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Warped Product Submanifolds of Kenmotsu Manifolds with Slant Fiber

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Abstract. Recently, we have discussed the warped product pseudo-slant submanifolds of the type $M_\theta \times_f M_\perp$ of Kenmotsu manifolds. In this paper, we study other type of warped product pseudo-slant submanifolds by reversing these two factors in Kenmotsu manifolds. The existence of such warped product immersions is proved by a characterization. Also, we provide an example of warped product pseudo-slant submanifolds. Finally, we establish a sharp estimation such as $||h||^2 \geq 2p\cos^2\theta\left(||\vec{\nabla}(\ln f)||^2 - 1\right)$ for the squared norm of the second fundamental form $||h||^2$, in terms of the warping function f, where $\vec{\nabla}(\ln f)$ is the gradient vector of the function $\ln f$. The equality case is also discussed.

1. Introduction

In 1972, Kenmotsu [19] introduced a new class of almost contact Riemannian manifolds which are known as Kenmotsu manifolds. It is well known that odd dimensional hyperbolic spaces admit Kenmotsu structures. Kenmotsu manifolds are locally isometric to warped product spaces with one dimensional base and Kaehler fiber.

On the other hand, B.-Y. Chen introduced the notion of warped product submanifolds in [10, 11]. The study of warped product submanifolds got momentum after Chen's papers and several articles appeared on warped product submanifolds in different structure of manifolds (for instance, see [3], [17], [22], [23], [25], [27], [30]). For the survey on warped product submanifolds we refers to [12–14, 16].

Next, pseudo-slant submanifolds of almost contact metric manifolds were studied by Carriazo in [8]. The warped products of these submanifolds were studied by Sahin under the name of hemi-slant warped product submanifolds of Kaehler manifolds [26]. Later, we extended this idea for cosymplectic manifolds [30].

Recently, we have studied warped product pseudo-slant submanifolds of the type $M_\theta \times_f M_\perp$ of a Kenmotsu manifold $\tilde{M}$, where M_θ and $M_\perp$ are proper slant and anti-invariant submanifolds of $\tilde{M}$, respectively. We derived an inequality for the squared norm of the second fundamental form in terms of the warping function. Also, the warped product submanifolds of Kenmotsu manifolds were studied in ([2], [3, 4], [20, 21], [24], [1], [29]) and references therein.

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In this paper, we study warped product submanifolds of the type $M_\perp \times_f M_\theta$ of a Kenmotsu manifold $\tilde{M}$, where $M_\perp$ and M_θ are anti-invariant and proper slant submanifold of $\tilde{M}$, respectively. The paper is organized as follows: In Section 2, we give some preliminaries formulas which we will use later. Section 3 is devoted to study of warped product pseudo-slant submanifolds of Kenmotsu manifolds and we prove the existence of warped pseudo-slant submanifolds with an example and a characterzation. In Section 4, we establish an inequality for the squared norm of second fundamental form in terms of the warping function and the slant angle. The equality case is also considered.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $\tilde{M}$ is said to be an *almost contact metric manifold* [6] if it admits a $(1,1)$ tensor field φ, a structure vector field ξ, a 1-form η and a Riemannian metric g, which satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \;\; \varphi\xi = 0, \;\; \eta \circ \varphi = 0, \;\; \eta(\xi) = 1, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \;\; \eta(X) = g(X, \xi), \tag{2}$$

for any vector fields X, Y on $\tilde{M}$. In addition, if

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \;\; \tilde{\nabla}_X \xi = X - \eta(X)\xi \tag{3}$$

where $\tilde{\nabla}$ denotes the operator of covariant differentiation with respect to g, then $(\tilde{M}, \varphi, \xi, \eta, g)$ is called a Kenmotsu manifold [19]. The covariant derivative of φ is defined as

$$(\tilde{\nabla}_X \varphi)Y = \tilde{\nabla}_X \varphi Y - \varphi \tilde{\nabla}_X Y \tag{4}$$

for any vector fields X, Y on $\tilde{M}$.

Let M be a Riemannian manifold isometrically immersed in $\tilde{M}$ and denoted by the same symbol g for the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(TM^\perp)$ the set of all vector fields normal to M, same notation for smooth sections of any other vector bundle $\mathfrak{E}$. Denotes by ∇ the Levi-Civita connection of M. Then the Gauss and Weingarten formulas are respectively given by

$$(a) \;\; \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \;\; (b) \;\; \tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{5}$$

for any vector fields $X, \; Y \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$, where $\nabla^\perp$ is the connection in the normal bundle $TM^\perp$, h is the second fundamental form of M and A_V is the Weingarten endomorphism associated with V. The second fundamental form h and the shape operator A are related by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{6}$$

For any $X \in \Gamma(TM)$, we write

$$\varphi X = TX + FX, \tag{7}$$

where TX is the tangential component of φX and FX is the normal component of φX. Similarly, for any vector field V normal to M, we put

$$\varphi V = BV + CV, \tag{8}$$

where BV and CV are the tangential and the normal components of φV, respectively.

Invariant and anti-invariant submanifolds are depend on the behavior of almost contact structure. A submanifold M tangent to the structure vector field ξ is said to be *invariant* (resp. *anti-invariant*) if $\varphi(T_pM) \subseteq T_pM, \; \forall \, p \in M$ (resp. $\varphi(T_pM) \subseteq T_pM^\perp, \; \forall \, p \in M$).

It is clear that if TX (resp. FX) is identically zero in (7), then M is an *anti-invariant* (resp. *invariant*) submanifold of a contact Riemannian manifold $\tilde{M}$.

We denote by H, the mean curvature vector defined as $H(p) = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$, where $\{e_1, \cdots, e_m\}$ is an orthonormal basis of the tangent space T_pM, for any $p \in M$.

Also, we set

$$h^r_{ij} = g(h(e_i, e_j), e_r) \text{ and } \|h\|^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)), \tag{9}$$

for $i, j = 1, \cdots, m$ and $r = m+1, \cdots, 2n+1$.

A submanifold M of a Riemannian manifold $\tilde{M}$ is said to be *totally umbilical* if $h(X, Y) = g(X, Y)H$ and *totally geodesic* if $h(X, Y) = 0$, for all $X, Y \in \Gamma(TM)$. Also, M is minimal in $\tilde{M}$, if $H = 0$.

There are some other classes of submanifolds of almost contact Riemannian manifolds which we define here:

1. A submanifold M tangent to ξ is said to be a *contact CR-submanifold* if there exists a pair of orthogonal distributions $\mathfrak{D} : p \to \mathfrak{D}_p$ and $\mathfrak{D}^\perp : p \to \mathfrak{D}^\perp_p$, $\forall\, p \in M$ such that
   (i) $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \langle\xi\rangle$, where $\langle\xi\rangle$ is a 1-dimensional distribution spanned by ξ.
   (ii) $\mathfrak{D}$ is invariant by φ, i.e., $\varphi\mathfrak{D} = \mathfrak{D}$
   (iii) $\mathfrak{D}^\perp$ is anti-invariant by φ, i.e., $\varphi\mathfrak{D}^\perp \subseteq TM^\perp$.
2. A submanifold M is called slant [7] if for each non-zero vector X tangent to M the angle $\theta(X)$ between φX and T_pM is a constant, i.e, it does not depend on the choice of $p \in M$ and $X \in T_pM - \langle\xi_p\rangle$.

For a slant submanifold M, if $\theta = 0$, then M is invariant and if $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

Now, we have the following characterization for a slant submanifold of an almost contact metric manifold.

Theorem 2.1. *[7] Let M be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in \Gamma(TM)$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi).$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2\theta$.

Following relations are straightforward consequence of the above theorem

$$g(TX, TY) = \cos^2\theta[g(X, Y) - \eta(X)\eta(Y)] \tag{10}$$

$$g(FX, FY) = \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)] \tag{11}$$

for any X, Y tangent to M.

We also have the following useful result for a slant submanifolds almost contact metric manifolds.

Theorem 2.2. *[33] Let M be a proper slant submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in \Gamma(TM)$. Then*

$$(a)\ BFX = \sin^2\theta(-X + \eta(X)\xi), \quad (b)\ CFX = -FTX$$

for any $X \in \Gamma(TM)$.

In [8], Carriazo introduced another class of submanifolds known as pseudo-slant (anti-slant) submanifolds which are the generalizations of slant and contact CR-submanifolds. He defined these submanifolds as follows:

Definition 2.3. *A Riemannian manifold M isometrically immersed in an almost contact manifold $\tilde{M}$ is said to be a pseudo-slant submanifold if there exists a pair of orthogonal distributions $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ such that $TM = \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta \oplus \langle\xi\rangle$, the distribution $\mathfrak{D}^\perp$ is anti-invariant i.e., $\varphi(\mathfrak{D}^\perp) \subseteq TM^\perp$ and the distribution $\mathfrak{D}^\theta$ is proper slant with slant angle $\theta \neq 0$.*

If we denote the dimension of $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ by q and p, respectively then it is clear that contact CR-submanifolds and slant submanifolds are particular classes of pseudo-slant submanifolds with slant angle $\theta = 0$ and $q = 0$, respectively. Also, the invariant (resp. anti-invariant) submanifold is a pseudo-slant submanifold with slant angle $\theta = 0$ and $q = 0$ (resp. $p = 0$). A pseudo-slant submanifold M is *proper pseudo-slant* if neither $q = 0$ nor $\theta = 0$ or $\frac{\pi}{2}$.

The normal bundle $TM^\perp$ of a pseudo-slant submanifold M is decomposed as

$$TM^\perp = \varphi\mathfrak{D}^\perp \oplus F\mathfrak{D}^\theta \oplus \nu \tag{12}$$

where ν is an invariant normal subbundle of $TM^\perp$.

3. Warped product pseudo-slant submanifolds

In [5], Bishop and O'Neill introduced the notion of warped products to study manifolds with negative curvature. They defined these manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y)$$

for any vector fields X, Y tangent to M, where $\star$ is the symbol for the tangent maps. On a warped product manifold $M = M_1 \times_f M_2$, M_1 is the base manifold and M_2 is the fiber. A warped product manifold is said to be *trivial* or simply a *Riemannian product manifold* if the warping function f is constant.

Now, we recall the following general result for a warped product manifold for later use.

Lemma 3.1. *[5] Let $M = M_1 \times_f M_2$ be a warped product manifold. Then*

(i) $\nabla_X Y \in TM_1$ *is the lift of* $\nabla_X Y$ *on* M_1

(ii) $\nabla_X Z = \nabla_Z X = X(\ln f)Z$

(iii) $\nabla_Z W = \nabla^{M_2}_Z W - g(Z, W)\vec{\nabla}(\ln f)$

or each X, $Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$ where ∇ and ∇^{M_2} denote the Levi-Civita connections on M and M_2, respectively, and $\vec{\nabla}(\ln f)$ is the gradient of $\ln f$.

From the above lemma it is clear that if $M = M_1 \times_f M_2$ be a warped product manifold, then M_1 is a totally geodesic submanifold of M and M_2 is a totally umbilical submanifold of M.

Let M be a Riemannian manifold of dimension k with the inner product g and $\{e_1, \cdots, e_k\}$ be an orthonormal frame on M. Then for a differentiable function f on M, the gradient $\vec{\nabla} f$ of a function f on M is defined by

$$g(\vec{\nabla} f, X) = X(f), \tag{13}$$

for any $X \in \Gamma(TM)$. As a consequence, we have

$$\|\vec{\nabla} f\|^2 = \sum_{i=1}^{k} (e_i(f))^2 \tag{14}$$

where $\vec{\nabla} f$ is the gradient of the function f on M.

In [1], we studied warped product submanifolds of the form $M_\theta \times_f M_\perp$ of a Kenmotsu manifold $\tilde{M}$, where M_θ and $M_\perp$ are proper slant and anti-invariant submanifolds of $\tilde{M}$, respectively. In this paper, we study warped product submanifolds of the form $M_\perp \times_f M_\theta$ of a Kenmotsu manifold and We call them warped product pseudo-slant submanifolds. For these types of warped products we have two possibilities that either the structure vector field ξ is tangential to M_θ or ξ is tangential to $M_\perp$. When ξ is tangent to M_θ, then it is easy to show that the warped product is trivial [1]. Therefore, throughout the paper, we consider the structure vector field ξ is tangent to $M_\perp$.

First, we give the following non trivial example of warped product pseudo-slant submanifolds.

Example 3.2. Consider a submanifold of $\mathbb{R}^7$ with the cartesian coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, t)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = 0, \quad 1 \leq i, j \leq 3.$$

It is easy to show $\mathbb{R}^7$ is an almost contact metric manifold with respect to the Euclidean metric tensor of $\mathbb{R}^7$. Let us consider a submanifold M of $\mathbb{R}^7$ defined by the immersion χ as follows

$$\chi(u_1, u_2, u_3, t) = (u_1 \cos u_3, u_2 \cos u_3, u_1 + u_2, u_1 - u_2, u_1 \sin u_3, u_2 \sin u_3, t).$$

Then the tangent space of M is spanned by vectors

$$\begin{aligned}
Z_1 &= \cos u_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2} + \sin u_3 \frac{\partial}{\partial x_3}, \\
Z_2 &= \cos u_3 \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2} + \sin u_3 \frac{\partial}{\partial y_3}, \\
Z_3 &= -u_1 \sin u_3 \frac{\partial}{\partial x_1} - u_2 \sin u_3 \frac{\partial}{\partial y_1} + u_1 \cos u_3 \frac{\partial}{\partial x_3} + u_2 \cos u_3 \frac{\partial}{\partial y_3}; \;\; Z_4 = \frac{\partial}{\partial t}.
\end{aligned}$$

Then, we find

$$\begin{aligned}
\varphi Z_1 &= -\cos u_3 \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} + \frac{\partial}{\partial x_2} - \sin u_3 \frac{\partial}{\partial y_3}, \\
\varphi Z_2 &= \cos u_3 \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2} + \sin u_3 \frac{\partial}{\partial x_3}, \\
\varphi Z_3 &= u_1 \sin u_3 \frac{\partial}{\partial y_1} - u_2 \sin u_3 \frac{\partial}{\partial x_1} - u_1 \cos u_3 \frac{\partial}{\partial y_3} + u_2 \cos u_3 \frac{\partial}{\partial x_3}; \;\; \varphi Z_4 = 0.
\end{aligned}$$

It is clear that φZ_3 is orthogonal to TM. Therefore, the anti-invariant distribution is $\mathfrak{D}^\perp = span\{Z_3\}$ and $\mathfrak{D}^\theta = span\{Z_1, Z_2\}$ is a proper slant distribution with slant angle $\theta = \arccos(\frac{1}{3}) = 70°52'$ such that $\xi = \frac{\partial}{\partial t}$ is tangent to $\mathfrak{D}^\theta$. Thus M is a proper pseudo-slant submanifold such that $\xi = Z_4$ is tangent to M. It is easy to observe that both the distributions are integrable. If we denote the integral manifolds of $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ by $M_\perp$ and M_θ, respectively then the metric tensor g of M is given by

$$g = 3(du_1^2 + du_2^2) + dt^2 + (u_1^2 + u_2^2)du_3^2.$$

Thus M is a warped product pseudo-slant submanifold $M = M_\theta \times_f M_\perp$ with the warping function $f = \sqrt{u_1^2 + u_2^2}$.

Now, we have the following results which are useful to prove the main theorem of this section.

Lemma 3.3. *Let $M = M_\perp \times {}_f M_\theta$ be a warped product pseudo-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_\perp)$. Then, we have*

(i) $\xi(\ln f) = 1$;
(ii) $g(h(Z,W), FX) = g(h(X,Z), \varphi W)$

for any $X \in \Gamma(TM_\theta)$ and $Z, W \in \Gamma(TM_\perp)$.

Proof. For any $X \in \Gamma(TM_\theta)$ and $\xi \in \Gamma(TM_\perp)$, we have $\tilde{\nabla}_X \xi = X$. Then using (5) (a) and Lemma 3.1 (ii), we get $\xi(\ln f)X = X$ which implies that $\xi(\ln f) = 1$, for any non-zero vector field $X \in \Gamma(TM_\theta)$, which proves (i). For the second part of the Lemma, consider any $X \in \Gamma(TM_\theta)$ and any $Z, W \in \Gamma(TM_\perp)$, then

$$g(h(Z,W), FX) = g(\tilde{\nabla}_Z W, \varphi X) - g(\nabla_Z W, TX) = g((\tilde{\nabla}_Z \varphi)W, X) - g(\tilde{\nabla}_Z \varphi W, X) + g(\nabla_Z TX, W).$$

First and the last terms in the right hand side of above relation are identically zero by using (3) and Lemma 3.1 (ii). Thus from (5) (b) and (6), we obtain $g(h(Z,W), FX) = g(h(X,Z), \varphi W)$, which is (ii). Hence, the proof is complete. □

Lemma 3.4. *Let $M = M_\perp \times_f M_\theta$ be a warped product submanifold of a Kenmotsu manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_\perp)$, where $M_\perp$ and M_θ are anti-invariant and proper slant submanifolds of $\tilde{M}$, respectively. Then, we have*

(i) $g(h(X,Y), \varphi Z) = \big(\eta(Z) - Z(\ln f)\big) g(TX, Y) + g(h(X,Z), FY)$
(ii) $g(h(TX,Y), \varphi Z) = \cos^2\theta \big(Z(\ln f) - \eta(Z)\big) g(X,Y) + g(h(TX,Z), FY)$

for any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. For any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we have

$$g(h(X,Z), FY) = g(\tilde{\nabla}_X Z, \varphi Y) - g(\tilde{\nabla}_X Z, TY) = g((\tilde{\nabla}_X \varphi)Z, Y) - g(\tilde{\nabla}_X \varphi Z, Y) - g(\nabla_X Z, TY).$$

Using (3), (5) (b), (6) and Lemma 3.1 (ii), we obtain

$$g(h(X,Z), FY) = -\eta(Z) g(TX, Y) + g(h(X,Y), \varphi Z) + Z(\ln f) g(TX, Y)$$

which proves (i). For the second part of the lemma, if we interchange X by TX in (i) and use Theorem 2.1, then we get (ii), which proves the lemma completely. □

Also, if we interchange the vector field Y by TY in Lemma 3.4 (i)-(ii), for any $Y \in \Gamma(TM_\theta)$, then we have the following relations.

$$g(h(X,TY), \varphi Z) = \cos^2\theta \big(\eta(Z) - Z(\ln f)\big) g(X,Y) + g(h(X,Z), FTY) \tag{15}$$

and

$$g(h(TX,TY), \varphi Z) = \cos^2\theta \big(\eta(Z) - Z(\ln f)\big) g(TX,Y) + g(h(TX,Z), FTY). \tag{16}$$

A warped product submanifold $M = M_1 \times_f M_2$ of a Kenmotsu manifold $\tilde{M}$ is said to be *mixed totally geodesic*, if $h(X,Z) = 0$, for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, where M_1 and M_2 are Riemannian submanifolds of $\tilde{M}$. Now, we give the following characterization for a mixed totally geodesic warped product submanifold by using a result of [18].

Theorem 3.5. *Let M be a pseudo-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that ξ is orthogonal to slant distribution $\mathfrak{D}^\theta$. Then M is locally a mixed totally geodesic warped product submanifold if and only if*

$$A_{FX} Z = 0 \;\; \text{and} \;\; A_{\varphi Z} TX = \cos^2\theta \{\eta(Z) - Z(\mu)\} X \tag{17}$$

for any $Z \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$ and $X \in \Gamma(\mathfrak{D}^\theta)$ for some smooth function μ on M such that $Y(\mu) = 0$, for any $Y \in \Gamma(\mathfrak{D}^\theta)$.

Proof. Let $M = M_\perp \times_f M_\theta$ be a mixed totally geodesic warped product psedo-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then, for any $X, Y \in \Gamma(TM_\theta$ and $Z, W \in \Gamma(TM_\perp)$, we have $g(A_{FX}Z, Y) = g(h(Y, Z), FX) = 0$, i.e., $A_{FX}Z$ has no component in $\Gamma(TM_\theta)$. Also, from Lemma 3.3 (ii), we have $A_{FX}Z$ has no component in $\Gamma(TM_\perp)$ too, therefore $A_{FX}Z = 0$, which is first relation of (17). Similarly, $g(A_{\varphi Z}TX, W) = g(h(TX, W), \varphi Z) = 0$, i.e., $A_{\varphi Z}TX$ has no component in $\Gamma(TM_\perp)$. Then second relation of (17) follows from Lemma 3.4 (ii).

Conversely, if M is a pseudo-slant submanifold of a Kenmotsu manifold $\tilde{M}$ with anti-invariant and proper slant distributions $\mathfrak{D}^\perp \oplus \langle\xi\rangle$ and $\mathfrak{D}^\theta$, respectively such that (17) holds, then for any $Z, W \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$ and $X \in \Gamma(\mathfrak{D}^\theta)$, we have

$$g(\nabla_Z W, X) = g(\tilde{\nabla}_Z W, X) = g(\varphi\tilde{\nabla}_Z W, \varphi X).$$

Using (4), we derive

$$g(\nabla_Z W, X) = g(\tilde{\nabla}_Z \varphi W, \varphi X) - g((\tilde{\nabla}_Z \varphi)W, \varphi X).$$

Then from (3) and the orthogonality of vector fields, we obtain

$$g(\nabla_Z W, X) = -g(\varphi W, \tilde{\nabla}_Z TX) - g(\varphi W, \tilde{\nabla}_Z FX) = -g(h(Z, TX), \varphi W) + g(W, \varphi\tilde{\nabla}_Z FX).$$

Using (4) and (6), we arrive at

$$g(\nabla_Z W, X) = -g(A_{\varphi W}TX, Z) + g(\tilde{\nabla}_Z \varphi FX, W) - g((\tilde{\nabla}_Z \varphi)FX, W).$$

The first term in the right hand side is identically zero by using (17) and the orthogonality of vector fields. Thus, from (3) and (8), we get

$$g(\nabla_Z W, X) = g(\tilde{\nabla}_Z BFX, W) + g(\tilde{\nabla}_Z CFX, W).$$

By using Theorem 2.2, we find

$$g(\nabla_Z W, X) = -\sin^2\theta g(\tilde{\nabla}_Z X, W) - g(\tilde{\nabla}_Z FTX, W) = \sin^2\theta g(\tilde{\nabla}_Z W, X) - g(A_{FTX}W, Z).$$

Again using (5) and (17), we obtain

$$\cos^2\theta g(\nabla_Z W, X) = 0. \tag{18}$$

Since M is a proper pseudo-slant submanifold, therefore $\cos^2\theta \neq 0$ and hence from (18), we conclude that $\nabla_Z W \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$ i.e., the leaves of the distribution $\mathfrak{D}^\perp \oplus \langle\xi\rangle$ are totally geodesic in M. On the other hand, for any $X, Y \in \Gamma(\mathfrak{D}^\theta)$ and $Z \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$ we have

$$g(\tilde{\nabla}_X Y, Z) = g(\varphi\tilde{\nabla}_X Y, \varphi Z) + \eta(Z)g(\tilde{\nabla}_X Y, \xi) = g(\tilde{\nabla}_X \varphi Y, \varphi Z) - g((\tilde{\nabla}_X \varphi)Y, \varphi Z) - \eta(Z)g(Y, \tilde{\nabla}_X \xi).$$

Using (3) and (7), we obtain

$$g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X TY, \varphi Z) + g(\tilde{\nabla}_X FY, \varphi Z) - \eta(Z)g(X, Y) = g(h(X, TY), \varphi Z) - g(\varphi\tilde{\nabla}_X FY, Z) - \eta(Z)g(X, Y).$$

Then from (4) and (6), we get

$$g(\tilde{\nabla}_X Y, Z) = g(A_{\varphi Z}TY, X) - g(\tilde{\nabla}_X \varphi FY, Z) + g((\tilde{\nabla}_X \varphi)FY, Z) - \eta(Z)g(X, Y).$$

Hence by (3), (7) and (17), we derive

$$g(\tilde{\nabla}_X Y, Z) = \cos^2\theta\Big(\eta(Z) - (Z\mu)\Big)g(X, Y) - g(\tilde{\nabla}_X BFY, Z) - g(\tilde{\nabla}_X CFY, Z) + \eta(Z)g(FX, FY) - \eta(Z)g(X, Y).$$

From (11) and Theorem 2.2, we find that

$$\begin{aligned} g(\tilde{\nabla}_X Y, Z) &= \cos^2\theta\Big(\eta(Z) - (Z\mu)\Big)g(X,Y) + \sin^2\theta g(\tilde{\nabla}_X Y, Z) + g(\tilde{\nabla}_X FTY, Z) \\ &\quad + \sin^2\theta\eta(Z)g(X,Y) - \eta(Z)g(X,Y) \\ &= \sin^2\theta g(\tilde{\nabla}_X Y, Z) + g(A_{FTY}Z, X) - \cos^2\theta(Z\mu)g(X,Y). \end{aligned}$$

Using (17), we obtain

$$g(\tilde{\nabla}_X Y, Z) = \sin^2\theta g(\tilde{\nabla}_X Y, Z) - \cos^2\theta(Z\mu)g(X,Y). \tag{19}$$

Similarly, we have

$$g(\tilde{\nabla}_Y X, Z) = \sin^2\theta g(\tilde{\nabla}_Y X, Z) - \cos^2\theta(Z\mu)g(X,Y). \tag{20}$$

Then from (19) and (20), we find

$$\cos^2\theta g([X,Y],Z) = 0. \tag{21}$$

For a proper pseudo-slant submanifold $\cos^2\theta \neq 0$ and hence from (21), we conclude that the slant distribution $\mathfrak{D}^\theta$ is integrable on M. If M_θ be a leaf of the integrable distribution $\mathfrak{D}^\theta$ in M and if h^θ is the second fundamental form of M_θ in M, then for any $X, Y \in \Gamma(\mathfrak{D}^\theta)$ and $Z \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$, we have

$$g(h^\theta(X,Y),Z) = g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\varphi\tilde{\nabla}_X Y, \varphi Z) + \eta(Z)g(\tilde{\nabla}_X Y, \xi).$$

Using (4) and the orthogonality of vector fields, we obtain

$$g(h^\theta(X,Y),Z) = g(\tilde{\nabla}_X \varphi Y, \varphi Z) - g((\tilde{\nabla}_X\varphi)Y, \varphi Z) - \eta(Z)g(Y, \tilde{\nabla}_X\xi).$$

Then from (3) and (7), we derive

$$g(h^\theta(X,Y),Z) = g(\tilde{\nabla}_X TY, \varphi Z) + g(\tilde{\nabla}_X FY, \varphi Z) - \eta(Z)g(X,Y) = g(A_{\varphi Z}TY, X) - g(\varphi\tilde{\nabla}_X FY, Z) - \eta(Z)g(X,Y).$$

Again, from (4) and (17), we get

$$g(h^\theta(X,Y),Z) = \cos^2\theta\Big(\eta(Z) - (Z\mu)\Big)g(X,Y) - g(\tilde{\nabla}_X\varphi FY, Z) + g((\tilde{\nabla}_X\varphi)FY, Z) - \eta(Z)g(X,Y).$$

Using (3) and (8), we find that

$$g(h^\theta(X,Y),Z) = -\cos^2\theta(Z\mu)g(X,Y) - \sin^2\theta\eta(Z)g(X,Y) - g(\tilde{\nabla}_X BFY, Z) - g(\tilde{\nabla}_X CFY, Z) + \eta(Z)g(TX, TY).$$

Then from (11) and Theorem 2.2, we arrive at

$$g(h^\theta(X,Y),Z) = -\cos^2\theta(Z\mu)g(X,Y) + \sin^2\theta g(\tilde{\nabla}_X Y, Z) + g(\tilde{\nabla}_X FTY, Z).$$

Using (5) (b) and the symmetry of the shape operator A, we derive

$$\cos^2\theta g(h^\theta(X,Y),Z) = -\cos^2\theta(Z\mu)g(X,Y) - g(A_{FTY}Z, X).$$

Second term in the right hand side of above relation is identically zero by using (17) and thus we have

$$g(h^\theta(X,Y),Z) = -(Z\mu)g(X,Y).$$

From (13), we get

$$h^\theta(X,Y) = -\vec{\nabla}\mu g(X,Y) \tag{22}$$

where $\vec{\nabla}\mu$ is the gradient of the function μ. Thus from (22), we conclude that M_θ is totally umbilical in M with non-vanishing mean curvature vector $H^\theta = -\vec{\nabla}\mu$. Also, we can prove that H^θ is parallel corresponding to the normal connection $D^{\mathfrak{R}}$ of M_θ in M (for instance, see [30]). Thus, M_θ is an extrinsic sphere in M. Hence, by a result of Hiepko [18], we conclude that M is a warped product submanifold, which proves the theorem completely. □

4. An inequality for warped products $M_\perp \times_f M_\theta$

In this section, we establish a sharp inequality for the squared norm of the second fundamental form $||h||^2$, in terms of the gradient of the warping function and the slant angle. First, we construct the following frame fields for a warped product pseudo-slant pseudo-slant submanifold of a Kenmotsu manifold to develop the main result of this section.

Let $M = M_\perp \times_f M_\theta$ be a warped product pseudo-slant submanifold of dimension m of a $(2n+1)$-dimensional Kenmotsu manifold $\tilde{M}$ such that the structure vector field ξ is tangent to $M_\perp$, where $M_\perp$ and M_θ are anti-invariant and proper slant submanifolds of $\tilde{M}$, respectively. Let us consider the $\dim M_\perp = q+1$ and $\dim M_\theta = 2p$ and their tangent bundles by $\mathfrak{D}^\perp \oplus \langle\xi\rangle$ and $\mathfrak{D}^\theta$, respectively. We set the orthonormal frame fields of $\mathfrak{D}^\theta$ and $\mathfrak{D}^\perp \oplus \langle\xi\rangle$, respectively as $\{e_1, \cdots, e_p, e_{p+1} = \sec\theta\, Te_1, \cdots, e_{2p} = \sec\theta\, Te_p\}$ and $\{e_{2p+1} = e_1^*, \cdots, e_{2p+q} = e_q^*, e_m = e_{2p+q+1} = e_{q+1}^* = \xi\}$. Then the orthonormal frames of the normal subbundles $F\mathfrak{D}^\theta$, $\varphi\mathfrak{D}^\perp$ and ν, respectively are $\{e_{m+1} = \tilde{e}_1 = \csc\theta\, Fe_1, \cdots e_{m+p} = \tilde{e}_p = \csc\theta Fe_p, e_{m+p+1} = \tilde{e}_{p+1} = \csc\theta\sec\theta\, FTe_1, \cdots, e_{m+2p} = \tilde{e}_{2p} = \csc\theta\sec\theta\, FTe_p\}$, $\{e_{m+2p+1} = \tilde{e}_{2p+1} = \varphi e_1^*, \cdots, e_{m+2p+q} = \tilde{e}_{2p+q} = \varphi e_q^*\}$ and $\{e_{2m} = \tilde{e}_m, \cdots, e_{2n+1} = \tilde{e}_{2(n+1-m))}\}$. It is clear that the dimensions of the normal subspaces $F\mathfrak{D}^\theta$, $\varphi\mathfrak{D}^\perp$ and ν, respectively are $2p$, q and $2(n-m+1)$.

Theorem 4.1. *Let $M = M_\perp \times_f M_\theta$ be a mixed totally geodesic warped product pseudo-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_\perp)$, where $M_\perp$ and M_θ are anti-invariant and proper slant submanifolds of $\tilde{M}$, respectively. Then, we have:*

(i) The squared norm of the second fundamental form h of M satisfies

$$||h||^2 \geq 2p\cos^2\theta\Big(||\vec{\nabla}(\ln f)||^2 - 1\Big) \tag{23}$$

where $2p = \dim M_\theta$ and $\vec{\nabla}(\ln f)$ is gradient of the function $\ln f$ along $M_\perp$.

(ii) If equality sign in (i) holds identically, then $M_\perp$ is totally geodesic in $\tilde{M}$ and M_θ is a totally umbilical submanifold of $\tilde{M}$.

Proof. From the definition (9), we have

$$||h||^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2.$$

Using the constructed frame, we obtain

$$||h||^2 = \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{q+1} g(h(e_i^*, e_j^*), e_r)^2 + 2\sum_{r=m+1}^{2n+1}\sum_{i=1}^{q+1}\sum_{j=1}^{2p} g(h(e_i^*, e_j), e_r)^2 + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{2p} g(h(e_i, e_j), e_r)^2. \tag{24}$$

Since M is mixed totally geodesic then the second term in the right hand side of (24) is identically zero, thus we find

$$\begin{aligned}
||h||^2 &= \sum_{r=m+1}^{m+2p}\sum_{i,j=1}^{q+1} g(h(e_i^*, e_j^*), e_r)^2 + \sum_{r=m+2p+1}^{m+2p+q}\sum_{i,j=1}^{q+1} g(h(e_i^*, e_j^*), e_r)^2 + \sum_{r=2m}^{2n+1}\sum_{i,j=1}^{q+1} g(h(e_i^*, e_j^*), e_r)^2\\
&+ \sum_{r=m+1}^{m+2p}\sum_{i,j=1}^{2p} g(h(e_i, e_j), e_r)^2 + \sum_{r=m+2p+1}^{m+2p+q}\sum_{i,j=1}^{2p} g(h(e_i, e_j), e_r)^2 + \sum_{r=2m}^{2n+1}\sum_{i,j=1}^{2p} g(h(e_i, e_j), e_r)^2\\
&= \sum_{r=1}^{2p}\sum_{i,j=1}^{q+1} g(h(e_i^*, e_j^*), \tilde{e}_r)^2 + \sum_{r=2p+1}^{2p+q}\sum_{i,j=1}^{q+1} g(h(e_i^*, e_j^*), \tilde{e}_r)^2 + \sum_{r=m}^{2(n+1-m)}\sum_{i,j=1}^{q+1} g(h(e_i^*, e_j^*), \tilde{e}_r)^2\\
&+ \sum_{r=1}^{2p}\sum_{i,j=1}^{2p} g(h(e_i, e_j), \tilde{e}_r)^2 + \sum_{r=2p+1}^{2p+q}\sum_{i,j=1}^{2p} g(h(e_i, e_j), \tilde{e}_r)^2 + \sum_{r=m}^{2(n+1-m)}\sum_{i,j=1}^{2p} g(h(e_i, e_j), \tilde{e}_r)^2.
\end{aligned} \tag{25}$$

First term in the right hand side of (25) is identically zero by using Lemma 3.3 (ii). Also, we couldn't find the relations for the second fundamental form for the vectors of $\mathfrak{D}^\perp$ with $\varphi\mathfrak{D}^\perp$ or ν and for the vectors of $\mathfrak{D}^\theta$ with $F\mathfrak{D}^\theta$ or ν, therefore we shall leave the positive second, third, fourth and sixth terms in (25), then we derive

$$\begin{aligned}
||h||^2 &\geq \sum_{r=1}^{q}\sum_{i,j=1}^{2p} g(h(e_i,e_j),\varphi e_r^*)^2 \\
&= \sum_{r=1}^{q}\sum_{i,j=1}^{p} g(h(e_i,e_j),\varphi e_r^*)^2 + \sec^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{p} g(h(Te_i,e_j),\varphi e_r^*)^2 \\
&+ \sec^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{p} g(h(e_i,Te_j),\varphi e_r^*)^2 + \sec^4\theta\sum_{r=1}^{q}\sum_{i,j=1}^{p} g(h(Te_i,Te_j),\varphi e_r^*)^2.
\end{aligned} \tag{26}$$

Then by Lemma 3.4 and the relations (15)-(16), we arrive at

$$\begin{aligned}
||h||^2 &\geq \sum_{r=1}^{q}\sum_{i,j=1}^{p}\Big(\eta(e_r^*) - e_r^*(\ln f)\Big)^2 g(Te_i,e_j)^2 + 2\cos^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{p}\Big(\eta(e_r^*) - e_r^*(\ln f)\Big)^2 g(e_i,e_j)^2 \\
&+ \sum_{r=1}^{q}\sum_{i,j=1}^{p}\Big(\eta(e_r^*) - e_r^*(\ln f)\Big)^2 g(Te_i,e_j)^2.
\end{aligned} \tag{27}$$

First and the last terms in the right hand side of (27) are identically zero by using the orthonormality of vector fields and hence finally, we get

$$||h||^2 \geq 2p\cos^2\theta\sum_{r=1}^{q+1}\Big(\eta(e_r^*) - e_r^*(\ln f)\Big)^2 - 2p\cos^4\theta\Big(\eta(e_{r+1}^*) - e_{r+1}^*(\ln f)\Big)^2. \tag{28}$$

Since $e_{r+1}^* = \xi$, then $\eta(e_{r+1}^*) = 1$ and from Lemma 3.3 (i), we have $e_{r+1}^*(\ln f) = 1$. Hence the last term in the right hand side of (28) is identically zero, then we have

$$||h||^2 \geq 2p\cos^2\theta\Big\{\sum_{r=1}^{q+1}\eta(e_r^*)^2 - 2\sum_{r=1}^{q+1}\eta(e_r^*)e_r^*(\ln f) + \sum_{r=1}^{q+1}(e_r^*(\ln f))^2\Big\}. \tag{29}$$

Since $\eta(e_r^*) = 0$, $\forall\ r = 1,\cdots q$ and $\eta(e_r^*) = 1$, $e_r^*(\ln f) = 1$, for $r = q+1$. Then using these facts with (14) in (29), we get the inequality (23). To prove the equality case, for the non-vanishing h from the first term of (25) with Lemma 3.3 (ii), we have

$$g(h(\mathfrak{D}^\perp,\mathfrak{D}^\perp),F\mathfrak{D}^\theta) = 0 \ \Rightarrow h(\mathfrak{D}^\perp,\mathfrak{D}^\perp) \perp F\mathfrak{D}^\theta. \tag{30}$$

Also, from the leaving second and third terms of (25), we obtain

$$h(\mathfrak{D}^\perp,\mathfrak{D}^\perp) \perp \varphi\mathfrak{D}^\perp \ \text{ and } \ h(\mathfrak{D}_1,\mathfrak{D}_1) \perp \nu. \tag{31}$$

From (30) and (31), we get

$$h(\mathfrak{D}^\perp,\mathfrak{D}^\perp) = 0. \tag{32}$$

Since $M_\perp$ is totally geodesic in M [5, 10], using this fact with (32), we conclude that $M_\perp$ is totally geodesic in $\tilde{M}$. Similarly, from the remaining fourth and sixth terms in (25), we obtain

$$h(\mathfrak{D}^\theta,\mathfrak{D}^\theta) \perp F\mathfrak{D}^\theta \ \text{ and } \ h(\mathfrak{D}^\theta,\mathfrak{D}^\theta) \perp \nu$$

which means that

$$h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) \subseteq \varphi\mathfrak{D}^\perp. \tag{33}$$

Furthermore, for a mixed totally geodesic submanifold M, by Lemma 3.4 (ii) and (33), we have

$$g(\sigma(TX, Y), \varphi Z) = -\cos^2\theta\Big(\eta(Z) - Z(\ln f)\Big)g(X, Y) \tag{34}$$

for any $X, Y \in \Gamma(\mathfrak{D}^\theta)$ and $Z \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$. Hence, since M_θ is totally umbilical in M [5, 10], it follows with (34) that M_θ is totally umbilical in $\tilde{M}$. This, ends the proof of the theorem. □

As a special case, we have the following applications of our derived results.

Remark 4.2. *If we assume $\theta = 0$ in Theorem 3.5, then the warped product becomes $M = M_\perp \times_f M_T$ of a Kenmotsu manifold $\tilde{M}$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\tilde{M}$, respectively, which is a case of warped product contact CR-submanifolds which have been studied in [31]. Thus, Theorem 3.1 of [31] is a special case of Theorem 3.5.*

Remark 4.3. *Also, if we consider $\theta = 0$ in Theorem 4.1, then the warped product is of the form $M = M_\perp \times_f M_T$ of a Kenmotsu $\tilde{M}$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\tilde{M}$, respectively and hence the inequality (23) will be $||h||^2 \geq 2p\Big(||\vec{\nabla}(\ln f)||^2 - 1\Big)$. Thus, Theorem 3.2 of [31] is again a special case of Theorem 4.1.*

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