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Nonexceptional Functions and Normal Families of Zero-free Meromorphic Functions

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Abstract. Let k be a positive integer, let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, all of whose poles are multiple, and let h be a meromorphic function in D, all of whose poles are simple, $h \not\equiv 0, \infty$. If for each $f \in \mathcal{F}$, $f^{(k)}(z) - h(z)$ has at most k zeros in D, ignoring multiplicities, then $\mathcal{F}$ is normal in D. The examples are provided to show that the result is sharp.

1. Introduction and Main Results

Let D be a domain in $\mathbb{C}$ and $\mathcal{F}$ be a family of functions meromorphic in D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D, to a meromorphic function or the constant ∞ (see [6, 12, 14]).

Let f and h be two functions meromorphic in D on $\mathbb{C}$, and let $a \in \mathbb{C} \cup \{\infty\}$. If $f(z) - h(z) \neq 0$ in D, then we say that h is an exceptional function in D. If $f(z) - h(z)$ has at least a zero in D, then we say that h is a nonexceptional function in D. In particular, when $h(z) \equiv a$, we say that a is an exceptional(nonexceptional) value in D.

In 1959, Hayman [5, cf. 6] proved the following result known as "Hayman's Alternative".

Theorem A. Let k be a positive integer, and let f be a nonconstant meromorphic function in $\mathbb{C}$. Then $f(z)$ or $f^{(k)}(z) - 1$ has at least one zero. Moreover, if f is transcendental, then $f(z)$ or $f^{(k)}(z) - 1$ has infinitely many zeros.

The normality corresponding to Theorem A was conjectured by Hayman [7, Problem 5.11] and confirmed by Gu [4].

Theorem B. Let k be a positive integer, and let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D. If for each $f \in \mathcal{F}$, $f^{(k)}(z) \neq 1$ in D, then $\mathcal{F}$ is normal in D.

In [2], Chang improved Theorem B by allowing $f^{(k)}(z) - 1$ to have zeros but restricting their numbers, and proved the following result.

Theorem C. Let k be a positive integer, and let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D. If for each $f \in \mathcal{F}$, $f^{(k)}(z) - 1$ has at most k zeros in D, ignoring multiplicities, then $\mathcal{F}$ is normal in D.

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Recently, Deng, Fang, and Liu [3] considered the case that a nonexceptional value was replaced by a nonexceptional holomorphic function in Theorem C, and obtained the following theorem.

Theorem D. Let k be a positive integer, let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, and let h be a holomorphic function in D, $h \not\equiv 0$. If for each $f \in \mathcal{F}$, $f^{(k)}(z) - h(z)$ has at most k zeros in D, ignoring multiplicities, then $\mathcal{F}$ is normal in D.

It is natural to ask what can be said if a nonexceptional holomorphic function is replaced by a nonexceptional meromorphic function in Theorem D. In this paper, we study this problem and first prove the following result.

Theorem 1. Let k be a positive integer, let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, all of whose poles are multiple, and let h be a zero-free meromorphic function in D, all of whose poles are simple, $h \not\equiv \infty$. If for each $f \in \mathcal{F}$, $f^{(k)}(z) - h(z)$ has at most k zeros in D, ignoring multiplicities, then $\mathcal{F}$ is normal in D.

Example 1. Let k be a positive integer, $D = \{z : |z| < 1\}$, $h(z) = 1/z$, and $\mathcal{F} = \{f_j(z) = 1/(jz) : j \geq k! + 1\}$. Then, for each $f_j \in \mathcal{F}$, $f_j(z) \neq 0$ and $f_j^{(k)}(z) - h(z) = \frac{(-1)^k k! - jz^k}{jz^{k+1}}$ has exactly k zeros in D, ignoring multiplicities. But $\mathcal{F}$ fails to be normal in D. This shows that the condition in Theorem 1 that the poles of the functions in $\mathcal{F}$ are multiple cannot be weakened.

Example 2. Let k be a positive integer, $D = \{z : |z| < 1\}$, $h(z) = 1/z^2$, and $\mathcal{F} = \{f_j(z) = 1/(jz^2) : j \geq (k+1)!+1\}$. Then, for each $f_j \in \mathcal{F}$, $f_j(z) \neq 0$ and $f_j^{(k)}(z)-h(z) = \frac{(-1)^k (k+1)! - jz^k}{jz^{k+2}}$ has exactly k zeros in D, ignoring multiplicities. But $\mathcal{F}$ fails to be normal in D. This shows that the condition in Theorem 1 that the poles of h are simple cannot be removed.

Example 3. Let k be a positive integer, $D = \{z : |z| < 1\}$, $h(z) = 1/z$, and $\mathcal{F} = \{f_j(z) = 1/(jz^2) : j \geq (k+1)!+1\}$. Then, for each $f_j \in \mathcal{F}$, $f_j(z) \neq 0$ and $f_j^{(k)}(z) - h(z) = \frac{(-1)^k (k+1)! - jz^{k+1}}{jz^{k+2}}$ has exactly $k+1$ zeros in D, ignoring multiplicities. But $\mathcal{F}$ fails to be normal in D. This shows that the condition in Theorem 1 that $f^{(k)}(z) - h(z)$ has at most k zeros is best possible.

Since normality is a local property, combining Theorem D with Theorem 1, we can obtain the following theorem, which generalizes Theorem B, Theorem C, and Theorem D.

Theorem 2. Let k be a positive integer, let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, all of whose poles are multiple, and let h be a meromorphic function in D, all of whose poles are simple, $h \not\equiv 0, \infty$. If for each $f \in \mathcal{F}$, $f^{(k)}(z) - h(z)$ has at most k zeros in D, ignoring multiplicities, then $\mathcal{F}$ is normal in D.

2. Some Lemmas

Lemma 1.(see [11, 15]) Let $\alpha \in \mathbb{R}$ satisfy $-1 < \alpha < +\infty$, and let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D. Then, if $\mathcal{F}$ is not normal at some point $z_0 \in D$, there exist

(i) points $z_j \in D$, $z_j \to z_0$,

(ii) functions $f_j \in \mathcal{F}$, and

(iii) positive numbers $\rho_j \to 0$

such that

$$\frac{f_j(z_j + \rho_j\zeta)}{\rho_j^{\alpha}} = g_j(\zeta) \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant zero-free meromorphic function on $\mathbb{C}$ of order at most 2. In particular, if g is an entire function, then g is of order at most 1.

Lemma 2.(see [10]) Let k be a positive integer, let f be a transcendental meromorphic function of finite order, all of whose zeros are of multiplicity at least $k+1$, and let p be a polynomial, $p \not\equiv 0$. Then $f^{(k)}(z) - p(z)$ has infinitely many zeros.

Lemma 3.(see [2]) Let k be a positive integer, and let f be a nonconstant zero-free rational function. Then $f^{(k)}(z) - 1$ has at least $k+1$ distinct zeros in $\mathbb{C}$.

Lemma 4. Let k be a positive integer, let $\{f_n\}$ be a sequence of zero-free meromorphic functions in a domain D, and let $\{h_n\}$ be a sequence of holomorphic functions in D such that $h_n \to h$ locally uniformly in D, where $h(z) \neq 0$, $z \in D$. If, for every n, $f_n^{(k)}(z) - h_n(z)$ has at most k zeros in D, ignoring multiplicities, then $\{f_n\}$ is normal in D.

Proof. Suppose that $\{f_n\}$ is not normal at $z_0 \in D$. Without loss of generality, we may assume that $h(z_0) = 1$. Then by Lemma 1 there exist points $z_n \to z_0$, numbers $\rho_n \to 0^+$, and a subsequence of $\{f_n\}$, which we continue to denote by $\{f_n\}$, such that

$$\frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} = g_n(\zeta) \to g(\zeta)$$

spherically locally uniformly on $\mathbb{C}$, where g is a nonconstant zero-free meromorphic function of order at most two.

We claim that $g^{(k)}(\zeta) - 1$ has at most k distinct zeros.

Suppose that $g^{(k)}(\zeta) - 1$ has at least $k + 1$ distinct zeros ζ_i, $1 \leq i \leq k + 1$. Clearly, $g^{(k)}(\zeta) \not\equiv 1$, for otherwise g would be a nonconstant polynomial of degree k, which contradicts the fact that g is zero-free. Then by Hurwitz's theorem and noting that

$$g_n^{(k)}(\zeta) - h_n(z_n + \rho_n\zeta) = f_n^{(k)}(z_n + \rho_n\zeta) - h_n(z_n + \rho_n\zeta) \to g^{(k)}(\zeta) - 1$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of g, there exist $\zeta_{n,i}$, $i = 1, 2, \cdots, k + 1$, $\zeta_{n,i} \to \zeta_i$, such that, for n sufficiently large, $f_n^{(k)}(z_n + \rho_n\zeta_{n,i}) = h_n(z_n + \rho_n\zeta_{n,i})$. However $f_n^{(k)}(z) - h_n(z)$ has at most k distinct zeros in D, and $z_n + \rho_n\zeta_{n,i} \to z_0$, which is a contradiction. Hence $g^{(k)}(\zeta) - 1$ has at most k distinct zeros.

Now from Lemma 2 it follows that g is a rational function. But this contradicts Lemma 3, which shows that $\{f_n\}$ is normal in D.

This completes the proof of Lemma 4.

Lemma 5.(see [13]) Let k be a positive integer, let f be a transcendental meromorphic function, and let R be a rational function, $R \not\equiv 0$. Suppose that, with at most finitely many exceptions, all poles of f are multiple and all zeros of f have multiplicity at least $k + 1$. Then $f^{(k)}(z) - R(z)$ has infinitely many zeros.

Lemma 5 generalizes the main result of [1], where the case $k = 1$ was proved. Actually, for the case $k = 1$, the result remains valid without any assumption on the poles of f, see [9].

Using the idea of [2], we get the following lemma.

Lemma 6. Let f be a nonconstant zero-free rational function, all of whose poles are multiple. Then $f^{(k)}(z) - 1/(z - c)$ has at least $k + 1$ distinct zeros in $\mathbb{C}$, where c is a constant.

Proof. Since f is a nonconstant zero-free rational function, f is not a polynomial. Then by the assumption we know that f has at least one finite multiple pole. Thus we can write

$$f(z) = \frac{C_1}{\prod_{i=1}^{q}(z + z_i)^{p_i}}, \tag{2.1}$$

where C_1 is a nonzero constant, q and $p_i \geq 2$ (when $1 \leq i \leq q$) are positive integers, the z_i (when $1 \leq i \leq q$) are distinct complex numbers, $p = \sum_{i=1}^{q} p_i$. By induction, we deduce from (2.1) that

$$f^{(k)}(z) = \frac{P(z)}{\prod_{i=1}^{q}(z + z_i)^{p_i+k}}, \tag{2.2}$$

where $P(z)$ is a polynomial of degree $(q - 1)k$. Further, by simple calculation, $f^{(k)}(z) - \frac{1}{z-c}$ has at least one zero in $\mathbb{C}$.

Next we discuss two cases.

Case 1. Suppose that for all i $(1 \leq i \leq q)$, $z_i \neq -c$. Then we can write

$$f^{(k)}(z) - \frac{1}{z - c} = \frac{C_2 \prod_{i=1}^{s}(z + \omega_i)^{l_i}}{(z - c)\prod_{i=1}^{q}(z + z_i)^{p_i+k}}, \tag{2.3}$$

where C_2 is a nonzero constant, s and l_i are positive integers, the $-c$, ω_i (when $1 \le i \le s$), and z_i (when $1 \le i \le q$) are distinct complex numbers. From (2.2)-(2.3), we have

$$\prod_{i=1}^{q}(z+z_i)^{p_i+k} + C_2\prod_{i=1}^{s}(z+\omega_i)^{l_i} = (z-c)P(z). \tag{2.4}$$

Then by (2.4) it follows that $\sum_{i=1}^{s} l_i = \sum_{i=1}^{q}(p_i+k) = p+qk$, $C_2 = -1$, and so

$$\prod_{i=1}^{q}(1+z_it)^{p_i+k} - \prod_{i=1}^{s}(1+\omega_it)^{l_i} = t^{p+k-1}Q(t), \tag{2.5}$$

where $Q(t) = -t^{(q-1)k+1}(1/t-c)P(1/t)$ is a polynomial of degree less than $(q-1)k+1$. From (2.5), we get

$$\frac{\prod_{i=1}^{q}(1+z_it)^{p_i+k}}{\prod_{i=1}^{s}(1+\omega_it)^{l_i}} = 1 + \frac{t^{p+k-1}Q(t)}{\prod_{i=1}^{s}(1+\omega_it)^{l_i}} = 1 + O\left(t^{p+k-1}\right), \tag{2.6}$$

as $t \to 0$. Thus by taking logarithmic derivatives of both sides of (2.6), it follows that

$$\sum_{i=1}^{q}\frac{(p_i+k)\,z_i}{1+z_it} - \sum_{i=1}^{s}\frac{l_i\omega_i}{1+\omega_it} = O\left(t^{p+k-2}\right), \tag{2.7}$$

as $t \to 0$. Comparing the coefficients of (2.7) for t^j, $j = 0, 1, \cdots, p+k-3$, we have

$$\sum_{i=1}^{q}(p_i+k)\,z_i^j - \sum_{i=1}^{s} l_i\omega_i^j = 0, \qquad j = 1, 2, \cdots, p+k-2. \tag{2.8}$$

Let $z_{q+i} = \omega_i$ when $1 \le i \le s$. Noting that $\sum_{i=1}^{q}(p_i+k) = \sum_{i=1}^{s} l_i$ and using (2.8), we deduce that the system of linear equations

$$\sum_{i=1}^{q+s} z_i^j x_i = 0 \tag{2.9}$$

where $0 \le j \le p+k-2$, has a nonzero solution

$$\left(x_1, \cdots, x_q, x_{q+1}, \cdots, x_{q+s}\right) = \left(p_1+k, \cdots, p_q+k, -l_1, \cdots, -l_s\right).$$

If $p+k-1 \ge q+s$, then the determinant $\det\left(z_i^j\right)_{(q+s)\times(q+s)}$ of the coefficients of the system of the equations (2.9) where $0 \le j \le q+s-1$ is equal to zero, by Cramer's rule (see e.g. [8]). However, the z_i are distinct complex numbers when $1 \le i \le q+s$, and the determinant is a Vandermonde determinant, so cannot be zero (see e.g. [8]), which is a contradiction.

Hence we conclude that $p+k-1 < q+s$. It follows from this and the two facts $p_i \ge 2$ (when $1 \le i \le q$) and $p = \sum_{i=1}^{q} p_i$ that $s \ge k+1$.

Case 2. Suppose that for some i $(1 \le i \le q)$, say q, $z_q = -c$. Then we can write

$$f^{(k)}(z) - \frac{1}{z-c} = \frac{C_3\prod_{i=1}^{s}(z+\omega_i)^{l_i}}{\prod_{i=1}^{q}(z+z_i)^{p_i+k}}, \tag{2.10}$$

where C_3 is a nonzero constant, s and l_i are positive integers, the ω_i (when $1 \le i \le s$) and z_i (when $1 \le i \le q$) are distinct complex numbers. From (2.2) and (2.10), we have

$$(z+z_q)^{p_q-1+k}\prod_{i=1}^{q-1}(z+z_i)^{p_i+k} + C_3\prod_{i=1}^{s}(z+\omega_i)^{l_i} = P(z). \tag{2.11}$$

Then by (2.11) it follows that $\sum_{i=1}^{s} l_i = \sum_{i=1}^{q} (p_i + k) - 1 = p + qk - 1$, $C_3 = -1$, and so

$$(1 + z_q t)^{p_q - 1 + k} \prod_{i=1}^{q-1} (1 + z_i t)^{p_i + k} - \prod_{i=1}^{s} (1 + \omega_i t)^{l_i} = t^{p+k-1} Q_1(t), \tag{2.12}$$

where $Q_1(t) = -t^{(q-1)k} P(1/t)$ is a polynomial of degree less than $(q-1)k$. From (2.12), we get

$$\frac{(1 + z_q t)^{p_q - 1 + k} \prod_{i=1}^{q-1} (1 + z_i t)^{p_i + k}}{\prod_{i=1}^{s} (1 + \omega_i t)^{l_i}} = 1 + \frac{t^{p+k-1} Q(t)}{\prod_{i=1}^{s} (1 + \omega_i t)^{l_i}} = 1 + O\left(t^{p+k-1}\right), \tag{2.13}$$

as $t \to 0$. Thus by taking logarithmic derivatives of both sides of (2.13), it follows that

$$\frac{(p_q - 1 + k) z_q}{1 + z_q t} + \sum_{i=1}^{q-1} \frac{(p_i + k) z_i}{1 + z_i t} - \sum_{i=1}^{s} \frac{l_i \omega_i}{1 + \omega_i t} = O\left(t^{p+k-2}\right), \tag{2.14}$$

as $t \to 0$. Let

$$n_i = \begin{cases} p_i, & 1 \le i \le q-1, \\ p_i - 1, & i = q. \end{cases}$$

Then (2.14) can be rewritten

$$\sum_{i=1}^{q} \frac{(n_i + k) z_i}{1 + z_i t} - \sum_{i=1}^{s} \frac{l_i \omega_i}{1 + \omega_i t} = O\left(t^{p+k-2}\right),$$

as $t \to 0$. Using the same argument as in Case 1, we can also get $s \ge k + 1$.

This completes the proof of Lemma 6.

3. Proof of Theorem 1

By Lemma 4, it suffices to prove that $\mathcal{F}$ is normal at points at which $h(z)$ has poles. So we may assume that $D = \Delta = \{z : |z| < 1\}$, and that for $z \in \Delta$, making standard normalizations,

$$h(z) = \frac{1}{z} + a_0 + a_1 z + \cdots = \frac{b(z)}{z},$$

where $b(0) = 1$, and $h(z) \ne 0, \infty$ for $0 < |z| < 1$. Next we only need to show that $\mathcal{F}$ is normal at 0. Suppose not. Then we have by Lemma 1 (with $\alpha = k - 1$) that there exist $f_n \in \mathcal{F}$, $z_n \to 0$, and $\rho_n \to 0^+$ such that

$$\frac{f_n(z_n + \rho_n \zeta)}{\rho_n^{k-1}} = g_n(\zeta) \to g(\zeta)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where g is a nonconstant zero-free meromorphic function on $\mathbb{C}$, all of whose poles are multiple. Moreover, g is of order at most two.

We consider two cases.

Case 1. Suppose that $z_n/\rho_n \to \infty$. Consider

$$\phi_n(\zeta) = z_n^{1-k} f_n(z_n + z_n \zeta) = z_n^{1-k} f_n\left(z_n(1 + \zeta)\right).$$

Then

$$\phi_n^{(k)}(\zeta) = z_n f_n^{(k)}\left(z_n(1 + \zeta)\right).$$

Obviously, ϕ_n is zero-free, all poles of ϕ_n are multiple, and $b(z_n(1+\zeta))/(1+\zeta) \to 1/(1+\zeta) \neq 0$ as $n \to \infty$ on Δ. A simple calculation now shows that

$$\begin{aligned}
\phi_n^{(k)}(\zeta) - \frac{b(z_n(1+\zeta))}{1+\zeta} &= z_n f_n^{(k)}(z_n(1+\zeta)) - \frac{b(z_n(1+\zeta))}{1+\zeta} \\
&= z_n \left(f_n^{(k)}(z_n(1+\zeta)) - \frac{b(z_n(1+\zeta))}{z_n(1+\zeta)} \right) \\
&= z_n \left(f_n^{(k)}(z_n(1+\zeta)) - h(z_n(1+\zeta)) \right).
\end{aligned}$$

Since $f_n^{(k)}(z) - h(z)$ has at most k zeros in Δ, ignoring multiplicities, the family $\{\phi_n\}$ is normal on Δ by Lemma 4. Thus we may find a sequence $\{\phi_{n_i}\}$ and a function ϕ satisfying

$$\phi_{n_i}(\zeta) = z_{n_i}^{1-k} f_{n_i}(z_{n_i}(1+\zeta)) \to \phi(\zeta)$$

and

$$\begin{aligned}
g^{(k-1)}(\zeta) &= \lim_{i\to\infty} f_{n_i}^{(k-1)}(z_{n_i} + \rho_{n_i}\zeta) \\
&= \lim_{i\to\infty} f_{n_i}^{(k-1)}\left(z_{n_i}\left(1 + \frac{\rho_{n_i}}{z_{n_i}}\zeta\right)\right) \\
&= \lim_{i\to\infty} \phi_{n_i}^{(k-1)}\left(\frac{\rho_{n_i}}{z_{n_i}}\zeta\right) \\
&= \phi^{(k-1)}(0).
\end{aligned}$$

Thereby we know that $g^{(k-1)}(\zeta)$ is constant, implying $g^{(k)}(\zeta) \equiv 0$. It follows that $g(\zeta)$ is a nonconstant polynomial of degree at most $k-1$. This contradicts that $g(\zeta)$ is zero-free.

Case 2. So we may assume that $z_n/\rho_n \to \alpha$, a finite complex number. We have

$$g_n^{(k)}(\zeta) - \frac{\rho_n b(z_n + \rho_n\zeta)}{z_n + \rho_n\zeta} = \rho_n \left(f_n^{(k)}(z_n + \rho_n\zeta) - \frac{b(z_n + \rho_n\zeta)}{z_n + \rho_n\zeta}\right) \to g^{(k)}(\zeta) - \frac{1}{\alpha+\zeta}$$

uniformly on compact subsets of $\mathbb{C}\backslash\{-\alpha\}$ disjoint from the poles of g.

We claim that $g^{(k)}(\zeta) - \frac{1}{\alpha+\zeta}$ has at most k distinct zeros.

Suppose that $g^{(k)}(\zeta) - \frac{1}{\alpha+\zeta}$ has at least $k+1$ distinct zeros ζ_i, $1 \leq i \leq k+1$. Clearly, $g^{(k)}(\zeta) - \frac{1}{\alpha+\zeta} \not\equiv 0$ since all poles of $g^{(k)}$ are multiple. Now by Hurwitz's theorem, there exist $\zeta_{n,i}$, $i = 1, 2, \cdots, k+1$, $\zeta_{n,i} \to \zeta_i$, such that, for n sufficiently large,

$$f_n^{(k)}(z_n + \rho_n\zeta_{n,i}) - \frac{b(z_n + \rho_n\zeta_{n,i})}{z_n + \rho_n\zeta_{n,i}} = f_n^{(k)}(z_n + \rho_n\zeta_{n,i}) - h(z_n + \rho_n\zeta_{n,i}) = 0$$

However $f_n^{(k)}(z) - h(z)$ has at most k distinct zeros in Δ, and $z_n + \rho_j\zeta_{n,i} \to z_0$, which is a contradiction. Hence $g^{(k)}(\zeta) - \frac{1}{\alpha+\zeta}$ has at most k distinct zeros.

But, from Lemma 5 and Lemma 6, we see that there do not exist nonconstant meromorphic functions that have the above properties. This contradiction shows that $\mathcal{F}$ is normal in D and so the proof of Theorem 1 is complete.

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