta\\ \gamma+\theta=\beta,\gamma+\delta=\alpha}}\frac{\alpha!\beta!}{\gamma!\delta!\theta!}H_{\delta+\theta}$.

$$H_\alpha \cdot H_\beta = \sum_{\substack{\mathbf{0}\le\tau<\delta+\beta \\ \gamma+\tau=\delta+\beta,\gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} H_\tau = \sum_{\substack{\mathbf{0}\le\tau<\delta+\beta \\ \alpha+\tau=\beta+2\delta}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} H_\tau, \qquad \alpha,\beta\in\mathcal{I}$$

After another change of variables $\tau = \delta+\theta$ we finally obtain the chaos expansion of $H_\alpha \cdot H_\beta$ in $(L)^2$:

$$H_\alpha \cdot H_\beta = \sum_{\tau\in\mathcal{I}} \sum_{\substack{\gamma\in\mathcal{I},\delta\le\tau \\ \gamma+\tau-\delta=\beta,\gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!}\ H_\tau = H_{\alpha+\beta} + \sum_{\tau\in\mathcal{I}} \sum_{\substack{\gamma>\mathbf{0},\delta\le\tau \\ \gamma+\tau-\delta=\beta,\gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!}\ H_\tau.$$

Similarly, we can rearrange the sums for $F\cdot G$ to obtain

$$F\cdot G = F\lozenge G + \sum_{\tau\in\mathcal{I}}\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha g_\beta \sum_{\substack{\gamma>\mathbf{0},\delta\le\tau \\ \gamma+\tau-\delta=\beta,\gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!}\ H_\tau = \sum_{\tau\in\mathcal{I}}\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha g_\beta\, a_{\alpha,\beta,\tau}\, H_\tau, \tag{6}$$

where

$$a_{\alpha,\beta,\tau} = \sum_{\substack{\gamma\in\mathcal{I},\delta\le\tau, \\ \gamma+\tau-\delta=\beta,\gamma+\delta=\alpha}} \frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!} \tag{7}$$

Note the following facts: for each $\alpha,\beta,\tau\in\mathcal{I}$ fixed there exists a unique pair of multi-indices $\gamma,\delta\in\mathcal{I}$ such that $\delta\le\tau$ and $\gamma+\tau-\delta=\beta, \gamma+\delta=\alpha$. Moreover, both $\alpha+\beta$ and $|\alpha-\beta|$ are odd (resp. even) if and only if τ is odd (resp. even). Also, $\alpha+\beta\ge\tau\ge|\alpha-\beta|$. Thus,

$$a_{\alpha,\beta,\tau} = \frac{\alpha!\beta!}{(\frac{\alpha+\beta-\tau}{2})!(\frac{\alpha-\beta+\tau}{2})!(\frac{\beta-\alpha+\tau}{2})!}.$$

For example, if $\tau=(2,0,0,0,\ldots)$, then the coefficient next to H_τ in (6) is $f_{(0,0,0,\ldots)}g_{(2,0,0,\ldots)} + f_{(1,0,0,\ldots)}g_{(1,0,0,\ldots)} + f_{(2,0,0,\ldots)}g_{(0,0,0,\ldots)} + 3f_{(1,0,0,\ldots)}g_{(3,0,0,\ldots)} + 4f_{(2,0,0,\ldots)}g_{(2,0,0,\ldots)} + 3f_{(3,0,0,\ldots)}g_{(1,0,0,\ldots)} + 18f_{(3,0,0,\ldots)}g_{(3,0,0,\ldots)} + \cdots$.

Lemma 2.11. *Let $\alpha,\beta,\tau\in\mathcal{I}$ and $a_{\alpha,\beta,\tau}$ be defined as in* (7). *Then*

$$a_{\alpha,\beta,\tau} \le (2\mathbb{N})^{\alpha+\beta}.$$

Proof. From the estimate $\alpha! = \frac{(2\alpha)!}{2^{|\alpha|}} \ge \frac{(2\alpha)!}{(2\mathbb{N})^\alpha}$, which follows from Lemma 2.1 part 1°, we obtain

$$a_{\alpha,\beta,\tau} = \frac{\alpha!\beta!}{(\frac{\alpha+\beta-\tau}{2})!(\frac{\alpha-\beta+\tau}{2})!(\frac{\beta-\alpha+\tau}{2})!} \le \frac{\alpha!\beta!}{(\alpha+\beta-\tau)!(\alpha-\beta+\tau)!(\beta-\alpha+\tau)!(2\mathbb{N})^{-(\alpha+\beta-\tau)}}.$$

Without loss of generality we may assume that $\alpha\le\beta$. The case $\beta\le\alpha$ can be considered similarly.

First case, if $\alpha\le\beta\le\tau$. Then, $\beta\le\tau$ implies that $\frac{\alpha!}{(\alpha-\beta+\tau)!}\le 1$, while $\alpha\le\tau$ implies that $\frac{\beta!}{(\beta-\alpha+\tau)!}\le 1$. Thus

$$a_{\alpha,\beta,\tau} \le \frac{(2\mathbb{N})^{\alpha+\beta-\tau}}{(\alpha+\beta-\tau)!} \le (2\mathbb{N})^{\alpha+\beta}.$$

Second case, if $\alpha\le\tau\le\beta$. Then, $\alpha\le\tau$ implies again $\frac{\beta!}{(\beta-\alpha+\tau)!}\le 1$, while $\tau\le\beta$ now implies that $\frac{\alpha!}{(\alpha+\beta-\tau)!}\le 1$. Thus,

$$a_{\alpha,\beta,\tau} \le \frac{(2\mathbb{N})^{\alpha+\beta-\tau}}{(\alpha-\beta+\tau)!} \le (2\mathbb{N})^{\alpha+\beta}.$$

Third case, if $\tau \leq \alpha \leq \beta$. Then $\beta - \alpha + \tau \leq \beta$ and $\alpha - \beta + \tau \leq \tau$. Thus, we obtain

$$\begin{aligned}
a_{\alpha,\beta,\tau} &\leq \prod_{i\in\mathbb{N}} \frac{\alpha_i!\ \beta_i!}{(\alpha_i+\beta_i-\tau_i)!\ (\alpha_i-\beta_i+\tau_i)!\ (\beta_i-\alpha_i+\tau_i)!\ (2i)^{-(\alpha_i+\beta_i-\tau_i)}} \\
&= \prod_{i\in\mathbb{N}} \frac{(\alpha_i-\beta_i+\tau_i)!\cdot(\alpha_i-\beta_i+\tau_i+1)\ldots(\alpha_i-1)\cdot\alpha_i\ (\beta_i-\alpha_i+\tau_i)!\cdot(\beta_i-\alpha_i+\tau_i+1)\ldots(\beta_i-1)\cdot\beta_i}{(\alpha_i-\beta_i+\tau_i)!\,(\beta_i-\alpha_i+\tau_i)!\,(\alpha_i+\beta_i-\tau_i)!\,(2i)^{-(\alpha_i+\beta_i-\tau_i)}} \\
&\leq 1\cdot(2\mathbb{N})^{\alpha+\beta-\tau} \leq (2\mathbb{N})^{\alpha+\beta}
\end{aligned}$$

□

Theorem 2.12. *The following holds:*

1° *If $F \in X \otimes (S)_{\rho,r_1}$ and $G \in X \otimes (S)_{\rho,r_2}$, for some $r_1, r_2 \in \mathbb{N}_0$, then the ordinary product $F \cdot G$ is a well defined element in $X \otimes (S)_{\rho,q}$ for $q \leq \min\{r_1, r_2\} - 8$.*

2° *If $F \in X \otimes (S)_{\rho,r_1}$ and $G \in X \otimes (S)_{-\rho,-r_2}$, for $r_1 - r_2 > 8$, then their ordinary product $F \cdot G$ is well defined and belongs to $X \otimes (S)_{-\rho,-q}$ for $r_2 \leq q \leq r_1 - 8$.*

Proof. 1° Let $q = p - 8$, where $p \leq \min\{p_1, p_2\} - 8$. By Lemma 2.11, Lemma 2.1 and the Cauchy-Schwartz inequality we have

$$\begin{aligned}
\|F\cdot G\|^2_{X\otimes(S)_{\rho,q}} &= \sum_{\tau\in\mathcal{I}} \tau!^{1+\rho} \| \sum_{\alpha,\beta\in\mathcal{I}} f_\alpha g_\beta a_{\alpha,\beta,\tau} \|^2_X (2\mathbb{N})^{q\tau} \\
&\leq \sum_{\tau\in\mathcal{I}} \tau!^{1+\rho} \| \sum_{\substack{\alpha,\beta\in\mathcal{I}\\ \tau\leq\alpha+\beta}} f_\alpha g_\beta (2\mathbb{N})^{\alpha+\beta} \|^2_X (2\mathbb{N})^{(p-8)\tau} \\
&= \sum_{\tau\in\mathcal{I}} (2\mathbb{N})^{-2\tau} \| \sum_{\substack{\alpha,\beta\in\mathcal{I}\\ \tau\leq\alpha+\beta}} f_\alpha g_\beta\, \tau!^{\frac{1+\rho}{2}}\, (2\mathbb{N})^{\alpha+\beta}\, (2\mathbb{N})^{\frac{p-6}{2}\tau} \|^2_X \\
&\leq \sum_{\tau\in\mathcal{I}} (2\mathbb{N})^{-2\tau} \| \sum_{\substack{\alpha,\beta\in\mathcal{I}\\ \tau\leq\alpha+\beta}} f_\alpha g_\beta\, \alpha!^{\frac{1+\rho}{2}}\, \beta!^{\frac{1+\rho}{2}}\, (2\mathbb{N})^{\frac{1+\rho}{2}}\, (2\mathbb{N})^{\alpha+\beta}\, (2\mathbb{N})^{\frac{p-6}{2}(\alpha+\beta)} \|^2_X \\
&\leq \sum_{\tau\in\mathcal{I}} (2\mathbb{N})^{-2\tau} \| \sum_{\alpha,\beta\in\mathcal{I}} \alpha!^{\frac{1+\rho}{2}} f_\alpha (2\mathbb{N})^{\frac{p\alpha}{2}}\, (2\mathbb{N})^{-\beta} \beta!^{\frac{1+\rho}{2}}\, g_\beta\, (2\mathbb{N})^{\frac{p\beta}{2}}\, (2\mathbb{N})^{-\alpha} \|^2_X \\
&= \sum_{\tau\in\mathcal{I}} (2\mathbb{N})^{-2\tau} \Big(\sum_{\alpha,\beta\in\mathcal{I}} \alpha!^{1+\rho} \|f_\alpha\|^2_X (2\mathbb{N})^{p\alpha}\, (2\mathbb{N})^{-2\beta} \sum_{\alpha,\beta\in\mathcal{I}} \beta!^{1+\rho}\, \|g_\beta\|^2_X\, (2\mathbb{N})^{p\beta}\, (2\mathbb{N})^{-2\alpha} \Big) \\
&\leq \sum_{\tau\in\mathcal{I}} (2\mathbb{N})^{-2\tau} \Big(\sum_{\beta\in\mathcal{I}} (2\mathbb{N})^{-2\beta} \sum_{\alpha\in\mathcal{I}} \alpha!^{1+\rho} \|f_\alpha\|^2_X (2\mathbb{N})^{p\alpha} \Big) \Big(\sum_{\alpha\in\mathcal{I}} (2\mathbb{N})^{-2\alpha} \sum_{\beta\in\mathcal{I}} \beta!^{1+\rho}\, \|g_\beta\|^2_X\, (2\mathbb{N})^{p\beta} \Big) \\
&\leq M\, C_1 C_2 \sum_{\alpha\in\mathcal{I}} \alpha!^2 \|f_\alpha\|^2_X (2\mathbb{N})^{p\alpha} \sum_{\beta\in\mathcal{I}} \alpha!^2 \|g_\beta\|^2_X (2\mathbb{N})^{p\beta} \\
&= M\, C_1 C_2 \|F\|^2_{X\otimes(S)_{\rho,p}} \|G\|^2_{X\otimes(S)_{\rho,p}} < \infty,
\end{aligned}$$

where $M = \sum_{\tau\in\mathcal{I}} (2\mathbb{N})^{-2\tau} < \infty$, $C_1 = \sum_{\beta\in\mathcal{I}} (2\mathbb{N})^{-2\beta} < \infty$ and $C_2 = \sum_{\alpha\in\mathcal{I}} (2\mathbb{N})^{-2\alpha} < \infty$.

2° Let $\varphi \in (S)_{\rho,q}$ and $F \in X\otimes(S)_{\rho,r_1}$. Then by Theorem 2.12 part 1°, $F\cdot\varphi \in (S)_{\rho,s}$ for $s \leq \min\{r_1, q\} - 8 = r_1 - 8$. Also, $G \in (S)_{-\rho,r_2}$ implies that $G \in (S)_{-\rho,-c}$ for $c \geq r_2$. Thus for any c such that $r_2 \leq c \leq s \leq r_1 - 8$ we have $F\cdot\varphi \in (S)_{\rho,c}$ and $G \in (S)_{-\rho,-c}$. Now,

$$\begin{aligned}
\|F\cdot G\|^2_{-\rho,-q} &= \sup_{\|\varphi\|_q\leq 1} |\ll F\cdot G, \varphi \gg_\rho| = \sup_{\|\varphi\|_q\leq 1} |\ll G, F\cdot\varphi \gg_\rho| \\
&\leq \sup_{\|\varphi\|_q\leq 1} \|G\|_{-\rho,-c}\cdot\|F\cdot\varphi\|_{\rho,c} \leq \sup_{\|\varphi\|_q\leq 1} \|G\|_{-\rho,-c}\cdot\|F\|_{\rho,r_1}\cdot\|\varphi\|_{\rho,q}.
\end{aligned}$$

This implies

$$\|F \cdot G\|^2_{-\rho,-q} \leq M \cdot \|G\|_{-\rho,-r_2} \cdot \|F\|_{\rho,r_1},$$

for some $M > 0$. □

Remark 2.13. *Note, for $F, G \in X \otimes (L)^2$ the ordinary product $F \cdot G$ will not necessarily belong to $X \otimes (L)^2$ (for a counterexample see [11]), but due to the Hölder inequality it will belong to $X \otimes (L)^1$.*

3. Operators of the Malliavin Calculus

In the classical literature [2, 12, 13, 15] the Malliavin derivative and the Skorokhod integral are defined on a subspace of $(L)^2$ so that the resulting process after application of these operators necessarily remains in $(L)^2$. We will recall of these classical results and denote the corresponding domains with a ″zero″ in order to retain a nice symmetry between test and generalized processes. In [6, 7, 9, 10] we allowed values in the Kondratiev space $(S)_{-1}$ and thus obtained larger domains for all operators. These domains will be denoted by a ″minus″ sign to reflect the fact that they correspond to generalized processes. In this paper we introduce also domains for test processes. These domains will be denoted by a ″plus″ sign.

Definition 3.1. *Let a generalized stochastic process $u \in X \otimes (S)_{-\rho}$ be of the form $u = \sum_{\alpha \in \mathcal{I}} u_\alpha \otimes H_\alpha$. If there exists $p \in \mathbb{N}_0$ such that*

$$\sum_{\alpha \in \mathcal{I}} |\alpha|^{1+\rho} \, \alpha!^{1-\rho} \, \|u_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty, \tag{8}$$

then the Malliavin derivative *of u is defined by*

$$\mathbb{D}u = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_k \, u_\alpha \otimes \xi_k \otimes H_{\alpha-\varepsilon^{(k)}} = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} (\alpha_k + 1) \, u_{\alpha+\varepsilon^{(k)}} \otimes \xi_k \otimes H_\alpha, \tag{9}$$

where by convention $\alpha - \varepsilon^{(k)}$ does not exist if $\alpha_k = 0$, i.e. $H_{\alpha-\varepsilon^{(k)}} = \begin{cases} 0, & \alpha_k = 0 \\ H_{(\alpha_1,\alpha_2,\ldots,\alpha_{k-1},\alpha_k-1,\alpha_{k+1},\ldots,\alpha_m,0,0,\ldots)}, & \alpha_k \geq 1 \end{cases}$, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_m, 0, 0, \ldots) \in \mathcal{I}$.

For two processes $u = \sum_{\alpha \in \mathcal{I}} u_\alpha \otimes H_\alpha$, $v = \sum_{\alpha \in \mathcal{I}} v_\alpha \otimes H_\alpha$ and constants a, b the linearity property holds, i.e. $\mathbb{D}(au + bv) = a\mathbb{D}(u) + b\mathbb{D}(v)$. The set of generalized stochastic processes $u \in X \otimes (S)_{-\rho}$ which satisfy (8) constitutes the domain of the Malliavin derivative, denoted by $Dom^\rho_-(\mathbb{D})$. Thus the domain of the Malliavin derivative is given by

$$Dom^\rho_-(\mathbb{D}) = \bigcup_{p \in \mathbb{N}_0} Dom^\rho_{-p}(\mathbb{D}) = \bigcup_{p \in \mathbb{N}_0} \left\{ u \in X \otimes (S)_{-\rho} : \sum_{\alpha \in \mathcal{I}} |\alpha|^{1+\rho} \, \alpha!^{1-\rho} \, \|u_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty \right\}.$$

A process $u \in Dom^\rho_-(\mathbb{D})$ is called a Malliavin differentiable process.

Theorem 3.2. *The Malliavin derivative of a process $u \in X \otimes (S)_{-\rho}$ is a linear and continuous mapping*

$$\mathbb{D} : \quad Dom^\rho_{-p}(\mathbb{D}) \to X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-\rho,-p},$$

for $l > p + 1$ and $p \in \mathbb{N}_0$.

Proof. Let $u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha \in Dom_-^\rho(\mathbb{D})$. Then,

$$\begin{aligned}
\|\mathbb{D}u\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-\rho,-p}} &= \sum_{\alpha\in\mathcal{I}} \Big(\sum_{k\in\mathbb{N}} (\alpha_k+1)^2\, \|u_{\alpha+\varepsilon^{(k)}}\|^2_X\,(2k)^{-l}\Big)\,\alpha!^{1-\rho}(2\mathbb{N})^{-p\alpha}\\
&= \sum_{|\beta|\geq 1} \Big(\sum_{k\in\mathbb{N}} \beta_k^2\,\|u_\beta\|^2_X\,(2k)^{-l}\left(\frac{\beta!}{\beta_k}\right)^{1-\rho}(2k)^p\Big)(2\mathbb{N})^{-p\beta}\\
&= \sum_{|\beta|\geq 1} \Big(\sum_{k\in\mathbb{N}} \beta_k^{1+\rho}\,(2k)^{-(l-p)}\Big)\|u_\beta\|^2_X\,(\beta!)^{1-\rho}\,(2\mathbb{N})^{-p\beta}\\
&\leq \sum_{\beta\in\mathcal{I}} \Big(\sum_{k=1}^\infty \beta_k\Big)^{1+\rho}\Big(\sum_{k=1}^\infty (2k)^{-(l-p)}\Big)\|u_\beta\|^2_X\,(\beta!)^{1-\rho}(2\mathbb{N})^{-p\beta}\\
&= c\sum_{\beta\in\mathcal{I}} |\beta|^{1+\rho}(\beta!)^{1-\rho}\|u_\beta\|^2_X\,(2\mathbb{N})^{-p\beta} = c\|u\|^2_{Dom^\rho_{-p}(\mathbb{D})} < \infty,
\end{aligned}$$

where $c = \sum_{k\in\mathbb{N}}(2k)^{-(l-p)} < \infty$ for $l-p > 1$ and where we used $(\alpha-\varepsilon^{(k)})! = \frac{\alpha!}{\alpha_k}$, $\alpha_k > 0$ and the estimate $\sum_{k\in\mathbb{N}}\alpha_k^{1+\rho} \leq (\sum_{k\in\mathbb{N}}\alpha_k)^{1+\rho} = |\alpha|^{1+\rho}$. □

For all $\alpha\in\mathcal{I}$ we have $|\alpha| < \alpha!$. Thus, the smallest domain of the spaces $Dom_-^\rho(\mathbb{D})$ is obtained for $\rho = 0$ and the largest is obtained for $\rho = 1$. In particular we have $Dom_-^0(\mathbb{D}) \subset Dom_-^1(\mathbb{D})$. Moreover if $p \leq q$ then $Dom^\rho_{-p}(\mathbb{D}) \subseteq Dom^\rho_{-q}(\mathbb{D})$.

For square integrable stochastic process $u \in X\otimes(L)^2$ the domain is given by

$$Dom_0(\mathbb{D}) = \left\{u\in X\otimes(L)^2 : \sum_{\alpha\in\mathcal{I}} |\alpha|\,\alpha!\,\|u_\alpha\|^2_X < \infty\right\}.$$

Theorem 3.3. *The Malliavin derivative of a process $u\in Dom_0(\mathbb{D})$ is a linear and continuous mapping*

$$\mathbb{D}:\ Dom_0(\mathbb{D}) \to X\otimes L^2(\mathbb{R})\otimes(L)^2.$$

Proof. Let $u\in Dom_0(\mathbb{D})$, i.e. $\sum_{\alpha\in\mathcal{I}}|\alpha|\alpha!\|u_\alpha\|^2_X < \infty$. Then,

$$\|\mathbb{D}u\|^2_{X\otimes L^2(\mathbb{R})\otimes(L)^2} = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k^2\,(\alpha-\varepsilon^{(k)})!\ \|u_\alpha\|^2_X = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k\,\alpha!\,\|u_\alpha\|^2_X = \sum_{\alpha\in\mathcal{I}} |\alpha|\,\alpha!\ \|u_\alpha\|^2_X < \infty.$$

□

In general, for $\rho\in[0,1]$ the domain of $\mathbb{D}$ in $X\otimes(S)_\rho$ is

$$Dom_+^\rho = \bigcap_{p\in\mathbb{N}_0} Dom_p^\rho(\mathbb{D}) = \bigcap_{p\in\mathbb{N}_0}\left\{u\in X\otimes(S)_\rho : \sum_{\alpha\in\mathcal{I}} |\alpha|^{1-\rho}\,(\alpha!)^{1+\rho}\,\|u_\alpha\|^2_X(2\mathbb{N})^{p\alpha} < \infty\right\}.$$

Theorem 3.4. *Let $\rho\in[0,1]$. The Malliavin derivative of a test stochastic process $v\in X\otimes(S)_\rho$ is a linear and continuous mapping*

$$\mathbb{D}:\quad Dom_p^\rho(\mathbb{D}) \to X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p},$$

for $l < p-1$ and $p\in\mathbb{N}_0$.

Proof. Let $v = \sum_{\alpha\in\mathcal{I}} v_\alpha \otimes H_\alpha \in Dom_p^\rho(\mathbb{D})$. Then, from (9) and

$$\begin{aligned}
\|\mathbb{D}v\|^2_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}} &= \sum_{\alpha\in\mathcal{I}} \|\sum_{k\in\mathbb{N}} (\alpha_k+1)\, v_{\alpha+\varepsilon^{(k)}}\xi_k\|^2_{X\otimes S_l(\mathbb{R})}\, \alpha!^{1+\rho}(2\mathbb{N})^{p\alpha} \\
&= \sum_{\alpha\in\mathcal{I}} \Big(\sum_{k\in\mathbb{N}} (\alpha_k+1)^2\, \|v_{\alpha+\varepsilon^{(k)}}\|^2_X\, (2k)^l\Big)\, \alpha!^{1+\rho}(2\mathbb{N})^{p\alpha} \\
&= \sum_{|\beta|\geq 1} \Big(\sum_{k\in\mathbb{N}} \beta_k^2\, \|v_\beta\|^2_X\, (2k)^l \left(\frac{\beta!}{\beta_k}\right)^{1+\rho} (2k)^{-p}\Big)\, (2\mathbb{N})^{p\beta} \\
&= \sum_{|\beta|\geq 1} \Big(\sum_{k\in\mathbb{N}} \beta_k^{1-\rho}\ (2k)^{-(p-l)}\Big)\, \|v_\beta\|^2_X\, \beta!^{1+\rho}\, (2\mathbb{N})^{p\beta} \\
&\leq c^{1-\rho} \sum_{\beta\in\mathcal{I}} |\beta|^{1-\rho}(\beta!)^{1+\rho}\|v_\beta\|^2_X\ (2\mathbb{N})^{p\beta} < \infty,
\end{aligned}$$

the assertion follows, where we used

$$\sum_{k\in\mathbb{N}} \beta_k^{1-\rho}(2k)^{l-p} \leq \Big(\sum_{k\in\mathbb{N}} \beta_k\Big)^{1-\rho}\Big(\sum_{k\in\mathbb{N}}(2k)^{\frac{l-p}{1-\rho}}\Big)^{1-\rho} \leq |\beta|^{1-\rho}\cdot c^{1-\rho},$$

and $c = \sum_{k\in\mathbb{N}} (2k)^{\frac{l-p}{1-\rho}} \leq \sum_{k\in\mathbb{N}} (2k)^{l-p} < \infty$, for $p > l+1$. We also used $\beta_k\,(\beta-\varepsilon^{(k)})! = \beta!$, $\beta\in\mathcal{I}$ and $(2\mathbb{N})^{\varepsilon^{(k)}} = (2k)$, $k\in\mathbb{N}$. □

Note that $Dom_p^\rho(\mathbb{D}) \subseteq Dom_0(\mathbb{D}) \subseteq Dom_{-p}^\rho(\mathbb{D})$ for all $p\in\mathbb{N}$. Therefore, $Dom_+^\rho(\mathbb{D}) \subseteq Dom_0(\mathbb{D}) \subseteq Dom_-^\rho(\mathbb{D})$. Moreover, using the estimate $|\alpha| \leq (2\mathbb{N})^\alpha$ it follows that

$$X\otimes(S)_{-\rho,-(p-2)} \subseteq Dom_{-p}^\rho(\mathbb{D}) \subseteq X\otimes(S)_{-\rho,-p}, \quad p>3, \quad \text{and}$$

$$X\otimes(S)_{\rho,p+1} \subseteq Dom_p^\rho(\mathbb{D}) \subseteq X\otimes(S)_{\rho,p}, \quad p>0.$$

Remark 3.5. *For $u \in Dom_+^\rho(\mathbb{D})$ and $u\in Dom_0(\mathbb{D})$ it is usual to write*

$$\mathbb{D}_t u = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k\, u_\alpha\ \otimes\ \xi_k(t)\ \otimes H_{\alpha-\varepsilon^{(k)}},$$

in order to emphasise that the Malliavin derivative takes a random variable into a process, i.e. that $\mathbb{D}u$ is a function of t. Moreover, the formula

$$\mathbb{D}_t F(\omega) = \lim_{h\to 0}\frac{1}{h}\Big(F(\omega + h\cdot\kappa_{[t,\infty)}) - F(\omega)\Big), \quad \omega\in S'(\mathbb{R}),$$

justifies the name stochastic derivative *for the Malliavin operator. Since generalized functions do not have point values, this notation would be somewhat misleading for $u\in Dom_-^\rho(\mathbb{D})$. Therefore, for notational uniformity, we omit the index t in $\mathbb{D}_t$ that usually appears in the literature and write $\mathbb{D}$.*

The Skorokhod integral, as an extension of the Itô integral for non-adapted processes, can be regarded as the adjoint operator of the Malliavin derivative in $(L)^2$-sense. In [6] we have extended the definition of the Skorokhod integral from Hilbert space valued processes to the class of S'-valued generalized processes.

Definition 3.6. *Let $F = \sum_{\alpha\in\mathcal{I}} f_\alpha\otimes H_\alpha \in X\otimes S'(\mathbb{R})\otimes(S)_{-\rho}$, be a generalized $S'(\mathbb{R})$-valued stochastic process and let $f_\alpha \in X\otimes S'(\mathbb{R})$ be given by the expansion $f_\alpha = \sum_{k\in\mathbb{N}} f_{\alpha,k}\otimes\xi_k$, $f_{\alpha,k}\in X$. If there exist $p\geq 0$, $l\geq 0$ such that*

$$\sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \Big(\alpha!\,(\alpha_k+1)\Big)^{1-\rho} \|f_{\alpha,k}\|^2_X\,(2k)^{-l}\,(2\mathbb{N})^{-p\alpha} < \infty,$$

then the Skorokhod integral of F is given by

$$\delta(F) = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} f_{\alpha,k}\otimes H_{\alpha+\varepsilon^{(k)}} = \sum_{\alpha>\mathbf{0}}\sum_{k\in\mathbb{N}} f_{\alpha-\varepsilon^{(k)},k}\otimes H_\alpha. \tag{10}$$

A linear combination of two Skorokhod integrable processes F, G is again Skorokhod integrable process $aF + bG$, $a, b \in \mathbb{R}$ such that $\delta(aF + bG) = a\delta(F) + b\delta(G)$.

In general, the domain $Dom_-^\rho(\delta)$ of the Skorokhod integral is

$$Dom_-^\rho(\delta) = \bigcup_{\substack{(l,p)\in\mathbb{N}^2\\ p>l+1}} Dom^\rho_{(-l,-p)}(\delta) = \bigcup_{\substack{(l,p)\in\mathbb{N}^2\\ p>l+1}} \left\{F\in X\otimes S'(\mathbb{R})\otimes(S)_{-\rho} : \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\Big(\alpha!\,(\alpha_k+1)\Big)^{1-\rho}\|f_{\alpha,k}\|_X^2\,(2k)^{-l}\,(2\mathbb{N})^{-p\alpha}<\infty\right\}.$$

Theorem 3.7. *Let $\rho \in [0, 1]$. The Skorokhod integral δ of a $S_{-l}(\mathbb{R})$-valued stochastic process is a linear and continuous mapping*

$$\delta : Dom^\rho_{(-l,-p)}(\delta) \to X\otimes(S)_{-\rho,-p}, \quad p > l + 1.$$

Proof. This statement follows from

$$\begin{aligned}
\|\delta(F)\|^2_{X\otimes(S)_{-\rho,-p}} &= \sum_{|\alpha|\ge 1}\alpha!^{1-\rho}\|\sum_{k\in\mathbb{N}} f_{\alpha-\varepsilon^{(k)},k}\|_X^2\,(2\mathbb{N})^{-p\alpha} = \sum_{|\alpha|\ge1}\|\sum_{k\in\mathbb{N}}\alpha!^{\frac{1-\rho}{2}} f_{\alpha-\varepsilon^{(k)},k}\|_X^2\,(2\mathbb{N})^{-p\alpha}\\
&= \sum_{\beta\in\mathcal{I}}\|\sum_{k\in\mathbb{N}}(\beta+\varepsilon^{(k)})!^{\frac{1-\rho}{2}} f_{\beta,k}\,(2k)^{-\frac{p}{2}}\|_X^2\,(2\mathbb{N})^{-p\beta}\\
&= \sum_{\beta\in\mathcal{I}}\|\sum_{k\in\mathbb{N}}(\beta+\varepsilon^{(k)})!^{\frac{1-\rho}{2}} f_{\beta,k}\,(2k)^{-\frac{l}{2}}(2k)^{-\frac{p-l}{2}}\|_X^2\,(2\mathbb{N})^{-p\beta}\\
&\le \sum_{\beta\in\mathcal{I}}\Big(\sum_{k\in\mathbb{N}}(\beta+\varepsilon^{(k)})!^{1-\rho}\|f_{\beta,k}\|_X^2\,(2k)^{-l}\sum_{k\in\mathbb{N}}(2k)^{-(p-l)}\Big)(2\mathbb{N})^{-p\beta}\\
&\le c\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}(\beta!\,(\beta_k+1))^{1-\rho}\|f_{\beta,k}\|_X^2\,(2k)^{-l}\,(2\mathbb{N})^{-p\beta} = c\,\|F\|^2_{Dom^\rho_{(-l,-p)}(\delta)} < \infty,
\end{aligned}$$

where $c = \sum_{k\in\mathbb{N}}(2k)^{-(p-l)} < \infty$ for $p > l + 1$. □

Note that for $\rho = 1$ it holds that $Dom_-^1(\delta) = X\otimes S'(\mathbb{R})\otimes(S)_{-1}$.

Now we characterize the domains $Dom_+^\rho(\delta)$ and $Dom_0(\delta)$ of the Skorokhod integral for test processes from $X\otimes S(\mathbb{R})\otimes(S)_\rho$ and square integrable processes from $X\otimes L^2(\mathbb{R})\otimes(L)^2$. The form of the derivative is in all cases given by the expression (10).

For square integrable stochastic processes $T\in X\otimes L^2(\mathbb{R})\otimes(L)^2$ of the form $T = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} t_{\alpha,k}\otimes\xi_k\otimes H_\alpha$, $t_{\alpha,k}\in X$, we define

$$Dom_0(\delta) = \left\{T\in X\otimes L^2(\mathbb{R})\otimes(L)^2 : \sum_{\alpha\in\mathcal{I}}\Big(\sum_{k\in\mathbb{N}}(\alpha_k+1)^{\frac12}\alpha!^{\frac12}\|t_{\alpha,k}\|_X\Big)^2<\infty\right\}.$$

Theorem 3.8. *The Skorokhod integral δ of an $L^2(\mathbb{R})$-valued stochastic process is a linear and continuous mapping*

$$\delta : \quad Dom_0(\delta) \to X\otimes(L)^2.$$

Proof. Let $T = \sum\limits_{\alpha\in\mathcal{I}} \sum\limits_{k\in\mathbb{N}} t_{\alpha,k} \otimes \xi_k \otimes H_\alpha \in Dom_0(\delta)$. Then,

$$\begin{aligned}\|\delta(T)\|^2_{X\otimes(L)^2} &= \sum_{|\alpha|\geq 1} \|\sum_{k\in\mathbb{N}} t_{\alpha-\varepsilon^{(k)},k}\|^2_X \, \alpha! = \sum_{|\alpha|\geq 1} \|\sum_{k\in\mathbb{N}} \alpha!^{\frac{1}{2}} \, t_{\alpha-\varepsilon^{(k)},k}\|^2_X \\ &= \sum_{\beta\in\mathcal{I}} \|\sum_{k\in\mathbb{N}} (\beta+\varepsilon^{(k)})!^{\frac{1}{2}} \, t_{\beta,k}\|^2_X \leq \sum_{\beta\in\mathcal{I}} \Big(\sum_{k\in\mathbb{N}} (\beta+\varepsilon^{(k)})!^{\frac{1}{2}} \, \|t_{\beta,k}\|_X\Big)^2 \\ &= \sum_{\beta\in\mathcal{I}} \Big(\sum_{k\in\mathbb{N}} \beta!^{\frac{1}{2}} (\beta_k+1)^{\frac{1}{2}} \, \|t_{\beta,k}\|_X\Big)^2 = \|T\|^2_{Dom_0(\delta)} < \infty.\end{aligned}$$

□

In general, for any $\rho\in[0,1]$, the domain $Dom^\rho_+(\delta)$ of the Skorokhod integral in $X\otimes S(\mathbb{R})\otimes(S)_\rho$ is

$$Dom^\rho_+(\delta) = \bigcap_{\substack{(l,p)\in\mathbb{N}^2\\ l>p+1}} Dom^\rho_{(l,p)}(\delta) = \bigcap_{\substack{(l,p)\in\mathbb{N}^2\\ l>p+1}} \left\{F\in X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p} : \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} (\alpha_k+1)^{1+\rho}\alpha!^{1+\rho}\|f_{\alpha,k}\|^2_X (2k)^l (2\mathbb{N})^{p\alpha} < \infty\right\}.$$

Theorem 3.9. *The Skorokhod integral δ of an $S_l(\mathbb{R})$-valued stochastic test process is a linear and continuous mapping*

$$\delta : Dom^\rho_{(l,p)}(\delta) \to X\otimes(S)_{\rho,p}, \qquad l>p+1.$$

Proof. Let $U = \sum_{\alpha\in\mathcal{I}} u_\alpha\otimes H_\alpha \in Dom^\rho_{(l,p)}(\delta)$, $u_\alpha = \sum_{k=1}^\infty u_{\alpha,k}\otimes\xi_k \in X\otimes S_l(\mathbb{R})$, $u_{\alpha,k}\in X$, for $l>p+1$. Then we obtain

$$\begin{aligned}\|\delta(U)\|^2_{X\otimes(S)_{\rho,p}} &= \sum_{\beta\in\mathcal{I}} \|\sum_{k\in\mathbb{N}} (\beta+\varepsilon^{(k)})!^{\frac{1+\rho}{2}} \, u_{\beta,k} \, (2k)^{\frac{p}{2}}\|^2_X \; (2\mathbb{N})^{p\beta} \\ &\leq \sum_{\beta\in\mathcal{I}} \Big(\sum_{k\in\mathbb{N}} (\beta!(\beta_k+1))^{1+\rho} \, \|u_{\beta,k}\|^2_X \, (2k)^l \sum_{k\in\mathbb{N}} (2k)^{-(l-p)}\Big) \, (2\mathbb{N})^{p\beta} \leq c \, \|U\|^2_{Dom^\rho_{(l,p)}(\delta)} < \infty,\end{aligned}$$

where $c = \sum_{k\in\mathbb{N}} (2k)^{-(l-p)} < \infty$ for $l>p+1$. □

Using the estimates $\alpha_k+1\leq 2|\alpha|$, which holds for all $\alpha\in\mathcal{I}$ except for $\alpha=\mathbf{0}$, and $|\alpha|\leq(2\mathbb{N})^\alpha$, $\alpha\in\mathcal{I}$ we obtain

$$\begin{aligned}\sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha!^{1+\rho}\|f_{\alpha,k}\|^2_X(2k)^l(2\mathbb{N})^{p\alpha} &\leq \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} (\alpha_k+1)^{1+\rho}\alpha!^{1+\rho}\|f_{\alpha,k}\|^2_X(2k)^l(2\mathbb{N})^{p\alpha} \\ &\leq \sum_{k\in\mathbb{N}} \|f_{\mathbf{0},k}\|^2_X(2k)^l + 4\sum_{\alpha>\mathbf{0}}\sum_{k\in\mathbb{N}} |\alpha|^2\alpha!^{1+\rho}\|f_{\alpha,k}\|^2_X(2k)^l(2\mathbb{N})^{p\alpha} \\ &\leq \|f_{\mathbf{0}}\|^2_{X\otimes S_l(\mathbb{R})} + 4\sum_{\alpha>\mathbf{0}}\sum_{k\in\mathbb{N}} \alpha!^{1+\rho}\|f_{\alpha,k}\|^2_X(2k)^l(2\mathbb{N})^{(p+2)\alpha} \\ &\leq 4\|F\|^2_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p+2}}.\end{aligned}$$

Thus,

$$X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p+2} \subseteq Dom^\rho_{(l,p)}(\delta) \subseteq X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}, \quad \text{for } \; l>p+1 \; \text{ and}$$

$$X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-\rho,-(p-1)} \subseteq Dom^\rho_{(-l,-p)}(\delta) \subseteq X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-\rho,-p}, \quad \text{for } \; p>l+1.$$

The third main operator of the Malliavin calculus is the Ornstein-Uhlenbeck operator.

Definition 3.10. *The composition of the Malliavin derivative and the Skorokhod integral is denoted by $\mathcal{R} = \delta\circ\mathbb{D}$ and called the Ornstein-Uhlenbeck operator.*

Therefore, for $u \in X \otimes (S)_{-\rho}$ given in the chaos expansion form $u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha$, the Ornstein-Uhlenbeck operator is given by

$$\mathcal{R}(u) = \sum_{\alpha\in\mathcal{I}} |\alpha| u_\alpha \otimes H_\alpha. \tag{11}$$

The Orstein-Uhlenbeck operator is linear, i.e. by (11) $\mathcal{R}(au + bv) = a\mathcal{R}(u) + b\mathcal{R}(v)$, $a, b \in \mathbb{R}$ holds. Let

$$Dom_-^\rho(\mathcal{R}) = \bigcup_{p\in\mathbb{N}_0} Dom_{-p}^\rho(\mathcal{R}) = \bigcup_{p\in\mathbb{N}_0} \left\{ u \in X \otimes (S)_{-\rho} : \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \, \|u_\alpha\|_X^2 \, (\alpha!)^{1-\rho} (2\mathbb{N})^{-p\alpha} < \infty \right\}.$$

Theorem 3.11. *The operator $\mathcal{R}$ is a linear and continuous mapping*

$$\mathcal{R}: \ Dom_{-p}^\rho(\mathcal{R}) \to X \otimes (S)_{-\rho,-p}, \quad p \in \mathbb{N}_0.$$

Moreover, $Dom_-^\rho(\mathcal{R}) \subseteq Dom_-^\rho(\mathbb{D})$.

Proof. Let $v = \sum_{\alpha\in\mathcal{I}} v_\alpha \otimes H_\alpha \in Dom_{-p}^\rho(\mathcal{R})$, for some $p \in \mathbb{N}_0$. Then, from (11) it follows that

$$\|\mathcal{R}v\|^2_{X\otimes(S)_{-\rho,-p}} = \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \, \|v_\alpha\|_X^2 \, (\alpha!)^{1-\rho} \, (2\mathbb{N})^{-p\alpha} < \infty.$$

For $v \in Dom_-^\rho(\mathbb{D})$ we obtain

$$\sum_{\alpha\in\mathcal{I}} |\alpha|^{1+\rho} \, \|v_\alpha\|_X^2 \, (\alpha!)^{1-\rho} \, (2\mathbb{N})^{-p\alpha} \le \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \, \|v_\alpha\|_X^2 \, (\alpha!)^{1-\rho} \, (2\mathbb{N})^{-p\alpha},$$

and the last assertion follows. Note that for $\rho = 1$, $Dom_{-p}^1(\mathcal{R}) = Dom_{-p}^1(\mathbb{D})$. □

For square integrable processes we define

$$Dom_0(\mathcal{R}) = \left\{ w \in X \otimes (L)^2 : \ \sum_{\alpha\in\mathcal{I}} \alpha! \, |\alpha|^2 \, \|w_\alpha\|_X^2 \ < \infty \right\}.$$

Theorem 3.12. *The operator $\mathcal{R}$ is a linear and continuous operator*

$$\mathcal{R}: \quad Dom_0(\mathcal{R}) \ \to \ X \otimes (L)^2.$$

Moreover, $Dom_0(\mathcal{R}) \subseteq Dom_0(\mathbb{D})$.

Proof. Let $w = \sum_{\alpha\in\mathcal{I}} w_\alpha \otimes H_\alpha \in Dom_0(\mathcal{R})$. Then $\mathcal{R}(w) = \sum_{\alpha\in\mathcal{I}} |\alpha| \, w_\alpha \otimes H_\alpha$ and

$$\|\mathcal{R}(w)\|^2_{X\otimes(L)^2} = \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \, \|w_\alpha\|_X^2 \ = \|w\|^2_{Dom_0(\mathcal{R})} < \infty.$$

Now from $|\alpha| \le |\alpha|^2$ for $\alpha \in \mathcal{I}$ it follows that $Dom_0(\mathcal{R}) \subseteq Dom_0(\mathbb{D})$. □

For test processes, we define

$$Dom_+^\rho(\mathcal{R}) = \bigcap_{p\in\mathbb{N}_0} Dom_p^\rho(\mathcal{R}) = \bigcap_{p\in\mathbb{N}_0} \left\{ v \in X \otimes (S)_{\rho,p} : \sum_{\alpha\in\mathcal{I}} (\alpha!)^{1+\rho} \, |\alpha|^2 \, \|v_\alpha\|_X^2 \ (2\mathbb{N})^{p\alpha} \ < \infty \right\}.$$

Theorem 3.13. *The operator $\mathcal{R}$ is a linear and continuous mapping*

$$\mathcal{R}: \quad Dom_p^{\rho}(\mathcal{R}) \to X \otimes (S)_{\rho,p}, \qquad p \in \mathbb{N}.$$

Moreover, $Dom_p^{\rho}(\mathcal{R}) \subseteq Dom_p^{\rho}(\mathbb{D})$.

Proof. Let $v = \sum_{\alpha\in\mathcal{I}} v_\alpha \otimes H_\alpha \in Dom_p^{\rho}(\mathcal{R})$. Then,

$$\|\mathcal{R}v\|^2_{X\otimes(S)_{\rho,p}} = \sum_{\alpha\in\mathcal{I}} \|v_\alpha\|_X^2\, |\alpha|^{1+\rho}\, \alpha!^2\, (2\mathbb{N})^{p\alpha} = \|v\|^2_{Dom_p^{\rho}(\mathcal{R})} < \infty.$$

From

$$\sum_{\alpha\in\mathcal{I}} |\alpha|^{1-\rho}\, \alpha!^{1+\rho} \|v_\alpha\|_X^2\, (2\mathbb{N})^{p\alpha} \leq \sum_{\alpha\in\mathcal{I}} |\alpha|^2\, \alpha!^{1+\rho} \|v_\alpha\|_X^2\, (2\mathbb{N})^{p\alpha}$$

follows that $Dom_p^{\rho}(\mathcal{R}) \subseteq Dom_p^{\rho}(\mathbb{D})$. □

Note also that

$$X\otimes(S)_{\rho,p+2} \subseteq Dom_p^{\rho}(\mathcal{R}) \subseteq X\otimes(S)_{\rho,p}, \quad p\in\mathbb{N}, \qquad \text{and}$$

$$X\otimes(S)_{-\rho,-(p-2)} \subseteq Dom_{-p}^{\rho}(\mathcal{R}) \subseteq X\otimes(S)_{-\rho,-p}.$$

In [8] we have proven that the mappings $\delta : Dom_-^{\rho}(\delta) \to X\otimes(S)_{-\rho}$, $\mathcal{R} : Dom_-^{\rho}(\mathcal{R}) \to X\otimes(S)_{-\rho}$, for $\rho = 1$, are surjective on the subspace of centered random variables (random variables with zero expectation). In the next section we prove the same type of surjectivity of the mappings for $\rho \in [0,1)$ as well, i.e. that the mappings $\delta : Dom_+^{\rho}(\delta) \to X\otimes(S)_{\rho}$, $\mathcal{R} : Dom_+^{\rho}(\mathcal{R}) \to X\otimes(S)_{\rho}$, $\delta : Dom_0(\delta) \to X\otimes(L)^2$, $\mathcal{R} : Dom_0(\mathcal{R}) \to X\otimes(L)^2$ have the corresponding range of centered generalized random variables. The mappings $\mathbb{D} : Dom_-^{\rho}(\mathbb{D}) \to X\otimes S'(\mathbb{R})\otimes(S)_{-\rho}$, $\mathbb{D} : Dom_+^{\rho}(\mathbb{D}) \to X\otimes S(\mathbb{R})\otimes(S)_{\rho}$, $\mathbb{D} : Dom_0(\mathbb{D}) \to X\otimes L^2(\mathbb{R})\otimes(L)^2$ are surjective on the subspace of generalized stochastic processes satisfying a certain symmetry condition which will be discussed in detail.

4. Range of the Malliavin Operators

Theorem 4.1. *(The Ornstein-Uhlenbeck operator) Let g have zero generalized expectation. The equation*

$$\mathcal{R}u = g, \qquad Eu = \tilde{u}_0 \in X,$$

has a unique solution u represented in the form

$$u = \tilde{u}_0 + \sum_{\alpha\in\mathcal{I},|\alpha|>0} \frac{g_\alpha}{|\alpha|} \otimes H_\alpha.$$

Moreover, the following holds:

1° *If $g \in X\otimes(S)_{-\rho,-p}$, $p\in\mathbb{N}$, then $u \in Dom_{-p}^{\rho}(\mathcal{R})$.*

2° *If $g \in X\otimes(S)_{\rho,p}$, $p\in\mathbb{N}$, then $u \in Dom_{p}^{\rho}(\mathcal{R})$.*

3° *If $g \in X\otimes(L)^2$, then $u \in Dom_0(\mathcal{R})$.*

Proof. Let us seek for a solution in form of $u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha$. From $\mathcal{R}u = g$ it follows that

$$\sum_{\alpha\in\mathcal{I}} |\alpha| u_\alpha \otimes H_\alpha = \sum_{\alpha\in\mathcal{I}} g_\alpha \otimes H_\alpha,$$

i.e., $u_\alpha = \frac{g_\alpha}{|\alpha|}$ for all $\alpha \in \mathcal{I}$, $|\alpha| > 0$. From the initial condition we obtain $u_{(0,0,0,0,\ldots)} = Eu = \tilde{u}_0$.

1° Let $g \in X \otimes (S)_{-\rho,-p}$. Then $u \in Dom^{\rho}_{-p}(\mathcal{R})$ since

$$\begin{aligned}\|u\|^2_{Dom^{\rho}_{-p}(\mathcal{R})} &= \|u_\mathbf{0}\|^2_X + \sum_{|\alpha|>0} |\alpha|^2 \, (\alpha!)^{1-\rho} \, \|u_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} = \|u_\mathbf{0}\|^2_X + \sum_{|\alpha|>0} |\alpha|^2 \, (\alpha!)^{1-\rho} \, \frac{\|g_\alpha\|^2_X}{|\alpha|^2} \, (2\mathbb{N})^{-p\alpha} \\ &= \|u_\mathbf{0}\|^2_X + \sum_{|\alpha|>0} (\alpha!)^{1-\rho} \, \|g_\alpha\|^2_X \, (2\mathbb{N})^{-p\alpha} = \|u_\mathbf{0}\|^2_X + \|g\|^2_{X\otimes(S)_{-\rho,-p}} < \infty.\end{aligned}$$

2° Assume that $g \in X \otimes (S)_{\rho,p}$. Then $u \in Dom^{\rho}_{p}(\mathcal{R})$ since

$$\begin{aligned}\|u\|^2_{Dom^{\rho}_{p}(\mathcal{R})} &= \|u_\mathbf{0}\|^2_X + \sum_{|\alpha|>0} |\alpha|^2 \, (\alpha!)^{1+\rho} \, \|u_\alpha\|^2_X (2\mathbb{N})^{p\alpha} = \|u_\mathbf{0}\|^2_X + \sum_{|\alpha|>0} (\alpha!)^{1+\rho} \, \|g_\alpha\|^2_X (2\mathbb{N})^{p\alpha} \\ &= \|u_\mathbf{0}\|^2_X + \|g\|^2_{X\otimes(S)_{\rho,p}} < \infty.\end{aligned}$$

3° If g is square integrable, then $u \in Dom_0(\mathcal{R})$ since

$$\|u\|^2_{Dom_0(\mathcal{R})} = \|u_\mathbf{0}\|^2_X + \sum_{|\alpha|>0} |\alpha|^2 \, \alpha! \, \|u_\alpha\|^2_X = \|u_\mathbf{0}\|^2_X + \sum_{|\alpha|>0} \alpha! \, \|g_\alpha\|^2_X = \|g\|^2_{X\otimes(L)^2} < \infty.$$

□

Corollary 4.2. *Let $\rho \in [0,1]$. Each process $g \in X \otimes (S)_{\pm\rho}$, resp. $g \in X \otimes (L)^2$ can be represented as $g = Eg + \mathcal{R}(u)$, for some $u \in Dom^{\rho}_{\pm}(\mathcal{R})$, resp. $u \in Dom_0(\mathcal{R})$.*

In [10] we provided one way for solving equation $\mathbb{D}u = h$: Using the chaos expansion method we transformed equation (15) into a system of infinitely many equations of the form

$$u_{\alpha+\varepsilon^{(k)}} = \frac{1}{\alpha_k + 1} \, h_{\alpha,k}, \quad \text{for all} \quad \alpha \in \mathcal{I}, \ k \in \mathbb{N}, \tag{12}$$

from which we calculated u_α, by induction on the length of α.

Denote by $r = r(\alpha) = \min\{k \in \mathbb{N} : \alpha_k \neq 0\}$, for a nonzero multi-index $\alpha \in \mathcal{I}$, i.e. let r be the position of the first nonzero component of α. Then the first nonzero component of α is the rth component α_r, i.e. $\alpha = (0, \ldots, 0, \alpha_r, \ldots, \alpha_m, 0, \ldots)$. Denote by $\alpha_{\varepsilon^{(r)}}$ the multi-index with all components equal to the corresponding components of α, except the rth, which is $\alpha_r - 1$. With the given notation we call $\alpha_{\varepsilon^{(r)}}$ the *representative* of α and write $\alpha = \alpha_{\varepsilon^{(r)}} + \varepsilon^{(r)}$. For $\alpha \in \mathcal{I}$, $|\alpha| > 0$ the set

$$\mathcal{K}_\alpha = \{\beta \in \mathcal{I} : \alpha = \beta + \varepsilon^{(j)}, \text{ for those } j \in \mathbb{N}, \text{ such that } \alpha_j > 0\}$$

is a nonempty set, because it contains at least the representative of α, i.e. $\alpha_{\varepsilon^{(r)}} \in \mathcal{K}_\alpha$. Note that, if $\alpha = n\varepsilon^{(r)}$, $n \in \mathbb{N}$ then $Card(\mathcal{K}_\alpha) = 1$ and in all other cases $Card(\mathcal{K}_\alpha) > 1$. Further, for $|\alpha| > 0$, $\mathcal{K}_\alpha$ is a finite set because α has finitely many nonzero components and $Card(\mathcal{K}_\alpha)$ is equal to the number of nonzero components of α. For example, the first nonzero component of $\alpha = (0,3,1,0,5,0,0,\ldots)$ is the second one. It follows that $r = 2$, $\alpha_r = 3$ and the representative of α is $\alpha_{\varepsilon^{(r)}} = \alpha - \varepsilon^{(2)} = (0,2,1,0,5,0,0,\ldots)$. The multi-index α has three nonzero components, thus the set $\mathcal{K}_\alpha$ consists of three elements: $\mathcal{K}_\alpha = \{(0,2,1,0,5,0,\ldots), (0,3,0,0,5,0,\ldots), (0,3,1,0,4,0,\ldots)\}$.

In [10] we obtained the coefficients u_α of the solution of (12) as functions of the representative $\alpha_{\varepsilon^{(r)}}$ of a nonzero multi-index $\alpha \in \mathcal{I}$ in the form

$$u_\alpha = \frac{1}{\alpha_r} h_{\alpha_{\varepsilon^{(r)}}, r}, \quad \text{for } |\alpha| \neq 0, \ \alpha = \alpha_{\varepsilon^{(r)}} + \varepsilon^{(r)}.$$

Theorem 4.3. ([10]) *Let* $h = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} h_{\alpha,k}\otimes\xi_k\otimes H_\alpha \in X\otimes S_{-p}(\mathbb{R})\otimes(S)_{-1,-p}$, $p\in\mathbb{N}_0$, *with* $h_{\alpha,k}\in X$ *such that*

$$\frac{1}{\alpha_r}\, h_{\alpha_{\varepsilon^{(r)}},r} = \frac{1}{\alpha_j}\, h_{\beta,j}\,, \tag{13}$$

for the representative $\alpha_{\varepsilon^{(r)}}$ *of* $\alpha\in\mathcal{I}$, $|\alpha|>0$ *and all* $\beta\in\mathcal{K}_\alpha$, *such that* $\alpha=\beta+\varepsilon^{(j)}$, *for* $j\geq r$, $r\in\mathbb{N}$. *Then, equation* (15) *has a unique solution in* $X\otimes(S)_{-1,-2p}$. *The chaos expansion of the generalized stochastic process, which represents the unique solution of equation* (15) *is given by*

$$u = \widetilde{u}_0 + \sum_{\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}\in\mathcal{I}} \frac{1}{\alpha_r}\, h_{\alpha_{\varepsilon^{(r)}},r}\ \otimes H_\alpha. \tag{14}$$

Here we provide another way of solving equation $\mathbb{D}u=h$ using the Skorokod integral operator.

Theorem 4.4. *(The Malliavin derivative) Let h have the chaos expansion* $h=\sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} h_{\alpha,k}\otimes\xi_k\otimes H_\alpha$ *and assume that condition* (13) *holds. Then the equation*

$$\mathbb{D}u = h, \qquad Eu=\widetilde{u}_0, \quad \widetilde{u}_0\in X, \tag{15}$$

has a unique solution u represented in the form

$$u = \widetilde{u}_0 + \sum_{\alpha\in\mathcal{I},|\alpha|>0}\frac{1}{|\alpha|}\sum_{k\in\mathbb{N}} h_{\alpha-\varepsilon^{(k)},k}\ \otimes H_\alpha. \tag{16}$$

Moreover, the following holds:

1° *If* $h\in X\otimes S_{-p}(\mathbb{R})\otimes(S)_{-\rho,-q}$, $q>p+1$, *then* $u\in Dom^{\rho}_{-q}(\mathbb{D})$.
2° *If* $h\in X\otimes S_{p}(\mathbb{R})\otimes(S)_{\rho,q}$, $p>q+1$, *then* $u\in Dom^{\rho}_{q}(\mathbb{D})$.
3° *If* $h\in Dom_0(\delta)$, *then* $u\in Dom_0(\mathbb{D})$.

Proof. 1° The proof is similar as for case 2°, so we present the proof of 2°.

2° Let $h\in X\otimes S_p(\mathbb{R})\otimes(S)_{\rho,q}$. Then $h\in Dom^{\rho}_{(p,q-2)}(\delta)$. Now, applying the Skorokhod integral on both sides of (15) one obtains

$$\mathcal{R}u=\delta(h).$$

From the initial condition it follows that the solution u is given in the form $u=\widetilde{u}_0+\sum_{\alpha\in\mathcal{I},|\alpha|>0} u_\alpha\otimes H_\alpha$ and its coefficients are obtained from the system

$$|\alpha|\, u_\alpha = \sum_{k\in\mathbb{N}} h_{\alpha-\varepsilon^{(k)},k}, \qquad |\alpha|>0, \tag{17}$$

where by convention $\alpha-\varepsilon^{(k)}$ does not exist if $\alpha_k=0$. Condition (13) ensures that δ is injective i.e. $\delta(\mathbb{D}u)=\delta(h)$ implies $\mathbb{D}u=h$.

It remains to prove that the solution $u\in Dom^{\rho}_{q}(\mathbb{D})$. Clearly,

$$\begin{aligned}
\|u-\tilde{u}_0\|^2_{Dom^{\rho}_{q}(\mathbb{D})} &= \sum_{\alpha\in\mathcal{I}} |\alpha|^{1-\rho}\,(\alpha!)^{1+\rho}\,\|u_\alpha\|^2_X\,(2\mathbb{N})^{q\alpha} = \sum_{\alpha\in\mathcal{I},|\alpha|>0} |\alpha|^{1-\rho}\,\frac{(\alpha!)^{1+\rho}}{|\alpha|^2}\,\|\sum_{k\in\mathbb{N}} h_{\alpha-\varepsilon^{(k)},k}\|^2_X\,(2\mathbb{N})^{q\alpha}\\
&= \sum_{\beta\in\mathcal{I}} \|\sum_{k\in\mathbb{N}} h_{\beta,k}\,\frac{(\beta+\varepsilon^{(k)})!^{\frac{1+\rho}{2}}}{|\beta+\varepsilon^{(k)}|^{\frac{1+\rho}{2}}}\,(2k)^{\frac{q}{2}}\|^2_X\,(2\mathbb{N})^{q\beta} \leq \sum_{\beta\in\mathcal{I}}\|\sum_{k\in\mathbb{N}} h_{\beta,k}\,\beta!^{\frac{1+\rho}{2}}\,(2k)^{\frac{p}{2}}\,(2k)^{\frac{q-p}{2}}\|^2_X\,(2\mathbb{N})^{q\beta}\\
&\leq \sum_{\beta\in\mathcal{I}}\Big(\sum_{k\in\mathbb{N}}\|h_{\beta,k}\|^2_X\,\beta!^{1+\rho}\,(2k)^p\sum_{k\in\mathbb{N}}(2k)^{q-p}\Big)\,(2\mathbb{N})^{q\beta} \leq c\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}\|h_{\beta,k}\|^2_X\,\beta!^{1+\rho}\,(2k)^p\,(2\mathbb{N})^{q\beta}\\
&= c\ \|h\|^2_{X\otimes S_p(\mathbb{R})\otimes(S)_{\rho,q}} < \infty,
\end{aligned}$$

since $c = \sum_{k\in\mathbb{N}}(2k)^{q-p} < \infty$, for $p > q+1$. In the fourth step of the estimation we used that $\dfrac{(\beta+\varepsilon^{(k)})!}{|\beta+\varepsilon^{(k)}|} \leq \beta!$. Thus,

$$\|u\|^2_{Dom^\rho_q(\mathbb{D})} \leq 2\left(\|\tilde{u}_0\|^2_X + c\,\|h\|^2_{X\otimes S_p(\mathbb{R})\otimes(S)_{\rho,q}}\right) < \infty$$

3° In this case we have that u given in (16) satisfies

$$\|u\|^2_{Dom_0(\mathbb{D})} = \sum_{\alpha\in\mathcal{I}} |\alpha|\,\alpha!\,\|u_\alpha\|^2_X = \sum_{\alpha\in\mathcal{I},|\alpha|>0} \frac{\alpha!}{|\alpha|}\,\|\sum_{k\in\mathbb{N}} h_{\alpha-\varepsilon^{(k)},k}\|^2_X \leq \sum_{\alpha\in\mathcal{I}} \alpha!\|\sum_{k\in\mathbb{N}} h_{\alpha-\varepsilon^{(k)},k}\|^2_X = \|h\|^2_{Dom_0(\delta)} < \infty.$$

□

Corollary 4.5. *If $\mathbb{D}(u) = 0$, then $u = Eu$, i.e. u is constant almost surely.*

Remark 4.6. *The form of the solution* (16) *can be transformed to the form* (14) *obtained in* [10]*. First we express all $h_{\beta,k}$ in condition* (13) *in terms of $h_{\alpha_{\varepsilon^{(r)}},r}$, i.e.*

$$h_{\beta,k} = \frac{\alpha_j}{\alpha_r}\,h_{\alpha_{\varepsilon^{(r)}},r},$$

where $\beta \in \mathcal{K}_\alpha$ correspond to the nonzero components of α in the following way: $\beta = \alpha - \varepsilon^{(k)}$, $k \in \mathbb{N}$, and $r \in \mathbb{N}$ is the first nonzero component of α. Note that the set $\mathcal{K}_\alpha$ has as many elements as the multi-index α has nonzero components. Therefore, from the form of the coefficients (17) *obtained in Theorem 4.4 we have*

$$\frac{1}{|\alpha|}\sum_{\beta\in\mathcal{K}_\alpha} h_{\beta,k} = \frac{1}{|\alpha|}\sum_{j\in\mathbb{N},\,\alpha_j\neq 0} \frac{\alpha_j}{\alpha_r}\,h_{\alpha_{\varepsilon^{(r)}},r} = \frac{1}{|\alpha|}\,\frac{\sum_{j\in\mathbb{N}}\alpha_j}{\alpha_r}\,h_{\alpha_{\varepsilon^{(r)}},r} = \frac{1}{\alpha_r}\,h_{\alpha_{\varepsilon^{(r)}},r}.$$

Theorem 4.7. *(The Skorokhod integral) Let f be a process with zero expectation and chaos expansion representation of the form $f = \sum_{\alpha\in\mathcal{I},|\alpha|\geq 1} f_\alpha\otimes H_\alpha$, $f_\alpha \in X$. Then the integral equation*

$$\delta(u) = f, \tag{18}$$

has a unique solution u in the class of processes satisfying condition (13) *given by*

$$u = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} (\alpha_k+1)\,\frac{f_{\alpha+\varepsilon^{(k)}}}{|\alpha+\varepsilon^{(k)}|}\,\otimes\,\xi_k\,\otimes\,H_\alpha. \tag{19}$$

Moreover, the following holds:

1° *If $f \in X\otimes(S)_{-\rho,-p}$, then $u \in Dom^\rho_{(-l,-p)}(\delta)$, for $l > p+1$.*

2° *If $f \in X\otimes(S)_{\rho,p}$, $p\in\mathbb{N}$, then $u \in Dom^\rho_{(l,p)}(\delta)$, for $l < p-1$.*

3° *If $f \in X\otimes(L)^2$, then $u \in Dom_0(\delta)$.*

Proof. 1° Since the proof of 1° and 2° are analogous, we will conduct only the proof of one of them.

2° We seek for the solution in $Range^\rho_+(\mathbb{D})$. It is clear that $u \in Range^\rho_+(\mathbb{D})$ is equivalent to $u = \mathbb{D}(\widetilde{u})$, for some $\widetilde{u}$. Thus, equation (18) is equivalent to the system of equations

$$u = \mathbb{D}(\widetilde{u}), \qquad \mathcal{R}(\widetilde{u}) = f.$$

The solution to $\mathcal{R}(\widetilde{u}) = f$ is given by

$$\widetilde{u} = \widetilde{u}_0 + \sum_{\alpha\in\mathcal{I},\,|\alpha|\geq 1} \frac{f_\alpha}{|\alpha|}\otimes H_\alpha,$$

where $\tilde{u}_{(0,0,0,\dots)} = \widetilde{u}_0$ can be chosen arbitrarily. Now, the solution of the initial equation (18) is obtained after applying the operator $\mathbb{D}$, i.e.

$$u = \mathbb{D}(\widetilde{u}) = \sum_{\alpha\in\mathcal{I},|\alpha|\geq 1}\sum_{k\in\mathbb{N}} \alpha_k \frac{f_\alpha}{|\alpha|}\otimes\xi_k\otimes H_{\alpha-\varepsilon^{(k)}} = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}(\alpha_k+1)\,\frac{f_{\alpha+\varepsilon^{(k)}}}{|\alpha+\varepsilon^{(k)}|}\otimes\xi_k\otimes H_\alpha.$$

One can directly check that this u satisfies (13): Indeed with $u_{\alpha,k} = (\alpha_k+1)\frac{f_{\alpha+\varepsilon^{(k)}}}{|\alpha+\varepsilon^{(k)}|}$ we have $\frac{1}{\alpha_k}u_{\alpha-\varepsilon^{(k)},k} = \frac{f_\alpha}{|\alpha|}$ for all $k\in\mathbb{N}$.

It remains to prove the convergence of the solution (19) in in the space $Dom^{\rho}_{(l,p)}(\delta)$. First we prove that $\widetilde{u}\in Dom^{\rho}_{p}(\mathbb{D})$ and then $u\in Dom^{\rho}_{(l,p)}(\delta)$ for appropriate $l\in\mathbb{N}$. We obtain

$$\begin{aligned}\|\widetilde{u}\|^2_{Dom^{\rho}_{p}(\mathbb{D})} &= \sum_{\alpha\in\mathcal{I}}|\alpha|^{1-\rho}\,(\alpha!)^{1+\rho}\,\|u_\alpha\|^2_X\,(2\mathbb{N})^{p\alpha} = \|\tilde{u}_0\|^2_X + \sum_{\alpha\in\mathcal{I},|\alpha|>0}|\alpha|^{1-\rho}\,(\alpha!)^{1+\rho}\,\frac{\|f_\alpha\|^2_X}{|\alpha|^2}\,(2\mathbb{N})^{p\alpha}\\ &\leq \|\tilde{u}_0\|^2_X + \sum_{\alpha\in\mathcal{I},|\alpha|>0}(\alpha!)^{1+\rho}\,\|f_\alpha\|^2_X\,(2\mathbb{N})^{p\alpha} = \|\tilde{u}_0\|^2_X + \|f\|^2_{X\otimes(S)_{\rho,p}} < \infty\end{aligned}$$

and thus $\widetilde{u}\in Dom^{\rho}_{+}(\mathbb{D})$. Now,

$$\begin{aligned}\|u\|^2_{Dom^{\rho}_{(l,p)}(\delta)} &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}(\alpha!)^{1+\rho}\,(\alpha_k+1)^{3+\rho}\,\frac{\|f_{\alpha+\varepsilon^{(k)}}\|^2_X}{|\alpha+\varepsilon^{(k)}|^2}\,(2k)^l\,(2\mathbb{N})^{p\alpha} = \sum_{\alpha\in\mathcal{I},|\alpha|>0}\sum_{k\in\mathbb{N}}(\alpha!)^{1+\rho}\,\alpha_k^2\,\frac{\|f_\alpha\|^2_X}{|\alpha|^2}\,(2k)^l\,(2\mathbb{N})^{p(\alpha-\varepsilon^{(k)})}\\ &\leq \sum_{\alpha\in\mathcal{I},|\alpha|>0}(\alpha!)^{1+\rho}\,\|f_\alpha\|^2_X\,(2\mathbb{N})^{p\alpha}\left(\sum_{k\in\mathbb{N}}\frac{\alpha_k^2}{|\alpha|^2}\,(2k)^l\,(2k)^{-p}\right) \leq c\,\|f\|^2_{X\otimes(S)_{\rho,p}} < \infty,\end{aligned}$$

since $c=\sum_{k\in\mathbb{N}}(2k)^{l-p}<\infty$ for $p>l+1$. In the second step we used that $(\alpha-\varepsilon^{(k)})!\,\alpha_k = \alpha!$, and in the fourth step we used $\alpha_k\leq|\alpha|$.

3° In this case we have

$$\|\widetilde{u}\|^2_{Dom_0(\mathbb{D})} = \sum_{\alpha\in\mathcal{I}}|\alpha|\,\alpha!\,\|u_\alpha\|^2_X = \|\tilde{u}_0\|^2_X + \sum_{\alpha\in\mathcal{I},|\alpha|>0}|\alpha|\,\alpha!\,\frac{\|f_\alpha\|^2_X}{|\alpha|^2} \leq \|\tilde{u}_0\|^2_X + \sum_{\alpha\in\mathcal{I},|\alpha|>0}\alpha!\,\|f_\alpha\|^2_X = \|\tilde{u}_0\|^2_X + \|f\|^2_{X\otimes(L)^2} < \infty$$

and thus $\widetilde{u}\in Dom_0(\mathbb{D})$. Also,

$$\begin{aligned}\|u\|^2_{Dom_0(\delta)} &= \sum_{\alpha\in\mathcal{I}}\alpha!\|\sum_{k\in\mathbb{N}}(\alpha_k+1)^{\frac12}\,(\alpha_k+1)\frac{f_{\alpha+\varepsilon^{(k)}}}{|\alpha+\varepsilon^{(k)}|}\|^2_X = \sum_{|\beta|\geq1}\|\sum_{k\in\mathbb{N}}\beta_k^{\frac32}\,(\beta-\varepsilon^{(k)})!^{\frac12}\,\frac{f_\beta}{|\beta|}\|^2_X = \sum_{|\beta|\geq1}\|\sum_{k\in\mathbb{N}}\beta_k\,\beta!^{\frac12}\,\frac{f_\beta}{|\beta|}\|^2_X\\ &= \sum_{|\beta|\geq1}\frac{\beta!}{|\beta|^2}\,\|f_\beta\|^2_X\Big(\sum_{k\in\mathbb{N}}\beta_k\Big)^2 = \sum_{|\beta|\geq1}\beta!\,\|f_\beta\|^2_X = \|f\|^2_{X\otimes(L)^2} < \infty.\end{aligned}$$

□

Corollary 4.8. *Each process $f\in X\otimes(S)_{\pm\rho}$, resp. $f\in X\otimes(L)^2$ can be represented as $f = Ef + \delta(u)$ for some $u\in X\otimes S(\mathbb{R})\otimes(S)_{\pm\rho}$, resp. $u\in X\otimes L^2(\mathbb{R})\otimes(L)^2$.*

The latter result reduces to the celebrated Itô representation theorem (see e.g. [4]) in case when f is a square integrable adapted process.

5. Properties of the Malliavin Operators

In the classical $(L)^2$ setting it is known that the Skorokhod integral is the adjoint of the Malliavin derivative. We extend this result in the next theorem and prove their duality by pairing a generalized process with a test process. The classical result is revisited in part 3° of the theorem.

Theorem 5.1. *(Duality) Assume that either of the following holds:*

1° $F \in Dom_{-}^{\rho}(\mathbb{D})$ *and* $u \in Dom_{+}^{\rho}(\delta)$

2° $F \in Dom_{+}^{\rho}(\mathbb{D})$ *and* $u \in Dom_{-}^{\rho}(\delta)$

3° $F \in Dom_0(\mathbb{D})$ *and* $u \in Dom_0(\delta)$

Then the following duality relationship between the operators $\mathbb{D}$ *and* δ *holds:*

$$E\left(F \cdot \delta(u)\right) \;=\; E\left(\langle \mathbb{D}F,\, u\rangle\right)\,, \tag{20}$$

where (20) *denotes the equality of the generalized expectations of two objects in* $X \otimes (S)_{-\rho}$ *and* $\langle\cdot,\cdot\rangle$ *denotes the dual pairing of* $S'(\mathbb{R})$ *and* $S(\mathbb{R})$.

Proof. First we show that the duality relationship (20) between $\mathbb{D}$ and δ holds formally. Let $u \in Dom(\delta)$ be given in its chaos expansion form $u = \sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}} u_{\beta,j}\otimes\xi_j\otimes H_\beta$. Then $\delta(u) = \sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}} u_{\beta,j}\otimes H_{\beta+\varepsilon^{(j)}}$. Let $F \in Dom(\mathbb{D})$ be given as $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha$. Then $\mathbb{D}(F) = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} (\alpha_k+1)\, f_{\alpha+\varepsilon^{(k)}}\otimes\xi_k\otimes H_\alpha$. Therefore we obtain

$$\begin{aligned}
F\cdot\delta(u) &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}} f_\alpha u_{\beta,j}\otimes H_\alpha\cdot H_{\beta+\varepsilon^{(j)}}\\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}} f_\alpha u_{\beta,j}\otimes \sum_{\gamma\le\min\{\alpha,\beta+\varepsilon^{(j)}\}} \gamma!\cdot\binom{\alpha}{\gamma}\binom{\beta+\varepsilon^{(j)}}{\gamma} H_{\alpha+\beta+\varepsilon^{(j)}-2\gamma}.
\end{aligned}$$

The generalized expectation of $F\cdot\delta(u)$ is the zeroth coefficient in the previous sum, which is obtained when $\alpha+\beta+\varepsilon^{(j)} = 2\gamma$ and $\gamma \le \min\{\alpha, \beta+\varepsilon^{(j)}\}$, i.e. only for the choice $\beta = \alpha-\varepsilon^{(j)}$ and $\gamma=\alpha$, $j\in\mathbb{N}$. Thus,

$$E\left(F\cdot\delta(u)\right) = \sum_{\alpha\in\mathcal{I},|\alpha|>0}\sum_{j\in\mathbb{N}} f_\alpha u_{\alpha-\varepsilon^{(j)},j}\cdot\alpha! = \sum_{\alpha\in\mathcal{I}}\sum_{j\in\mathbb{N}} f_{\alpha+\varepsilon^{(j)}} u_{\alpha,j}\cdot(\alpha+\varepsilon^{(j)})!\,.$$

On the other hand,

$$\begin{aligned}
\langle\mathbb{D}(F),u\rangle &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} (\alpha_k+1)\, f_{\alpha+\varepsilon^{(k)}}\, u_{\beta,j}\,\langle\xi_k,\xi_j\rangle H_\alpha\cdot H_\beta\\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}} (\alpha_j+1)\, f_{\alpha+\varepsilon^{(j)}}\, u_{\beta,j} \sum_{\gamma\le\min\{\alpha,\beta\}}\gamma!\cdot\binom{\alpha}{\gamma}\binom{\beta}{\gamma}\cdot H_{\alpha+\beta-2\gamma}
\end{aligned}$$

and its generalized expectation is obtained for $\alpha=\beta=\gamma$. Thus

$$E\left(\langle\mathbb{D}(F),u\rangle\right) = \sum_{\alpha\in\mathcal{I}}\sum_{j\in\mathbb{N}} (\alpha_j+1)\, f_{\alpha+\varepsilon^{(j)}} u_{\alpha,j}\cdot\alpha! = \sum_{\alpha\in\mathcal{I}}\sum_{j\in\mathbb{N}} f_{\alpha+\varepsilon^{(j)}} u_{\alpha,j}\cdot(\alpha+\varepsilon^{(j)})! = E\left(F\cdot\delta(u)\right).$$

1° Let $\rho\in[0,1]$ be fixed. Let $F\in Dom_{-p}^{\rho}(\mathbb{D})$ and $u \in Dom_{(r,s)}^{\rho}(\delta)$, for some $p\in\mathbb{N}$ and all $r,s\in\mathbb{N}$, $r>s+1$. Then $\mathbb{D}F \in X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-\rho,-p}$ for $l>p+1$. Since r is arbitrary, we may assume that $r=l$ and denote by $\langle\cdot,\cdot\rangle$ the dual pairing between $S_{-l}(\mathbb{R})$ and $S_l(\mathbb{R})$. Moreover, $\langle\mathbb{D}F,u\rangle$ is well defined in $X\otimes(S)_{-\rho,-p}$. On the other hand, $\delta(u)\in X\otimes(S)_{\rho,s}$ and thus by Theorem 2.12, $F\cdot\delta(u)$ is also defined as an element in $X\otimes(S)_{-\rho,-k}$, for $k \in [p, s-8]$, $s > p+8$. Since s was arbitrary, one can take any $k \geq p$. This means that both objects, $F \cdot \delta(u)$ and $\langle \mathbb{D}F, u\rangle$ exist in $X \otimes (S)_{-\rho,-k}$, for $k \geq p$. Taking generalized expectations of $\langle \mathbb{D}F, u\rangle$ and $F \cdot \delta(u)$ we showed that the zeroth coefficients of the formal expansions are equal. Therefore the duality formula is valid for this case.

2° Let $F \in Dom_p^\rho(\mathbb{D})$ and $u \in Dom^\rho_{(-r,-s)}(\delta)$, for some $r, s \in \mathbb{N}$, $s > r+1$, and all $p \in \mathbb{N}$. Then $\mathbb{D}F \in X \otimes S_l(\mathbb{R}) \otimes (S)_{\rho,p}$, $l < p-1$, but since p is arbitrary, so is l. Now, $\langle \mathbb{D}F, u\rangle$ is a well defined object in $X \otimes (S)_{-\rho,-s}$. On the other hand, $\delta(u) \in X \otimes (S)_{-\rho,-s}$ and thus by Theorem 2.12, $F \cdot \delta(u)$ is also well defined and belongs to $X \otimes (S)_{-\rho,-k}$, for $k \in [s, p-8]$, $p > s+8$. Thus, both processes $F \cdot \delta(u)$ and $\langle \mathbb{D}F, u\rangle$ belong to $X \otimes (S)_{-\rho,-k}$ for $k \geq s$.

3° For $F \in Dom_0(\mathbb{D})$ and $u \in Dom_0(\delta)$ the dual pairing $\langle \mathbb{D}F, u\rangle$ represents the inner product in $L^2(\mathbb{R})$ and the product $F\delta(u)$ is an element in $X \otimes (L)^1$ (see Remark 2.13). The classical duality formula is clearly valid for this case. □

The next theorem states a higher order duality formula, which connects the kth order iterated Skorokhod integral and the Malliavin derivative operator of kth order, $k \in \mathbb{N}$. For the definition of higher order iterated operators we refer to [8].

Theorem 5.2. *Let $f \in Dom_+^\rho(\mathbb{D}^{(k)})$ and $u \in Dom_-^\rho(\delta^{(k)})$, or let $f \in Dom_-^\rho(\mathbb{D}^{(k)})$ and $u \in Dom_+^\rho(\delta^{(k)})$, $k \in \mathbb{N}$. Then the duality formula*

$$E\left(f \cdot \delta^{(k)}(u)\right) = E\left(\langle \mathbb{D}^{(k)}(f), u\rangle\right)$$

holds, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $S'(\mathbb{R})^{\otimes k}$ and $S(\mathbb{R})^{\otimes k}$.

Proof. The assertion follows by induction and applying Theorem 5.1 successively k times. □

Remark 5.3. *The previous theorems are special cases of a more general identity. It can be proven, under suitable assumptions that make all the products well defined, that the following formulae hold:*

$$F\,\delta(u) = \delta(Fu) + \langle \mathbb{D}(F), u\rangle, \tag{21}$$

$$F\,\delta^{(k)}(u) = \sum_{i=0}^{k} \binom{k}{i} \delta^{(k-i)}(\langle \mathbb{D}^{(i)}F, u\rangle), \quad k \in \mathbb{N}.$$

Taking the expectation in (21) *and using the fact that $\delta(Fu) = 0$, the duality formula* (20) *follows.*

Example 5.4. *Let $\psi \in L^2(\mathbb{R})$. In [6] we have shown that the stochastic exponentials $\exp^\lozenge\{\delta(\psi)\}$ are eigenvalues of the Malliavin derivative, i.e. $\mathbb{D}(\exp^\lozenge\{\delta(\psi)\}) = \psi \cdot \exp^\lozenge\{\delta(\psi)\}$. We will prove that they are also eigenvalues of the Ornstein-Uhlenbeck operator. Indeed, using* (21) *we obtain*

$$\begin{aligned} \mathcal{R}(\exp^\lozenge\{\delta(\psi)\}) &= \delta(\psi \cdot \exp^\lozenge\{\delta(\psi)\}) = \delta(\psi)\exp^\lozenge\{\delta(\psi)\} - \langle \mathbb{D}(\exp^\lozenge\{\delta(\psi)\}), \psi\rangle \\ &= \delta(\psi)\exp^\lozenge\{\delta(\psi)\} - \langle \psi \cdot \exp^\lozenge\{\delta(\psi)\}, \psi\rangle \\ &= (\delta(\psi) - \|\psi\|^2_{L^2(\mathbb{R})})\exp^\lozenge\{\delta(\psi)\}. \end{aligned}$$

In the next theorem we prove a weaker type of duality instead of (20) which holds if $F \in Dom_-^0(\mathbb{D})$ and $u \in Dom_-^0(\delta)$ are both generalized processes. Recall that $\ll \cdot, \cdot \gg_r$ denotes the scalar product in $(S)_{0,r}$.

Lemma 5.5. *Let $u \in Dom^0_{-q}(\mathbb{D})$ and $\varphi \in S_{-n}(\mathbb{R})$, $n < q-1$. Then $u \cdot \varphi \in Dom^0_{(-n,-q)}(\delta)$.*

Proof. Let $u = \sum_{\alpha \in \mathcal{I}} u_\alpha H_\alpha$ and $\varphi = \sum_{k \in \mathbb{N}} \varphi_k\, \xi_k$. Then, $u \cdot \varphi = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} u_\alpha\, \varphi_k\, \xi_k\, H_\alpha$ and

$$\begin{aligned} \|u \cdot \varphi\|^2_{Dom^0_{(-n,-q)}(\delta)} &= \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha!\,(\alpha_k + 1)\|u_\alpha\|^2_X\, \varphi_k^2\,(2k)^{-n}(2\mathbb{N})^{-q\alpha} = \sum_{\alpha \in \mathcal{I}} \alpha!\,\|u_\alpha\|^2_X \Big(\sum_{k \in \mathbb{N}} (\alpha_k + 1)(2k)^{-n}\,\varphi_k^2\Big)(2\mathbb{N})^{-q\alpha} \\ &\leq \Big(\|u_{\mathbf{0}}\|^2_X + 2\sum_{|\alpha|>0} \alpha!\,|\alpha|\|u_\alpha\|^2_X(2\mathbb{N})^{-q\alpha}\Big) \cdot \sum_{k \in \mathbb{N}} \varphi_k^2(2k)^{-n} = \Big(\|u_{\mathbf{0}}\|^2_X + 2\|u\|^2_{Dom^0_{-q}(\mathbb{D})}\Big) \cdot \|\varphi\|^2_{-n} < \infty. \end{aligned}$$

We used the estimate $\alpha_k + 1 \leq 2\,|\alpha|$, for $|\alpha| > 0$, $k \in \mathbb{N}$. □

Theorem 5.6. *(Weak duality) Let $\rho = 0$ and consider the Hida spaces. Let $F \in Dom^0_{-p}(\mathbb{D})$ and $u \in Dom^0_{-q}(\mathbb{D})$ for $p, q \in \mathbb{N}$. For any $\varphi \in S_{-n}(\mathbb{R})$, $n < q-1$, it holds that*

$$\ll \langle \mathbb{D}F, \varphi\rangle_{-r}, u \gg_{0,-r} = \ll F, \delta(\varphi u) \gg_{0,-r},$$

for $r > \max\{q, p+1\}$.

Proof. Let $F = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha \in Dom^0_{-p}(\mathbb{D})$, $u = \sum_{\alpha\in\mathcal{I}} u_\alpha H_\alpha \in Dom^0_{-q}(\mathbb{D})$ and $\varphi = \sum_{k\in\mathbb{N}} \varphi_k \xi_k \in S_{-n}(\mathbb{R})$. Then, for $k > p+1$, $\mathbb{D}F \in X \otimes S_{-k}(\mathbb{R}) \otimes (S)_{0,-p} \subseteq X \otimes S_{-r}(\mathbb{R}) \otimes (S)_{0,-r}$ if $r > p+1$. Also, by Lemma 5.5 it follows that $\varphi u \in Dom^0_{(-n,-q)}(\delta)$ and since $q > n+1$, this implies that $\delta(\varphi u) \in X \otimes (S)_{0,-q} \subseteq X \otimes (S)_{0,-r}$, for $r \geq q$. Therefore we let $r > \max\{p+1, q\}$. Thus,

$$\begin{aligned}
\langle \mathbb{D}F, \varphi\rangle_{-r} &= \langle \sum_{k\in\mathbb{N}} \sum_{\alpha\in\mathcal{I}} (\alpha_k + 1) f_{\alpha+\varepsilon^{(k)}} H_\alpha \otimes \xi_k, \sum_{k\in\mathbb{N}} \varphi_k \xi_k \rangle_{-r} \\
&= \sum_{k\in\mathbb{N}} \varphi_k \sum_{\alpha\in\mathcal{I}} (\alpha_k + 1) f_{\alpha+\varepsilon^{(k)}} H_\alpha \ (2k)^{-r},
\end{aligned}$$

and consequently

$$\begin{aligned}
\ll \langle \mathbb{D}F, \varphi\rangle_{-r}, u \gg_{0,-r} &= \ll \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \varphi_k (\alpha_k+1) f_{\alpha+\varepsilon^{(k)}} (2k)^{-r} H_\alpha, \sum_{\alpha\in\mathcal{I}} u_\alpha H_\alpha \gg_{0,-r} \\
&= \sum_{\alpha\in\mathcal{I}} \alpha! u_\alpha \sum_{k\in\mathbb{N}} \varphi_k (\alpha_k+1) f_{\alpha+\varepsilon^{(k)}} (2k)^{-r} (2\mathbb{N})^{-r\alpha}.
\end{aligned}$$

On the other hand, $\varphi u = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} u_\alpha \varphi_k \xi_k \otimes H_\alpha$ and $\delta(\varphi u) = \sum_{\alpha>\mathbf{0}} \sum_{k\in\mathbb{N}} u_{\alpha-\varepsilon^{(k)}} \varphi_k H_\alpha$. Thus,

$$\begin{aligned}
\ll F, \delta(\varphi u) \gg_{0,-r} &= \ll \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha, \sum_{\alpha>\mathbf{0}} \sum_{k\in\mathbb{N}} u_{\alpha-\varepsilon^{(k)}} \varphi_k H_\alpha \gg_{0,-r} \\
&= \sum_{\alpha>\mathbf{0}} \alpha! f_\alpha \sum_{k\in\mathbb{N}} u_{\alpha-\varepsilon^{(k)}} \varphi_k (2\mathbb{N})^{-r\alpha} \\
&= \sum_{\beta\in\mathcal{I}} \sum_{k\in\mathbb{N}} (\beta + \varepsilon^{(k)})! f_{\beta+\varepsilon^{(k)}} u_\beta \varphi_k (2\mathbb{N})^{-r(\beta+\varepsilon^{(k)})} \\
&= \sum_{\beta\in\mathcal{I}} \sum_{k\in\mathbb{N}} \beta! (\beta_k+1) f_{\beta+\varepsilon^{(k)}} u_\beta \varphi_k (2k)^{-r} (2\mathbb{N})^{-r\beta},
\end{aligned}$$

which completes the proof. □

The following theorem states the product rule for the Ornstein-Uhlenbeck operator. Its special case for $F, G \in Dom_0(\mathcal{R})$ and $F \cdot G \in Dom_0(\mathcal{R})$ states that (22) holds (see e.g. [2]). We extend the classical $(L)^2$ case to multiplying a generalized process with a test process. The product rule also holds if we multiply two generalized processes, but in this case the ordinary product has to be replaced by the Wick product.

Theorem 5.7. *(Product rule for $\mathcal{R}$)*

1° *Let $F \in Dom_+^\rho(\mathcal{R})$ and $G \in Dom_-^\rho(\mathcal{R})$, or vice versa. Then $F \cdot G \in Dom_-^\rho(\mathcal{R})$ and*

$$\mathcal{R}(F \cdot G) = F \cdot \mathcal{R}(G) + G \cdot \mathcal{R}(F) - 2 \cdot \langle \mathbb{D}F, \mathbb{D}G \rangle, \tag{22}$$

holds, where $\langle \cdot, \cdot \rangle$ is the dual pairing between $S'(\mathbb{R})$ and $S(\mathbb{R})$.

2° *Let $F, G \in Dom_-^\rho(\mathcal{R})$. Then $F \cdot G \in Dom_-^\rho(\mathcal{R})$ and*

$$\mathcal{R}(F \lozenge G) = F \lozenge \mathcal{R}(G) + \mathcal{R}(F) \lozenge G. \tag{23}$$

Proof. 1° Let $F = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes H_\alpha \in Dom_+^\rho(\mathcal{R})$ and $G = \sum_{\beta \in \mathcal{I}} g_\beta \otimes H_\beta \in Dom_-^\rho(\mathcal{R})$. Then, $\mathcal{R}(F) = \sum_{\alpha \in \mathcal{I}} |\alpha| \, f_\alpha \otimes H_\alpha$ and $\mathcal{R}(G) = \sum_{\beta \in \mathcal{I}} |\beta| \, g_\beta \otimes H_\beta$.

The left hand side of (22) can be written in the form

$$\begin{aligned}
\mathcal{R}(F \cdot G) &= \mathcal{R}\left(\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_\alpha \, g_\beta \sum_{\gamma \leq \min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} H_{\alpha+\beta-2\gamma} \right) \\
&= \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_\alpha \, g_\beta \sum_{\gamma \leq \min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} |\alpha + \beta - 2\gamma| \, H_{\alpha+\beta-2\gamma} \\
&= \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_\alpha \, g_\beta \sum_{\gamma \leq \min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} \, (|\alpha| + |\beta| - 2|\gamma|) \, H_{\alpha+\beta-2\gamma}.
\end{aligned}$$

On the other hand, the first two terms on the right hand side of (22) are

$$\mathcal{R}(F) \cdot G = \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_\alpha \, g_\beta \otimes \sum_{\gamma \leq \min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} |\alpha| \, H_{\alpha+\beta-2\gamma} \tag{24}$$

and

$$F \cdot \mathcal{R}(G) = \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_\alpha \, g_\beta \otimes \sum_{\gamma \leq \min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} |\beta| \, H_{\alpha+\beta-2\gamma}. \tag{25}$$

Since $F \in Dom_+^\rho(\mathcal{R}) \subseteq Dom_+^\rho(\mathbb{D})$ and $G \in Dom_-^\rho(\mathcal{R}) \subseteq Dom_-^\rho(\mathbb{D})$ we have $\mathbb{D}(F) = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_k \, f_\alpha \otimes \xi_k \otimes H_{\alpha-\varepsilon^{(k)}}$ and $\mathbb{D}(G) = \sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}} \beta_j \, g_\beta \otimes \xi_j \otimes H_{\beta-\varepsilon^{(k)}}$. Thus, the third term on the right hand side of (22) is

$$\begin{aligned}
\langle \mathbb{D}(F), \mathbb{D}(G) \rangle &= \Big\langle \sum_{\alpha \in \mathcal{I}, |\alpha|>0} \sum_{k \in \mathbb{N}} \alpha_k \, f_\alpha \otimes \xi_k \otimes H_{\alpha-\varepsilon^{(k)}}, \sum_{\beta \in \mathcal{I}, |\beta|>0} \sum_{j \in \mathbb{N}} \beta_j \, g_\beta \otimes \xi_j \otimes H_{\beta-\varepsilon^{(j)}} \Big\rangle \\
&= \sum_{|\alpha|>0} \sum_{|\beta|>0} \sum_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}} \alpha_k \, \beta_j \, f_\alpha \, g_\beta \, \langle \xi_k, \xi_j \rangle \otimes H_{\alpha-\varepsilon^{(k)}} \cdot H_{\beta-\varepsilon^{(j)}} \\
&= \sum_{|\alpha|>0} \sum_{|\beta|>0} \sum_{k \in \mathbb{N}} \alpha_k \beta_k f_\alpha g_\beta \otimes \sum_{\gamma \leq \min\{\alpha-\varepsilon^{(k)}, \beta-\varepsilon^{(k)}\}} \gamma! \binom{\alpha - \varepsilon^{(k)}}{\gamma} \binom{\beta - \varepsilon^{(k)}}{\gamma} H_{\alpha+\beta-2\varepsilon^{(k)}-2\gamma},
\end{aligned}$$

where we used the fact that $\langle \xi_k, \xi_j \rangle = 0$ for $k \neq j$ and $\langle \xi_k, \xi_j \rangle = 1$ for $k = j$. Now we put $\theta = \gamma + \varepsilon^{(k)}$ and use the identities

$$\alpha_k \cdot \binom{\alpha - \varepsilon^{(k)}}{\gamma} = \alpha_k \cdot \binom{\alpha - \varepsilon^{(k)}}{\theta - \varepsilon^{(k)}} = \theta_k \cdot \binom{\alpha}{\theta}, \quad k \in \mathbb{N},$$

and $\theta_k \cdot (\theta - \varepsilon^{(k)})! = \theta!$. Thus we obtain

$$\begin{aligned}\langle \mathbb{D}(F), \mathbb{D}(G)\rangle &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}} f_\alpha\, g_\beta \sum_{\theta\leq\min\{\alpha,\beta\}} \theta_k^2\,(\theta-\varepsilon^{(k)})!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2\theta}\\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}} f_\alpha\, g_\beta \sum_{\theta\leq\min\{\alpha,\beta\}} \theta_k\theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2\theta}\\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\theta\leq\min\{\alpha,\beta\}} \left(\sum_{k\in\mathbb{N}}\theta_k\right)\theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2\theta}\\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\theta\leq\min\{\alpha,\beta\}} |\theta|\,\theta!\binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2\theta}.\end{aligned}$$

Combining all previously obtained results we now have

$$\begin{aligned}\mathcal{R}(F\cdot G) &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\gamma\leq\min\{\alpha,\beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}\ (|\alpha|+|\beta|-2|\gamma|)\ H_{\alpha+\beta-2\gamma}\\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\gamma\leq\min\{\alpha,\beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} |\alpha|\ H_{\alpha+\beta-2\gamma} + \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\gamma\leq\min\{\alpha,\beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} |\beta|\ H_{\alpha+\beta-2\gamma}\\
&\qquad - 2\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\gamma\leq\min\{\alpha,\beta\}} |\gamma|\,\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2\gamma}\\
&= \mathcal{R}(F)\cdot G + F\cdot\mathcal{R}(G) - 2\cdot\langle\mathbb{D}(F),\mathbb{D}(G)\rangle\end{aligned}$$

and thus (22) holds.

Assume that $F \in Dom^{\rho}_{-p}(\mathcal{R})$ and $G \in Dom^{\rho}_{q}(\mathcal{R})$. Then $\mathcal{R}(F) \in X \otimes (S)_{-\rho,-p}$ and $\mathcal{R}(G) \in X \otimes (S)_{\rho,q}$. From Theorem 2.12 it follows that $F \cdot \mathcal{R}(G)$ and $G \cdot \mathcal{R}(F)$ are both well defined and belong to $X \otimes (S)_{-\rho,-s}$, for $s \in [p, q-8]$, $q-p > 8$. Similarly, $\langle\mathbb{D}(F), \mathbb{D}(G)\rangle$ belongs to $X \otimes (S)_{-\rho,-p}$, since $\mathbb{D}(F) \in X \otimes S_{-l_1}(\mathbb{R}) \otimes (S)_{-\rho,-p}$, where $l_1 > p+1$ and $\mathbb{D}(G) \in X \otimes S_{l_2}(\mathbb{R}) \otimes (S)_{\rho,q}$, where $l_2 < q-1$ and the dual pairing is obtained for any $l \in [l_1, l_2]$. Thus, the right hand side of (22) is in $X \otimes (S)_{-\rho,-s}$, $s \geq p$. Hence, $F \cdot G \in Dom^{\rho}_{-s}(\mathcal{R})$.

2° From

$$G\lozenge\mathcal{R}(F) = \sum_{\gamma\in\mathcal{I}}\sum_{\alpha+\beta=\gamma} |\alpha| f_\alpha g_\beta\, H_\gamma \quad\text{and}\quad F\lozenge\mathcal{R}(G) = \sum_{\gamma\in\mathcal{I}}\sum_{\alpha+\beta=\gamma} f_\alpha |\beta| g_\beta\, H_\gamma,$$

it follows that

$$G\lozenge\mathcal{R}(F) + F\lozenge\mathcal{R}(G) = \sum_{\gamma\in\mathcal{I}} |\gamma| \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta\, H_\gamma = \mathcal{R}(F\lozenge G).$$

If $F \in Dom^{\rho}_{-p}(\mathcal{R})$ and $G \in Dom^{\rho}_{-q}(\mathcal{R})$, then $\mathcal{R}(F) \in X \otimes (S)_{-\rho,-p}$ and $\mathcal{R}(G) \in X \otimes (S)_{-\rho,-q}$. From Theorem 2.9 it follows that $\mathcal{R}(F)\lozenge G \in X \otimes (S)_{-\rho,-(p+q+4)}$ and $\mathcal{R}(G)\lozenge F \in X \otimes (S)_{-\rho,-(p+q+4)}$. Thus, the right hand side of (23) is in $X \otimes (S)_{-\rho,-(p+q+4)}$, i.e. $F\lozenge G \in Dom^{\rho}_{-r}(\mathcal{R})$ for $r = p+q+4$. □

Corollary 5.8. *Let $F \in Dom^{\rho}_{+}(\mathcal{R})$ and $G \in Dom^{\rho}_{-}(\mathcal{R})$, or vice versa (including also the possibility $F, G \in Dom_0(\mathcal{R})$). Then the following property holds:*

$$E(F\cdot\mathcal{R}(G)) = E\left(\langle\mathbb{D}F, \mathbb{D}G\rangle\right).$$

Proof. From the chaos expansion form of $\mathcal{R}(F\cdot G)$ it follows that $E\mathcal{R}(F\cdot G)=0$. Moreover, taking the expectations on both sides of (24) and (25) we obtain

$$E\left(\mathcal{R}(F)\cdot G\right)=E\left(F\cdot\mathcal{R}(G)\right).$$

Now, from Theorem 5.7 it follows that

$$0=2E\left(F\cdot\mathcal{R}(G)\right)-2E(\langle\mathbb{D}F,\mathbb{D}G\rangle),$$

and the assertion follows. □

In the classical literature ([2, 15]) it is proven that the Malliavin derivative satisfies the product rule with respect to ordinary multiplication, i.e. if $F,G\in Dom_0(\mathbb{D})$ such that $F\cdot G\in Dom_0(\mathbb{D})$ then (26) holds. The following theorem recapitulates this result and extends it for multiplication of a generalized process with a test processes, and extends it also for Wick multiplication.

Theorem 5.9. *(Product rule for $\mathbb{D}$)*

1° *Let $F\in Dom_-^{\rho}(\mathbb{D})$ and $G\in Dom_+^{\rho}(\mathbb{D})$ or vice versa. Then $F\cdot G\in Dom_-^{\rho}(\mathbb{D})$ and the product rule*

$$\mathbb{D}(F\cdot G)=F\cdot\mathbb{D}G\,+\,\mathbb{D}F\cdot G \tag{26}$$

holds.

2° *Let $F,G\in Dom_-^{\rho}(\mathbb{D})$. Then $F\lozenge G\in Dom_-^{\rho}(\mathbb{D})$ and*

$$\mathbb{D}(F\lozenge G)=F\lozenge\mathbb{D}G\,+\,\mathbb{D}F\lozenge G.$$

Proof. 1°

$$\begin{aligned}\mathbb{D}(F\cdot G)&=\mathbb{D}\Big(\sum_{\alpha\in\mathcal{I}}f_\alpha H_\alpha\cdot\sum_{\beta\in\mathcal{I}}g_\beta H_\beta\Big)\\&=\mathbb{D}\left(\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}f_\alpha\,g_\beta\sum_{\gamma\le\min\{\alpha,\beta\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}H_{\alpha+\beta-2\gamma}\right)\\&=\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}f_\alpha g_\beta\sum_{\gamma\le\min\{\alpha,\beta\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}(\alpha_k+\beta_k-2\gamma_k)\,\xi_k H_{\alpha+\beta-2\gamma-\varepsilon^{(k)}}\end{aligned}$$

On the other side we have

$$\begin{aligned}F\cdot\mathbb{D}(G)&=\sum_{\alpha\in\mathcal{I}}f_\alpha H_\alpha\cdot\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}\beta_k\,g_\beta\xi_k\,H_{\beta-\varepsilon^{(k)}}\\&=\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}f_\alpha g_\beta\sum_{\gamma\le\min\{\alpha,\beta-\varepsilon^{(k)}\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta-\varepsilon^{(k)}}{\gamma}\beta_k\,\xi_k H_{\alpha+\beta-2\gamma-\varepsilon^{(k)}}\end{aligned}$$

and

$$G\cdot\mathbb{D}(F)=\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}f_\alpha g_\beta\sum_{\gamma\le\min\{\alpha-\varepsilon^{(k)},\beta\}}\gamma!\binom{\alpha-\varepsilon^{(k)}}{\gamma}\binom{\beta}{\gamma}\alpha_k\,\xi_k H_{\alpha+\beta-2\gamma-\varepsilon^{(k)}}.$$

Summing up the chaos expansions for $F\cdot\mathbb{D}(G)$ and $G\cdot\mathbb{D}(F)$ and applying the identities

$$\alpha_k\binom{\alpha-\varepsilon^{(k)}}{\gamma}=\alpha_k\cdot\frac{(\alpha-\varepsilon^{(k)})!}{\gamma!\,(\alpha-\varepsilon^{(k)}-\gamma)!}=\frac{\alpha!}{\gamma!\,(\alpha-\gamma)!}\cdot(\alpha_k-\gamma_k)=\binom{\alpha}{\gamma}(\alpha_k-\gamma_k)$$

and

$$\beta_k \binom{\beta - \varepsilon^{(k)}}{\gamma} = \binom{\beta}{\gamma} (\beta_k - \gamma_k),$$

for all $\alpha, \beta \in \mathcal{I}$, $k \in \mathbb{N}$ and $\gamma \in \mathcal{I}$ such that $\gamma \leq \min\{\alpha, \beta\}$ and the expression $(\alpha_k - \gamma_k) + (\beta_k - \gamma_k) = \alpha_k + \beta_k - 2\gamma_k$ we obtain (26).

Assume that $F \in Dom^{\rho}_{-p}(\mathbb{D})$, $G \in Dom^{\rho}_{q}(\mathbb{D})$. Then $\mathbb{D}(F) \in X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-\rho,-p}$, $l > p+1$, and $\mathbb{D}(G) \in X \otimes S_k(\mathbb{R}) \otimes (S)_{\rho,q}$, $k < q-1$. From Theorem 2.12 it follows that all products on the right hand side of (26) are well defined, moreover $F \cdot \mathbb{D}(G) \in X \otimes S_k(\mathbb{R}) \otimes (S)_{-\rho,-r}$, $\mathbb{D}(F) \cdot G \in X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-\rho,-r}$, for $r \in [p, q-8]$, $q > p+8$. Thus the right hand sifde of (26) can be embedded into $X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-\rho,-r}$, $r \geq p$. Thus, $F \cdot G \in Dom^{\rho}_{-r}(\mathbb{D})$.

2° By definition of the Malliavin derivative and the Wick product it can be easily verified that

$$\begin{aligned} \mathbb{D}(F) \lozenge G + F \lozenge \mathbb{D}(G) &= \sum_{\gamma \in \mathcal{I}} \sum_{k=1}^{\infty} \sum_{\alpha+\beta-\varepsilon^{(k)}=\gamma} \alpha_k f_\alpha g_\beta H_\gamma + \sum_{\gamma \in \mathcal{I}} \sum_{k=1}^{\infty} \sum_{\alpha+\beta-\varepsilon^{(k)}=\gamma} \beta_k f_\alpha g_\beta H_\gamma \\ &= \sum_{\gamma \in \mathcal{I}} \sum_{k=1}^{\infty} \sum_{\alpha+\beta=\gamma} \gamma_k f_\alpha g_\beta H_{\gamma-\varepsilon^{(k)}} = \mathbb{D}(F \lozenge G). \end{aligned}$$

If $F \in Dom^{\rho}_{-p}(\mathbb{D})$ and $G \in Dom^{\rho}_{-q}(\mathbb{D})$, then $\mathbb{D}(F) \in X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-\rho,-p}$, $l > p+1$, and $\mathbb{D}(G) \in X \otimes S_{-k}(\mathbb{R}) \otimes (S)_{-\rho,-q}$, $k > q+1$. From Theorem 2.9 it follows that $\mathbb{D}(F) \lozenge G$ and $F \lozenge \mathbb{D}(G)$ both belong to $X \otimes S_{-m}(\mathbb{R}) \otimes (S)_{-\rho,-(p+q+4)}$, $m = \max\{l, k\}$. Thus, $F \lozenge G \in Dom^{\rho}_{-r}(\mathbb{D})$ for $r = p+q+4$. □

A generalization of Theorem 5.9 for higher order derivatives, i.e. the Leibnitz formula is given in the next theorem.

Theorem 5.10. *Let $F, G \in Dom^{\rho}_{-}(\mathbb{D}^{(k)})$, $k \in \mathbb{N}$, then $F \lozenge G \in Dom^{\rho}_{-}(\mathbb{D}^{(k)})$ and the Leibnitz rule holds:*

$$\mathbb{D}^{(k)}\,(F \lozenge G) = \sum_{i=0}^{k} \binom{k}{i} \mathbb{D}^{(i)}(F) \lozenge \mathbb{D}^{(k-i)}(G),$$

where $\mathbb{D}^{(0)}(F) = F$ and $\mathbb{D}^{(0)}(G) = G$.

Moreover, if $G \in Dom^{\rho}_{+}(\mathbb{D}^{(k)})$, then $F \cdot G \in Dom^{\rho}_{-}(\mathbb{D}^{(k)})$ and

$$\mathbb{D}^{(k)}\,(F \cdot G) = \sum_{i=0}^{k} \binom{k}{i} \mathbb{D}^{(i)}(F) \cdot \mathbb{D}^{(k-i)}(G). \tag{27}$$

Proof. The Leibnitz rule (27) follows by induction and applying Theorem 5.9. Clearly, (27) holds also if $F, G \in Dom_0(\mathbb{D}^{(k)})$ and $F \cdot G \in Dom_0(\mathbb{D}^{(k)})$. □

Theorem 5.11. *Assume that either of the following hold:*

1° *$F \in Dom^{\rho}_{-}(\mathbb{D})$, $G \in Dom^{\rho}_{+}(\mathbb{D})$ and $u \in Dom^{\rho}_{+}(\delta)$,*

2° *$F, G \in Dom^{\rho}_{+}(\mathbb{D})$ and $u \in Dom^{\rho}_{-}(\delta)$,*

3° *$F, G \in Dom_0(\mathbb{D})$ and $u \in Dom_0(\delta)$.*

Then the second integration by parts formula holds:

$$E(F\langle \mathbb{D}G, u\rangle) + E(G\langle \mathbb{D}F, u\rangle) = E(F\,G\,\delta(u)).$$

Proof. The assertion follows directly from the duality formula (20) and the product rule (26). Assume the first case holds when $F \in Dom^{\rho}_{-}(\mathbb{D})$, $G \in Dom^{\rho}_{+}(\mathbb{D})$ and $u \in Dom^{\rho}_{+}(\delta)$. Then $F \cdot G \in Dom^{\rho}_{-}(\mathbb{D})$, too, and we have

$$\begin{aligned} E(F\,G\,\delta(u)) &= E(\langle \mathbb{D}(F\cdot G), u\rangle) = E(\langle F\cdot \mathbb{D}(G) + G\cdot \mathbb{D}(F), u\rangle) \\ &= E(F\,\langle \mathbb{D}(G), u\rangle) + E(G\,\langle \mathbb{D}(F), u\rangle). \end{aligned}$$

The second and third case can be proven in an analogous way. □

The next theorem states the chain rule for the Malliavin derivative. The classical $(L)^2$-case has been known throughout the literature as a direct consequence of the definition of Malliavin derivatives as Fréchet derivatives. Here we provide an alternative proof suited to the setting of chaos expansions.

Theorem 5.12. *(Chain rule) Let ϕ be a twice continuously differentiable function with bounded derivatives.*

1° *If $F \in Dom^{\rho}_{+}(\mathbb{D})$, resp. $F \in Dom_0(\mathbb{D})$, then $\phi(F) \in Dom^{\rho}_{+}(\mathbb{D})$, resp. $\phi(F) \in Dom_0(\mathbb{D})$, and the chain rule holds:*

$$\mathbb{D}\,(\phi(F)) \;=\; \phi'(F)\;\cdot\;\mathbb{D}(F). \tag{28}$$

2° *If $F \in Dom^{\rho}_{-}(\mathbb{D})$ and ϕ is analytic, then $\phi^{\lozenge}(F) \in Dom^{\rho}_{-}(\mathbb{D})$ and*

$$\mathbb{D}\,(\phi^{\lozenge}(F)) \;=\; \phi'^{\lozenge}(F)\lozenge\mathbb{D}(F). \tag{29}$$

Proof. 1° First we prove that (28) holds true when ϕ is a polynomial of degree n, $n \in \mathbb{N}$. Then we use the Stone-Weierstrass theorem and approximate a continuously differentiable function ϕ by a polynomial $\widetilde{p_n}$ of degree n, and since we assumed that ϕ is regular enough, $\widetilde{p}'_n$ will also approximate ϕ'.

By Theorem 5.9 we obtain by induction on $k \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{D}(F^{k+1}) &= \mathbb{D}(F\cdot F^k) \\ &= \mathbb{D}(F)\cdot F^k + F\cdot \mathbb{D}(F^k) = \mathbb{D}(F)\cdot F^k + F\cdot kF^{k-1}\cdot \mathbb{D}(F) \\ &= (k+1)F^k\cdot \mathbb{D}(F). \end{aligned}$$

Since $\mathbb{D}$ is a linear operator, we have for any polynomial $p_n(x) = \sum_{k=0}^{n} a_k x^k$ with real coefficients a_k, $k \in \mathbb{N}$:

$$\mathbb{D}(p_n(F)) = \sum_{k=0}^{n} a_k \mathbb{D}(F^k) = \sum_{k=1}^{n} a_k\; kF^{(k-1)}\cdot \mathbb{D}(F) = p_n'(F)\cdot \mathbb{D}(F). \tag{30}$$

Let $\phi \in C^2(\mathbb{R})$ and $F \in Dom^{\rho}_{p}(\mathbb{D})$, $p \in \mathbb{N}$. Then, by the Stone–Weierstrass theorem, there exists a polynomial $\widetilde{p_n}$ such that

$$\|\phi(F) - \widetilde{p_n}(F)\|_{X\otimes(S)_{\rho,p}} = \|\phi(F) - \sum_{k=0}^{n} a_k F^k\|_{X\otimes(S)_{\rho,p}} \to 0$$

and

$$\|\phi'(F) - \widetilde{p_n}'(F)\|_{X\otimes(S)_{\rho,p}} = \|\phi'(F) - \sum_{k=1}^{n} a_k k F^{k-1}\|_{X\otimes(S)_{\rho,p}} \to 0$$

as $n \to \infty$.

From (30) and the fact that $\mathbb{D}$ is a bounded operator, Theorem 3.2, we obtain (for $l < p-1$)

$$\begin{aligned} \|\mathbb{D}(\phi(F)) - \phi'(F)\cdot \mathbb{D}(F)\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}} &= \|\mathbb{D}(\phi(F)) - \mathbb{D}(\widetilde{p_n}(F)) + \mathbb{D}(\widetilde{p_n}(F)) - \phi'(F)\cdot \mathbb{D}(F)\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}} \\ &\le \|\mathbb{D}(\phi(F)) - \mathbb{D}(\widetilde{p_n}(F))\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}} + \|\mathbb{D}(\widetilde{p_n}(F)) - \phi'(F)\cdot \mathbb{D}(F)\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}} \\ &= \|\mathbb{D}(\phi(F) - \widetilde{p_n}(F))\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}} + \|\widetilde{p_n}'(F)\cdot \mathbb{D}(F) - \phi'(F)\cdot \mathbb{D}(F)\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}} \\ &\le \|\mathbb{D}\|\cdot \|(\phi(F) - \widetilde{p_n}(F))\|_{X\otimes(S)_{\rho,p}} + \|\widetilde{p_n}'(F) - \phi'(F)\|\cdot \|\mathbb{D}(F)\|_{X\otimes(S)_{\rho,p}} \to 0, \end{aligned}$$

as $n \to \infty$. From this follows (28) as well as the estimate

$$\|\mathbb{D}(\phi(F))\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}} \le \|\phi'(F)\|_{X\otimes(S)_{\rho,p}} \cdot \|\mathbb{D}(F)\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{\rho,p}} < \infty,$$

and thus $\phi(F) \in Dom_p^\rho(\mathbb{D})$.

2° The proof of (29) for the Wick version can be conducted in a similar manner. According to Theorem 5.9 we have

$$\mathbb{D}(F^{\Diamond k}) = k\, F^{\Diamond(k-1)} \Diamond \mathbb{D}(F).$$

If ϕ is an analytic function given by $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$, then $\phi'(x) = \sum_{k=1}^{\infty} a_k k x^{k-1}$, and consequently

$$\phi^{\Diamond}(F) = \sum_{k=0}^{\infty} a_k F^{\Diamond k}, \quad \phi'^{\Diamond}(F) = \sum_{k=1}^{\infty} a_k k F^{\Diamond(k-1)}.$$

Thus,

$$\mathbb{D}(\phi^{\Diamond}(F)) = \sum_{k=0}^{\infty} a_k \mathbb{D}(F^{\Diamond k}) = \sum_{k=0}^{\infty} a_k k F^{\Diamond(k-1)} \Diamond \mathbb{D}(F) = \phi'^{\Diamond}(F) \Diamond \mathbb{D}(F).$$

and the identity (29) follows. □

References

[1] Benth, F. E., The Malliavin derivative of generalized random variables, Centre for Mathematical Physics and Stochastics, University of Aarhus, 1998, 1–18.

[2] Da Prato, G., An introduction to infinite-dimensional analysis, Universitext, Springer, 2006.

[3] T. Hida, T., Kuo, H.-H., Pothoff, J., Streit, L., White noise - an infinite dimensional calculus, Kluwer Academic Publishers, 1993.

[4] Holden, H., Øksendal, B., Ubøe, J., Zhang, T., Stochastic partial differential equations. A modeling, White noise functional approach, 2nd Edition, Springer, 2010.

[5] Levajković, T., Mena, H., Equations involving Malliavin derivative: a chaos expansion approach, Pseudo-differential operators and generalized functions, 199–216, Oper. Theory Adv. Appl., 245, Birkhäuser/Springer, Cham, 2015.

[6] Levajković, T., Pilipović, S., Seleši, D., Chaos expansions: Applications to a generalized eigenvalue problem for the Malliavin derivative, Integral Transforms Spec. Funct. 22(2) (2011), 97–105.

[7] Levajković, T., Pilipović, S., Seleši, D., The stochastic Dirichlet problem driven by the Ornstein-Uhlenbeck operator: Approach by the Fredholm alternative for chaos expansions. Stoch. Anal. Appl. 29(2011), 317–331.

[8] Levajković, T., Pilipović, S., Seleši, D., Fundamental equations with higher order Malliavin operators, in Stochastics: An International Journal of Probability and Stochastic Processes Stochastics 88(1), (2016), 106–127.

[9] Levajković, T., Seleši, D., Chaos expansion methods for stochastic differential equations involving the Malliavin derivative Part I, Publ. Inst. Math. (Beograd) (N.S.) 90(104) (2011), 65–85.

[10] Levajković, T., Seleši, D., Chaos expansion methods for stochastic differential equations involving the Malliavin derivative Part II, Publ. Inst. Math. (Beograd) (N.S.) 90(104) (2011), 85–98.

[11] Levajković, T., Pilipović, S.,Seleši, D., Chaos expansion methods in Malliavin calculus: A survey of recent results, Novi Sad Journal of Mathematics, 45(1) (2015), 45–103.

[12] Lototsky, S., Rozovsky, B., Stochastic Differential Equations: A Wiener Chaos Approach, Book chapter in The Shiryaev Festschrift "From Stochastic Calculus to Mathematical Finance", (Ed: Yu. Kabanov et al.), Springer Berlin, (2006), 433–507.

[13] Lototsky, S., Rozovsky, B., Wiener chaos solutions of linear stochastic evolution equations, Ann. Probab. 34(2) (2006), 638–662.

[14] Mikulevicius, R., Rozovskii, B., On unbiased stochastic Navier-Stokes equations, Probab. Theory Related Fields 154 (2012), 787–834.

[15] Nualart, D., The Malliavin Calculus and related topics, Probability and its Applications, 2nd edition, Springer-Verlag, New York, 2006.

[16] Pilipović, S., Seleši, D., Expansion theorems for generalized random processes, Wick products and applications to stochastic differential equations, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 10(1) (2007), 79–110.

[17] Pilipović, S., Seleši, D., On the stochastic Dirichlet problem – Part I: The stochastic weak maximum principle, Potential Anal. 32(4) (2010), 363–387.

[18] Pilipović, S., Seleši, D., On the stochastic Dirichlet problem – Part II: Solvability, stability and the Colombeau case, Potential Anal. 33(3) (2010), 263–289.

[19] Seleši, D., Hilbert space valued generalized random processes - part I, Novi Sad J. Math. 37(1) (2007), 129–154.

[20] Seleši, D., Hilbert space valued generalized random processes - part II, Novi Sad J. Math. 37(2) (2007), 93–108.