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Fejér-Type Inequalities for Lipschitzian Functions and their Applications

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Abstract. In this paper, we shall establish some Fejér-type inequalities for L-Lipschitzian functions. These inequalities can connect with Fejér inequality (1). Also, some applications to convex function, γ-th moment, mathematical expectation of a random variable and Euler's Beta function are provided.

1. Introduction

Throughout this paper, let $I := [a, b]$ in $\mathbb{R}$ with $a < b$.

The inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b g(s)\,ds \le \int_a^b f(s)\,g(s)\,ds \le \frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds \tag{1}$$

which holds for all convex functions $f : I \to \mathbb{R}$ and integrable, symmetric functions $g : I \to \mathbb{R}^+ \cup \{0\}$ is known as Fejér inequality [4]. If we choose $g(s) \equiv 1$, then inequality (1) reduces to Hermite-Hadamard inequality [5].

Recently, many authors improved, generalized and extended Hermite-Hadamard and Fejér inequalities (see [1]-[3], [6], [9], [10], [13] and [15]-[18]) or applied them to other inequalities (see [7], [8], [11], [12] and [14]).

The followings are the theorems we interest:

In [17, Theorem 5-6, Remark 6] and [18, Theorem 1], Yang and Tseng offered the following three theorems which refined (1) .

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be convex and let $g : I \to \mathbb{R}^+ \cup \{0\}$ be integrable and symmetric about $\frac{a+b}{2}$. If $P_g : I \to \mathbb{R}$ is defined by*

$$P_g(s) := \int_a^b f\left(sx + (1-s)\frac{a+b}{2}\right) g(x)\,dx, \tag{2}$$

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then P_g *is convex, increasing on* $[0,1]$ *and , for all* $s \in [0,1]$,

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx = P_g(0) \le P_g(s) \le P_g(1) = \int_a^b f(x)\,g(x)\,dx.$$

Theorem 1.2. *Let* f, g *be defined as Theorem 1.1. If* $Q_g : I \to \mathbb{R}$ *is defined by*

$$\begin{aligned} Q_g(s) \quad : \quad &= \frac{1}{2}\int_a^b \left[f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right)g\left(\frac{a+x}{2}\right)\right. \\ &\left. f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right)g\left(\frac{b+x}{2}\right)\right]dx, \end{aligned} \tag{3}$$

then Q_g *is convex, increasing on* $[0,1]$ *and , for all* $s \in [0,1]$,

$$\int_a^b f(x)\,g(x)\,dx = Q_g(0) \le Q_g(s) \le Q_g(1) = \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.$$

Theorem 1.3. *Let* f, g *be defined as Theorem 1.1. If* $G_g : I \to \mathbb{R}$ *is defined by*

$$G_g(s) := \frac{1}{\int_a^b g(x)\,dx}\int_a^b\int_a^b f(sx + (1-s)y)\,g(x)\,g(y)\,dxdy \tag{4}$$

then we obtain the following results:

(1) G_g *is convex on* $[0,1]$, *symmetric about* $\frac{1}{2}$, *decreasing on* $\left[0, \frac{1}{2}\right]$ *and increasing on* $\left[\frac{1}{2}, 1\right]$,

$$\sup_{s\in[0,1]} G_g(s) = G_g(0) = G_g(1) = \int_a^b f(x)\,g(x)\,dx$$

and

$$\inf_{s\in[0,1]} G_g(s) = G_g\left(\frac{1}{2}\right) = \frac{1}{\int_a^b g(x)\,dx}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)g(x)\,g(y)\,dxdy$$

(2)

$$P_g(s) \le G_g(s) \quad (s \in (0,1)).$$

Besides, Dragomir et al. [2] and Matić and Pečarić [9] provided the following theorem where are Hadamard-type inequalities for L-Lipschitzian functions.

Theorem 1.4. *Let* $f : I \to \mathbb{R}$ *be a L-Lipschitzian function on the interval I of real numbers. Then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{L(b-a)}{4} \tag{5}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{L(b-a)}{4}. \tag{6}$$

Remark 1.5.

(1) In Theorem 1.1, let $g(x) \equiv 1$. *Then Theorem 1.1 reduces to [1, Theorem 1].*
(2) In Theorem 1.2, let $g(x) \equiv 1$. *Then Theorem 1.2 reduces to [16, Theorem 1-2].*
(3) In Theorem 1.3, let $g(x) \equiv \frac{1}{b-a}$. *Then Theorem 1.3 reduces to [1, Theorem 2].*

The aim of this paper is to establish some Fejér-type inequalities for Lipschitzian functions through mappings (2), (3) and (4). The study results can generalize inequalities (5) and (6). Besides, some applications to convex function, γ-th moment, mathematical expectation of a random variable and Euler's Beta function are presented afterwards.

2. Main Results

Throughout this section, let $f : I \to \mathbb{R}$ be a L-Lipschitzian function on I. For all $u, v \in I$, we have

$$\left| f(u) - f(v) \right| \leq L \left| u - v \right|$$

where L is positive constant.

In order to prove our main results, let us consider the following lemma.

Lemma 2.1. *Let $g : I \to \mathbb{R}^{+} \cup \{0\}$ be integrable, nondecreasing on $[a, \eta]$ and nonincreasing on $(\eta, b]$. Then, we have*

$$\int_a^b s(x)\, g(x)\, dx \leq \frac{2\gamma - \eta - a}{2} \int_a^{\eta} g(x)\, dx + \frac{b + \eta - 2\delta}{2} \int_{\eta}^{b} g(x)\, dx \tag{7}$$

where

$$s(x) := \begin{cases} \gamma - x, & if\ \gamma \geq x,\ x \in [a, \eta] \\ x - \delta, & if\ x \geq \delta,\ x \in (\eta, b] \end{cases} .$$

Proof. Let $x \in [a, \eta]$. Since $g(x)$ is nondecreasing on $[a, \eta]$,

$$\int_a^x g(t)\, dt \quad (x \in [a, \eta])$$

is a convex function, whence

$$\begin{aligned} & \int_a^{\eta} (\gamma - x)\, g(x)\, dx \\ \leq\ & (\gamma - \eta) \int_a^{\eta} g(x)\, dx + \frac{1}{\eta - a} \int_a^{\eta} \left[(x - a) \int_a^{\eta} g(t)\, dt \right] dx \\ =\ & \frac{2\gamma - \eta - a}{2} \int_a^{\eta} g(x)\, dx. \end{aligned} \tag{8}$$

Now let $x \in (\eta, b]$. Similarly, we have

$$\int_{\eta}^{b} (x - \delta)\, g(x)\, dx \leq \frac{b + \eta - 2\delta}{2} \int_{\eta}^{b} g(x)\, dx. \tag{9}$$

Inequality (7) follows from inequalities (8) and (9). This completes the proof. □

Now, we are ready to state and prove the main inequalities of Fejér's type.

Theorem 2.2. *Let $f : I \to \mathbb{R}$ satisfy L-Lipschitzian condition and $g : I \to \mathbb{R}^{+} \cup \{0\}$ be integrable, nondecreasing on $\left[a, \frac{a+b}{2}\right]$ and symmetric about $\frac{a+b}{2}$. Then we obtain*
(1) The mapping P_g is α-Lipschitzian on $[0, 1]$.
(2) We have the inequalities:

$$\left| P_g(t) - \int_a^b f(x)\, g(x)\, dx \right| \leq \frac{1 - t}{2} L (b - a) \int_a^{\frac{a+b}{2}} g(x)\, dx, \tag{10}$$

$$\left| P_g(t) - f\left(\frac{a + b}{2}\right) \int_a^b g(x)\, dx \right| \leq \frac{t}{2} L (b - a) \int_a^{\frac{a+b}{2}} g(x)\, dx \tag{11}$$

and

$$\begin{aligned}
&\left|P_g(t) - t\int_a^b f(x)\,g(x)\,dx - (1-t)\,f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\right| \\
\le\ &\frac{t(1-t)}{2}L(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx
\end{aligned} \tag{12}$$

for all $t \in [0,1]$, *where* $\alpha = \frac{L}{2}(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx$ *and* P_g *is defined by* (2) .

Proof. (1) Since $u, v \in [0,1]$ and f satisfies L-Lipschitzian condition, we have

$$\begin{aligned}
&\left|P_g(u) - P_g(v)\right| \\
\le\ &\int_a^b \left|f\left(ux + (1-u)\frac{a+b}{2}\right) - f\left(vx + (1-v)\frac{a+b}{2}\right)\right| g(x)\,dx \\
\le\ &\int_a^b L\,|u-v| \left|\frac{a+b}{2} - x\right| g(x)\,dx = J.
\end{aligned}$$

Now, by using g which is symmetric about $\frac{a+b}{2}$ and Lemma 2.1, we obtain

$$\begin{aligned}
J &= 2L\,|u-v| \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right) g(x)\,dx \\
&\le \frac{L}{2}(b-a)\left(\int_a^{\frac{a+b}{2}} g(x)\,dx\right)|u-v|\,.
\end{aligned}$$

Thus, for all $u, v \in [0,1]$, we get

$$\left|P_g(u) - P_g(v)\right| \le \frac{L}{2}(b-a)\left(\int_a^{\frac{a+b}{2}} g(x)\,dx\right)|u-v|\,, \tag{13}$$

which yields that mapping P_g is α-Lipschitzian on $[0,1]$.

(2) Inequalities (10) and (11) follow from (13) by choosing $u = t$, $v = 1$ and $u = t$, $v = 0$ respectively. Inequality (12) follows by adding t times (10) and $(1-t)$ times (11). This completes the proof. □

Remark 2.3. *In* (13), *let* $u = 1$, $v = 0$ *and* $g(x) \equiv 1$ $(x \in I)$. *Then inequality* (13) *reduces to* (6) *which was proved by Matić and Pečarić [9].*

Theorem 2.2 implies the following corollary which is important in applications for convex function:

Corollary 2.4. *Let* $f : I \to \mathbb{R}$ *be a differentiable convex mapping on* I *and* $L = \sup_{x\in[a,b]} \left|f'(x)\right| < \infty$. *Then we have the inequalities:*

$$0 \le \int_a^b f(x)\,g(x)\,dx - P_g(t) \le \frac{1-t}{2}L(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx, \tag{14}$$

and

$$0 \le P_g(t) - f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \frac{t}{2}L(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx \tag{15}$$

for all $t \in [0,1]$.

Theorem 2.5. *Let f, g and α be defined as Theorem 2.2. Then we have*
(1) The mapping Q_g is α-Lipschitzian on $[0,1]$.
(2) We have the inequalities:

$$\left|Q_g(t) - \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\right| \le \frac{1-t}{2}L(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx, \tag{16}$$

$$\left|Q_g(t) - \int_a^b f(x)g(x)\,dx\right| \le \frac{t}{2}L(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx \tag{17}$$

and

$$\begin{aligned} &\left|Q_g(t) - t\int_a^b f(x)g(x)\,dx - (1-t)\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\right| \\ \le\; & \frac{t(1-t)}{2}L(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx \end{aligned} \tag{18}$$

for all $t\in[0,1]$ and Q_g is defined by (3).

Proof. (1) Since $u,v\in[0,1]$ and f satisfies L-Lipschitzian condition, we have

$$\begin{aligned} &\left|Q_g(u) - Q_g(v)\right| \\ \le\; & \frac{1}{2}\int_a^b \left[\left|f\left(\frac{1+u}{2}a + \frac{1-u}{2}x\right) - f\left(\frac{1+v}{2}a + \frac{1-v}{2}x\right)\right| g\left(\frac{a+x}{2}\right)\right. \\ & \left. + \left|f\left(\frac{1+u}{2}b + \frac{1-u}{2}x\right) - f\left(\frac{1+v}{2}b + \frac{1-v}{2}x\right)\right| g\left(\frac{x+b}{2}\right)\right] dx \\ \le\; & \frac{1}{2}\int_a^b \left[L\left|\frac{u-v}{2}a + \frac{v-u}{2}x\right| g\left(\frac{a+x}{2}\right)\right. \\ & \left. + L\left|\frac{u-v}{2}b + \frac{v-u}{2}x\right| g\left(\frac{x+b}{2}\right)\right] dx \\ =\; & J_1 + J_2. \end{aligned}$$

By utilising g which is symmetric about $\frac{a+b}{2}$ and Lemma 2.1, we obtain

$$\begin{aligned} & J_1 + J_2 \\ =\; & \int_a^{\frac{a+b}{2}} L\left|\frac{u-v}{2}a + \frac{v-u}{2}(2x-a)\right| g(x)\,dx \\ & + \int_{\frac{a+b}{2}}^b L\left|\frac{u-v}{2}b + \frac{v-u}{2}(2x-b)\right| g(x)\,dx \\ =\; & L|u-v|\left(\int_a^{\frac{a+b}{2}} (x-a)g(x)\,dx + \int_{\frac{a+b}{2}}^b (b-x)g(x)\,dx\right) \\ \le\; & \frac{L|u-v|(b-a)}{2}\int_a^{\frac{a+b}{2}} g(x)\,dx. \end{aligned}$$

Thus, for all $u,v\in[0,1]$, we get

$$\left|Q_g(u) - Q_g(v)\right| \le \frac{L}{2}(b-a)\left(\int_a^{\frac{a+b}{2}} g(x)\,dx\right)|u-v|, \tag{19}$$

which yields that mapping Q_g is α-Lipschitzian on $[0,1]$.

(2) Inequalities (16) and (17) follow from (19) by choosing $u = t$, $v = 1$ and $u = t$, $v = 0$ respectively. Inequality (18) follows by adding t times (16) and $(1-t)$ times (17). This completes the proof. □

Remark 2.6. *In* (19), *let* $u = 1$, $v = 0$ *and* $g(x) \equiv 1$ $(x \in I)$. *Then inequality* (19) *reduces to* (5) *which was proved by Dragomir et al. [2].*

Theorem 2.5 implies the following corollary which is important in applications for convex function:

Corollary 2.7. *Let f and L be defined as Corollary 2.4. Then we have the inequalities:*

$$0 \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx - Q_g(t) \leq \frac{1-t}{2}L(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx \tag{20}$$

and

$$0 \leq Q_g(t) - \int_a^b f(x)\,g(x)\,dx \leq \frac{t}{2}L(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx, \tag{21}$$

for all $t \in [0,1]$.

Theorem 2.8. *Let f and g be defined as Theorem 2.2. Then we have*
(1) The mapping G_g is β-Lipschitzian on $[0,1]$.
(2) We have the inequalities:

$$\left|G_g(t) - \int_a^b f(x)\,g(x)\,dx\right| \leq \frac{7L(b-a)}{8}\max\{t, 1-t\}\int_a^{\frac{a+b}{2}} g(x)\,dx, \tag{22}$$

$$\begin{aligned} &\left|G_g(t) - \frac{1}{\int_a^b g(x)\,dx}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)g(x)\,g(y)\,dxdy\right| \\ \leq\ & \frac{7L(b-a)}{16}|2t-1|\int_a^{\frac{a+b}{2}} g(x)\,dx \end{aligned} \tag{23}$$

and

$$\left|G_g(t) - P_g(t)\right| \leq \frac{1-t}{2}L(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx \tag{24}$$

where $\beta = \frac{7L(b-a)}{8}\int_a^{\frac{a+b}{2}} g(x)\,dx$ *and* G_g *is defined by* (4).

Proof. (1) Since $u, v \in [0,1]$ and f satisfies L-Lipschitzian condition, we have

$$\begin{aligned} &\left|G_g(u) - G_g(v)\right| \\ \leq\ & \frac{1}{\int_a^b g(x)\,dx}\int_a^b\int_a^b \left|f(ux+(1-u)y) - f(vx+(1-v)y)\right| g(x)\,g(y)\,dxdy \\ \leq\ & \frac{1}{\int_a^b g(x)\,dx}L|u-v|\int_a^b\int_a^b \left|y-x\right| g(x)\,g(y)\,dxdy. \end{aligned}$$

The double integral in the inequality above can be illustrated as follows:

$$\begin{aligned}
&\int_a^b\int_a^b |y-x|\, g(x)\, g(y)\, dxdy\\
=\;& \int_a^{\frac{a+b}{2}} g(y)\int_a^{\frac{a+b}{2}} |y-x|\, g(x)\, dxdy + \int_{\frac{a+b}{2}}^b g(y)\int_a^{\frac{a+b}{2}} (y-x)\, g(x)\, dxdy\\
&+\int_a^{\frac{a+b}{2}} g(y)\int_{\frac{a+b}{2}}^b (x-y)\, g(x)\, dxdy + \int_{\frac{a+b}{2}}^b g(y)\int_{\frac{a+b}{2}}^b |y-x|\, g(x)\, dxdy\\
=\;& J_1+J_2+J_3+J_4.
\end{aligned}$$

Now, by using Lemma 2.1, we have

$$\begin{aligned}
J_1 \;=\;& \int_a^{\frac{a+b}{2}} g(y)\left(\int_a^y (y-x)\, g(x)\, dx + \int_y^{\frac{a+b}{2}} (x-y)\, g(x)\, dx\right) dy \qquad (25)\\
\le\;& \int_a^{\frac{a+b}{2}} g(y)\left(\frac{y-a}{2}\int_a^y g(x)\, dx + \frac{\frac{a+b}{2}-y}{2}\int_y^{\frac{a+b}{2}} g(x)\, dx\right) dy\\
\le\;& \frac{1}{2}\int_a^{\frac{a+b}{2}} g(x)\, dx\\
&\cdot\left(\int_a^{\frac{3a+b}{4}}\left(\frac{a+b}{2}-y\right) g(y)\, dy + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} (y-a)\, g(y)\, dy\right)\\
\le\;& \frac{3}{8}(b-a)\int_a^{\frac{a+b}{2}} g(x)\, dx\cdot\int_a^{\frac{a+b}{2}} g(x)\, dx,
\end{aligned}$$

$$\begin{aligned}
J_2 \;\le\;& \int_a^{\frac{a+b}{2}} g(x)\, dx\int_{\frac{a+b}{2}}^b \left(y-\frac{3a+b}{4}\right) g(y)\, dy \qquad (26)\\
\le\;& \frac{b-a}{2}\int_a^{\frac{a+b}{2}} g(x)\, dx\cdot\int_{\frac{a+b}{2}}^b g(x)\, dx,
\end{aligned}$$

$$\begin{aligned}
J_3 \;\le\;& \int_{\frac{a+b}{2}}^b g(x)\, dx\int_a^{\frac{a+b}{2}} \left(\frac{a+3b}{4}-y\right) g(y)\, dy \qquad (27)\\
\le\;& \frac{b-a}{2}\int_{\frac{a+b}{2}}^b g(x)\, dx\cdot\int_a^{\frac{a+b}{2}} g(x)\, dx
\end{aligned}$$

and

$$\begin{aligned}
J_4 \;=\;& \int_{\frac{a+b}{2}}^b g(y)\left(\int_{\frac{a+b}{2}}^y (y-x)\, g(x)\, dx + \int_y^b (x-y)\, g(x)\, dx\right) dy \qquad (28)\\
\le\;& \int_{\frac{a+b}{2}}^b g(y)\left(\frac{y-\frac{a+b}{2}}{2}\int_{\frac{a+b}{2}}^y g(x)\, dx + \frac{b-y}{2}\int_y^b g(x)\, dx\right) dy
\end{aligned}$$

$$\leq \frac{1}{2}\int_{\frac{a+b}{2}}^{b} g(x)\,dx \cdot \left(\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} (b-y)\,g(y)\,dy + \int_{\frac{a+3b}{4}}^{b}\left(y-\frac{a+b}{2}\right)g(y)\,dy\right)$$
$$\leq \frac{3}{8}(b-a)\int_{\frac{a+b}{2}}^{b} g(x)\,dx \cdot \int_{\frac{a+b}{2}}^{b} g(x)\,dx.$$

Then, by using g which is symmetric about $\frac{a+b}{2}$ and inequalities (25)-(28), we obtain:

$$J_1+J_2+J_3+J_4 \leq \frac{7(b-a)}{8}\left(\int_a^{\frac{a+b}{2}} g(x)\,dx\right)^2.$$

Thus, for all $u, v \in [0,1]$, we get

$$\left|G_g(u)-G_g(v)\right| \leq \frac{7L(b-a)}{8}\left(\int_a^{\frac{a+b}{2}} g(x)\,dx\right)|u-v| \tag{29}$$

which yields that mapping G_g is β-Lipschitzian on $[0,1]$.

(2) Inequalities (22) and (23) follow from (29) by choosing $u=t$, $v=1$ or 0 and $u=t$, $v=\frac{1}{2}$ respectively. Now, we shall prove inequality (24). Since f is L-Lipschitzian, we can write

$$\left|f(tx+(1-t)y)-f\left(tx+(1-t)\frac{a+b}{2}\right)\right| \leq L(1-t)\left|y-\frac{a+b}{2}\right| \tag{30}$$

for all $t\in[0,1]$ and $x, y \in I$. By multipling inequality (30) by $g(x)\,g(y)$ and then integrating the result on $I\times I$, we have

$$\left|\int_a^b\int_a^b f(tx+(1-t)y)\,g(x)\,g(y)\,dxdy - \int_a^b g(y)\,dy\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)g(x)\,dx\right|$$
$$\leq L(1-t)\int_a^b g(x)\,dx\int_a^b\left|y-\frac{a+b}{2}\right|g(y)\,dy.$$

Note that, by Lemma 2.1, we get

$$\int_a^b\left|y-\frac{a+b}{2}\right|g(y)\,dy \leq \frac{b-a}{2}\int_a^{\frac{a+b}{2}} g(y)\,dy.$$

Therefore, we can get

$$\left|G_g(t)-P_g(t)\right| \leq \frac{1-t}{2}L(b-a)\int_a^{\frac{a+b}{2}} g(x)\,dx$$

for all $t\in[0,1]$ which proves inequality (24). This completes the proof. □

Theorem 2.8 implies the following corollary which is important in applications for convex function:

Corollary 2.9. *Let f and L be defined as Corollary 2.4. Then we have the inequalities:*

$$0 \leq \int_a^b f(x) g(x) dx - G_g(t) \leq \frac{7L(b-a)}{8} \max\{t, 1-t\} \int_a^{\frac{a+b}{2}} g(x) dx, \tag{31}$$

$$\begin{aligned} 0 &\leq G_g(t) - \frac{1}{\int_a^b g(x) dx} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) g(x) g(y) dxdy \\ &\leq \frac{7L(b-a)}{16} |2t-1| \int_a^{\frac{a+b}{2}} g(x) dx \end{aligned} \tag{32}$$

and

$$0 \leq G_g(t) - P_g(t) \leq \frac{1-t}{2} L(b-a) \int_a^{\frac{a+b}{2}} g(x) dx \tag{33}$$

for all $t \in [0,1]$.

3. Applications

In this section, we apply our results to γ-th moment of a random variable and Euler's Beta function.

3.1. The Applicaions to γ-th Moment of a Random Variable

Throughout this subsection, let $0 < a < b$, $\gamma \geq 0$ and let X be a continuous random variable having the continuous probability density function $g : I \to \mathbb{R}^+ \cup \{0\}$ which is nondecreasing on $\left[a, \frac{a+b}{2}\right]$, symmetric about $\frac{a+b}{2}$, and then γ-th moment of X about the origin is defined as follows.

$$E_\gamma(X) = \int_a^b u^\gamma g(u) du$$

which is assumed to be finite.

To prove the results of this section, we need the following lemma [6, Lemma 1].

Lemma 3.1. *Let $f : I \to \mathbb{R}$ be differentiable with $\|f'\|_\infty < \infty$. Then f is L-Lipschitzian function on $[a,b]$ where $L = \|f'\|_\infty$.*

Now, we present some applications of our result to γ-th moment of a random variable.

Proposition 3.2. *The inequality*

$$\left| E_\gamma(X) - (1-\delta)\left(\frac{a+b}{2}\right)^\gamma - \delta\left(\frac{a^\gamma + b^\gamma}{2}\right) \right| \leq \frac{\gamma b^{\gamma-1}}{4}(b-a) \tag{34}$$

holds, for $\gamma \geq 0$, $0 \leq \delta \leq 1$ and $0 < a < b$.

Proof. Let $f(x) = x^\gamma \ (x \in I, \gamma \geq 0)$, we observe that $\|f'\|_\infty = \gamma b^{\gamma-1}$ in Lemma 3.1. Since

$$\int_a^b f(x) g(x) dx = E_\gamma(X), \quad \int_a^b g(x) dx = 1$$

and

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx = \left(\frac{a+b}{2}\right)^\gamma, \quad \frac{f(a)+f(b)}{2} \int_a^b g(x) dx = \frac{a^\gamma + b^\gamma}{2}.$$

The result follows immediately from inequalities (13) and (19). □

Remark 3.3. *In Proposition 3.2, let $\delta = 0$ and $\delta = 1$. Then we have*

$$\left|E_\gamma(X) - \left(\frac{a+b}{2}\right)^\gamma\right| \le \frac{\gamma b^{\gamma-1}}{4}(b-a) \tag{35}$$

and

$$\left|E_\gamma(X) - \frac{a^\gamma + b^\gamma}{2}\right| \le \frac{\gamma b^{\gamma-1}}{4}(b-a) \tag{36}$$

holds, for $\gamma \ge 0$, $0 < a < b$.

Remark 3.4. *In Proposition 3.2, let $\delta = 1/2$ and $\delta = 1/3$. Then we have*

$$\left|E_\gamma(X) - \frac{1}{2}\left[\frac{a^\gamma + b^\gamma}{2} + \left(\frac{a+b}{2}\right)^\gamma\right]\right| \le \frac{\gamma b^{\gamma-1}}{4}(b-a) \tag{37}$$

$$\left|E_\gamma(X) - \frac{1}{6}\left[a^\gamma + 4\left(\frac{a+b}{2}\right)^\gamma + b^\gamma\right]\right| \le \frac{\gamma b^{\gamma-1}}{4}(b-a) \tag{38}$$

holds, for $\gamma \ge 0$, $0 < a < b$.

Proposition 3.5. *The inequality*

$$\left|E_\gamma(X) - \int_a^b \int_a^b \left(\frac{x+y}{2}\right)^\gamma g(x)\, g(y)\, dxdy\right| \le \frac{7\gamma b^{\gamma-1}}{32}(b-a) \tag{39}$$

holds, for $\gamma \ge 0$, $0 < a < b$.

Proof. The proof is similar to Proposition 3.2 by using inequality (29). □

3.2. The Applicaions to Euler's Beta Function

Throughout this subsection, let us recall the *Beta function* of Euler, that is

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\, dt \quad (p,q > -1).$$

We have for $r \ge 1$, that

$$B(p+r,p) := \int_0^1 t^{p-1}(1-t)^{p-1}\, t^r dt. \tag{40}$$

In [3], for $r \ge 1$, $t \in [0,1]$, that

$$H_B^{[r]}(t,p) := \frac{1}{B(p,p)} \int_0^1 u^{p-1}(1-u)^{p-1}\left[tu + \frac{1}{2}(1-t)\right]^r du \tag{41}$$

and

$$F_B^{[r]}(t,p) := \frac{1}{B^2(p,p)} \int_0^1 \int_0^1 u^{p-1} v^{p-1}(1-u)^{p-1}(1-v)^{p-1}\left[tu + (1-t)v\right]^r dudv. \tag{42}$$

Define $g : (0,1) \to \mathbb{R}$ given by

$$g(t) := \frac{1}{B(p,p)} t^{p-1}(1-t)^{p-1}.$$

It is clear that

$$g(t) = g(1-t) \text{ for all } t \in (0,1)$$

and

$$\int_0^1 g(t)\,dt = 1.$$

Based on (40), (41) and (42), we can obtain the following propositions of Beta function:

Proposition 3.6. *The inequality*

$$\left|\frac{B(p+r,p)}{B(p,p)} - \frac{1-\delta}{2^r} - \frac{\delta}{2}\right| \le \frac{r}{4} \tag{43}$$

holds, for all $r \ge 1$, $0 \le \delta \le 1$ *and* $p > -1$.

Proof. Define $f : [0,1] \to \mathbb{R}$ given by $f(t) := t^r$, $r \ge 1$, we observe that $\|f'\|_\infty = r$ in Lemma 3.1. Since

$$\int_a^b f(t)\,g(t)\,dt = \frac{1}{B(p,p)}\int_0^1 t^{p-1}(1-t)^{p-1}\,t^r dt = \frac{B(p+r,p)}{B(p,p)}.$$

The result follows immediately from inequalities (13) and (19). □

Proposition 3.7. *The inequality*

$$\left|H_B^{[r]}(t,p) - t\frac{B(p+r,p)}{B(p,p)} - (1-t)\frac{1}{2^r}\right| \le \frac{rt(1-t)}{4} \tag{44}$$

holds, for all, $r \ge 1$, $0 \le t \le 1$ *and* $p > -1$.

Proof. The proof is similar to Proposition 3.6 by using inequality (12). □

By using inequalities (22) and (24), we can state the following propositions.

Proposition 3.8. *The inequality*

$$\left|F_B^{[r]}(t,p) - \frac{B(p+r,p)}{B(p,p)}\right| \le \frac{7r}{16}\max\{t, 1-t\} \tag{45}$$

holds, for all, $r \ge 1$, $0 \le t \le 1$ *and* $p > -1$.

Proposition 3.9. *The inequality*

$$\left|F_B^{[r]}(t,p) - H_B^{[r]}(t,p)\right| \le \frac{r(1-t)}{4} \tag{46}$$

holds, for all, $r \ge 1$, $0 \le t \le 1$ *and* $p > -1$.

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