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Ricci Soliton and η-Ricci Soliton on Generalized Sasakian Space Form

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Abstract. The aim of the present paper is to study Ricci soliton, η-Ricci soliton and various types of curvature tensors on Generalized Sasakian space form. We have also studied conformal Killing vector field, torse forming vector field on Generalized Sasakian space form. We have also established suitable examples of kenmotsu manifold, Sasakian manifold and cosymplectic manifold respectively.

1. Introduction

In 1982, the notion of the Ricci flow was introduced by Hamilton [6] to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold given by

$$\frac{\partial}{\partial t}g(t) = -2S.$$

A Riemannian manifold (M, g) is called a Ricci soliton if there exists a smooth vector field V such that the Ricci tensor S satisfies the following equation

$$S + \frac{1}{2}£_V g + \lambda g = 0, \tag{1}$$

for some constant λ and the Lie derivative $£_V g$. Ricci soliton appears as a self-similar solution to Hamiltons's Ricci flow and often arise as limit of dilation of singularities in the Ricci flow [6]. They are natural generalization of Einstein metrics. Note that a soliton is called shrinking, steady and expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ respectively.

If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1) becomes

$$\nabla\nabla f = S + \lambda g.$$

According to Perelman [8], we know that a Ricci soliton on a compact manifold is a gradient Ricci soliton.

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Ramesh Sharma [10], Mukut Mani Tripathi [11], Cornelia Livia Bejan and Mircea Crasmareanu [2] have studied the Ricci soliton in contact metric manifolds.

J.C. Cho and M. Kimura introduced the notion of η-Ricci soliton [5] which was treated by C. Călin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [4]. A Riemannian manifold (M, g) is called a η-Ricci soliton if there exist a smooth vector field ξ such that the Ricci tensor satisfies the following equation

$$2S + \pounds_{\xi} g + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{2}$$

where $\pounds_{\xi}$ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor and λ, μ are real constants. If $\mu = 0$, then η-Ricci soliton becomes Ricci soliton.

In this paper we have studied Ricci soliton, η-Ricci soliton on generalized Sasakian space form. We have also proved that a generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ admitting a Ricci soliton (g, V, λ) is Ricci-semi symmetric iff V is conformal Killing vector field. Also we have studied torse forming vector field on this space form. Later we consider η-Ricci soliton on generalized Sasakian space form satisfying some certain curvature conditions.

2. Generalized Sasakian Space Form

The sectional curvature of a Riemannian manifold (M, g) plays an important role in differential geometry. The curvature tensor of a Riemannian manifold with constant sectional curvature c is given by the following equation

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\},$$

for all $X, Y, Z \in \chi(M)$. Then the Riemannian manifold with constant sectional curvature c is called a real-space form. A Riemannian manifold with constant sectional curvature c is said to be elliptic, hyperbolic or flat according as $c > 0$, $c < 0$ or $c = 0$.

Similarly we can define constant holomorphic sectional curvature in the complex manifold. A Kähler manifold M^n is of constant holomorphic sectional curvature c iff

$$R(X, Y)Z = \frac{c}{4}\{g(X, Z)Y - g(Y, Z)X + g(F(X), Z)F(Y)$$
$$-g(F(Y), Z)F(X) + 2g(F(X), Y)F(Z)\},$$

for all $X, Y, Z \in \chi(M)$. Then the complex manifold with constant holomorphic sectional curvature c is called a complex-space form.

A $(2n + 1)$-dimensional smooth manifold M with an almost contact structure (ϕ, ξ, η, g) satisfies the following conditions [3]

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{3}$$

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi), \tag{4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{5}$$

P. Alegre, D. E. Blair and A. Carriazo introduced the concept of generalized Sasakian space form in [1]. An almost contact metric manifold M with an almost contact metric structure (ϕ, ξ, η, g) is called a generalized

Sasakian space form if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X$$
$$+2g(X,\phi Y)\phi Z\} + f_3\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}. \tag{6}$$

for all vector fields X, Y, Z on M.

If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ then M is a Sasakian space form.
If $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$ then M is a Kenmotsu space form.
If $f_1 = f_2 = f_3 = \frac{c}{4}$ then M is a cosymplectic space form.

In a $(2n+1)$-dimensional generalized Sasakian space form $M^{(2n+1)}(f_1, f_2, f_3)$, we have the following relations [1]

$$\nabla_X \xi = (f_3 - f_1)\phi X, \tag{7}$$

$$(\nabla_X \phi)(Y) = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{8}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = (f_3 - f_1)g(\phi X, Y), \tag{9}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{10}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{11}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{12}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{13}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{14}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{15}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{16}$$

Next we discuss about three important curvature tensors in differential geometry which plays an important role in the general theory of relativity.

The projective curvature tensor is an important tensor from the differential geometric point of view. This has one-one correspondence between each coordinate neighbourhood of an n-dimensional Riemannian manifold and a domain of Euclidean space such that there is one-one correspondence between geodesics of Riemannian manifold with straight line in Euclidean space.

The projective curvature tensor in a (2n+1)-dimensional generalized Sasakian space-form $M^{2n+1}(f_1, f_2, f_3)$ is defined by [14]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[g(QY,Z)X - g(QX,Z)Y], \tag{17}$$

for any X, Y, $Z \in \chi(M)$ and Q is a Ricci operator.
The generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ is called ξ-projectively flat if $P(X,Y)\xi = 0$ for any vector fields X, Y on M.

A concircular transformation on an n-dimensional Riemannian manifold M is a transformation which transforms every geodesic circle of M into a geodesic circle. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature (i.e. torsion) is identically zero [12].

The concircular curvature tensor in a $(2n+1)$-dimensional generalized Sasakian space-form $M^{2n+1}(f_1, f_2, f_3)$ is defined by [15]

$$L(X,Y)Z = R(X,Y)Z - \frac{r}{(2n+1)2n}[g(Y,Z)X - g(X,Z)Y], \tag{18}$$

for any $X, Y, Z \in \chi(M)$, Q is a Ricci operator and r is the scalar curvature.
Now the generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ is called ξ-concircularly flat if $L(X,Y)\xi = 0$ for any vector fields X, Y on M.

In 1970 Pokhariyal and Mishra [9] have introduced another type of curvature tensor named as W_2-curvature tensor and is defined as

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{2n}(g(X,Z)QY - g(Y,Z)QX). \tag{19}$$

Next we give the following definitions on a generalized Sasakian space form obtained from [7], [13].

Definition 2.1: The Ricci tensor S on a generalized Sasakian space form is called η-parallel if it satisfies the relation

$$(\nabla_Z S)(\phi X, \phi Y) = 0, \tag{20}$$

for all $X, Y, Z \in \chi(M)$. M. Kon introduced Ricci η-parallelity for the sasakian manifold in [7].

Definition 2.2: A vector field V is a conformal Killing vector field on a generalized Sasakian space form if $\pounds_V g = \rho g$ where ρ is scalar function and $\pounds_V$ denotes the Lie derivative along V.

Definition 2.3: A vector field ξ is a torse-forming vector field [13] on a generalized Sasakian space form if $\nabla_X \xi = fX + \gamma(X)\xi$ where f is a smooth function and γ is a 1-form.

3. Examples of Three Types of Contact Manifold:

In this section we have constructed three examples of Kenmotsu manifold, Sasakian manifold and cosymplectic manifold.

Example 3.1: We consider the three dimensional manifold $M = \{(x,y,z) \in R^3 : y \neq 0\}$ where (x,y,z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^y \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}, e_3 = e^y \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_2)$ for any $Z \in \chi(M^3)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = e_3, \phi(e_2) = 0, \phi(e_3) = -e_1$. Then using the linearity property of ϕ and g yields that,

$$\eta(e_2) = 1, \phi^2(Z) = -Z + \eta(Z)e_2, g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_2 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now we have

$$[e_1, e_3] = 0, [e_1, e_2] = -e_1, [e_2, e_3] = e_3.$$

The Riemannian connection ∇ is given by the Koszul's formula which is

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).$$

By Koszul's formula we get

$$\nabla_{e_1}e_1 = e_2, \nabla_{e_2}e_1 = 0, \nabla_{e_3}e_1 = 0, \nabla_{e_1}e_2 = -e_1, \nabla_{e_2}e_2 = 0,$$

$$\nabla_{e_3}e_2 = -e_3, \nabla_{e_1}e_3 = 0, \nabla_{e_2}e_3 = 0, \nabla_{e_3}e_3 = e_2.$$

From the above it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a Kenmotsu manifold.

Example 3.2: Let $M = \{(x,y,z) \in R^3 : y, z \neq 0\}$ where (x,y,z) are the standard coordinates in R^3. The linearly independent vector fields are given by

$$e_1 = \frac{\partial}{\partial y}, e_2 = z^2\left(\frac{\partial}{\partial z} + 2y\frac{\partial}{\partial x}\right), e_3 = \frac{\partial}{\partial x}.$$

Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_2)$ for any $Z \in \chi(M^3)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = e_3, \phi(e_2) = 0, \phi(e_3) = -e_1$. Then using the linearity property of ϕ and g we have

$$\eta(e_2) = 1, \phi^2(Z) = -Z + \eta(Z)e_2, g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_2 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, after some calculations we have

$$[e_1, e_3] = 0, [e_1, e_2] = 2z^2 e_3, [e_2, e_3] = 0.$$

By Koszul's formula we get,

$$\nabla_{e_1}e_1 = 0, \nabla_{e_2}e_1 = -z^2 e_3, \nabla_{e_3}e_1 = -z^2 e_2, \nabla_{e_1}e_2 = z^2 e_3, \nabla_{e_2}e_2 = 0,$$

$$\nabla_{e_3}e_2 = z^2 e_1, \nabla_{e_1}e_3 = -z^2 e_2, \nabla_{e_2}e_3 = z^2 e_1, \nabla_{e_3}e_3 = 0.$$

Hence one can see that $M^3(\phi, \xi, \eta, g)$ is a z^2-Sasakian manifold.

Example 3.3: We consider $M = \{(x,y,z) \in R^3\}$ where (x,y,z) are the standard coordinates in R^3. The vector fields

$$e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. The Riemannian metric g is defined by

$$g_{ij} = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_2)$ for any $Z \in \chi(M^3)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = e_3, \phi(e_2) = 0, \phi(e_3) = -e_1$. Using the linearity property of ϕ and g gives that

$$\eta(e_2) = 1, \phi^2(Z) = -Z + \eta(Z)e_2, g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_2 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. By definition of Lie bracket, we obtain

$$[e_1, e_3] = 0, [e_1, e_2] = 0, [e_2, e_3] = 0.$$

By Koszul's formula it can be easily verified that $M^3(\phi, \xi, \eta, g)$ is a cosymplectic manifold.

4. Ricci Soliton on Generalized Sasakian Space Form:

In this section we study Ricci soliton on generalized Sasakian space form.

Theorem 4.1: On a generalized Sasakian space form any parallel symmetric $(0, 2)$-tensor field is a constant multiple of the metric with $f_1 \neq f_3$ and the Ricci soliton is shrinking, expanding or steady if $f_1 > f_3$, $f_1 < f_3$ or $f_1 = f_3$ respectively..

Proof: Consider α a symmetric $(0, 2)$ tensor field which is parallel with respect to the Levi Civita connection i.e. $\nabla\alpha = 0$.
Then

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0.$$

Considering $X = Z = W = \xi$ we have

$$\alpha(Y, \xi) - \alpha(\xi, \xi)\eta(Y) = 0. \tag{21}$$

Differentiating (21) covariantly with respect to X we have

$$\alpha(Y, \phi X) = \alpha(\xi, \xi)g(Y, \phi X). \tag{22}$$

Taking $X = \phi X$ we have

$$\alpha(X, Y) = \alpha(\xi, \xi)g(X, Y). \tag{23}$$

Now suppose that a generalized Sasakian space form satisfies the Ricci soliton. Then by the equation (1) we get $\nabla\lambda g = 0$. So, $£_V g + 2S$ is parallel, we can write $£_V g + 2S = T$ is a constant multiple of the metric tensor g, i.e.,

$$(£_V g + 2S)(X, Y) = T(X, Y) = T(\xi, \xi)g(X, Y). \tag{24}$$

So, from the definition of Ricci soliton we have

$$\lambda = 2n(f_3 - f_1). \tag{25}$$

Hence the theorem is proved.

Example 4.2: In [1] P. Alegre, D. Blair and A. Carriazo showed that $\mathbb{R} \times \mathbb{C}^m$ is a generalized Sasakian space form with

$$f_1 = -\frac{(f')^2}{f^2}, \;\; f_2 = 0, \;\; f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t)$, $t \in \mathbb{R}$ and f' denotes derivative of f with respect to t.
Now we apply Ricci soliton on this generalized Sasakian space form and obtain different conditions to show the different states of Ricci soliton.

Case 1: If we consider $f(t) = e^t$ then $f_1 < f_3$. Therefore the Ricci soliton is expanding.
Case 2: If we take $f(t) =$ constant then $f_1 = f_3$. So the Ricci soliton is steady.
Case 3: If we put $f(t) = \cos t$ where $t \in [0, \frac{\pi}{4}]$ then $f_1 > f_3$. So the Ricci soliton is shrinking.

Theorem 4.3: If V is point-wise collinear with ξ on a generalized Sasakian space form satisfying Ricci soliton, then V is a constant multiple of ξ, (M, g) is an Einstein manifold and the Ricci soliton is shrinking, expanding or steady if $f_1 > f_3$, $f_1 < f_3$ or $f_1 = f_3$ respectively.

Proof: Let V be pointwise collinear with ξ i.e. $V = b\xi$, where b is a function. Then by the equation (1) we get

$$bg(\nabla_X\xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y\xi, X) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{26}$$

Putting $Y = \xi$ in the equation (26) we get

$$(Xb) + (\xi b)\eta(X) + 2S(X, \xi) + 2\lambda\eta(X) = 0. \tag{27}$$

Replacing X by ξ in the above equation we get

$$\xi b = 2n(f_3 - f_1) - \lambda. \tag{28}$$

Putting the value of ξb in the equation (27) we get

$$(db)(X) + [\lambda + 2n(f_1 - f_3)]\eta(X) = 0. \tag{29}$$

Taking exterior differentiation to (29) we get

$$[\lambda + 2n(f_1 - f_3)]d\eta = 0. \tag{30}$$

Since $d\eta \neq 0$ we get

$$\lambda = 2n(f_3 - f_1). \tag{31}$$

So, from the equation (29) we have

$$(db)(X) = 0 \Rightarrow b = \text{constant}. \tag{32}$$

So, the Ricci soliton is shrinking, expanding or steady if $f_1 > f_3$, $f_1 < f_3$ or $f_1 = f_3$ respectively. Also from (7) and (26) we can easily see that (M, g) is an Einstein manifold.

Remark 4.4: If we put $V = \xi$ in the equation (1) we obtain $(\pounds_\xi g)(X, Y) = 0$. Then a generalized Sasakian space form becomes Einstein manifold.

Theorem 4.5: If the Ricci tensor S is η-parallel on a generalized Sasakian space form then the scalar curvature is constant provided that f_1, f_3 are constants.

Proof: From (13) we have

$$S(\phi X, \phi Y) = (2nf_1 + 3f_2 - f_3)g(\phi X, \phi Y). \tag{33}$$

Differentiating the equation (33) we obtain

$$(\nabla_Z S)(\phi X, \phi Y) = d(2nf_1 + 3f_2 - f_3)(Z)g(\phi X, \phi Y)$$

$$-(2nf_1 + 3f_2 - f_3)(f_3 - f_1)g(\phi Z, X)\eta(Y) + g(\phi Z, Y)\eta(X).$$

Since the Ricci tensor S is η-parallel on a generalized Sasakian space form, so we have

$$(\nabla_Z S)(\phi X, \phi Y) = 0,$$

for all vector fields X, Y, Z on M.

Thus we get

$$d(2nf_1 + 3f_2 - f_3)(Z)g(\phi X, \phi Y) - (2nf_1 + 3f_2 - f_3)(f_3 - f_1)g(\phi Z, X)\eta(Y)$$

$$+g(\phi Z, Y)\eta(X) = 0.$$

Putting $X = Y = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the generalized Sasakian space form and taking $1 \le i \le (2n+1)$ we find

$$2nf_1 + 3f_2 - f_3 = \text{constant} = k(\text{say}).$$

Now we have from (16)

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3,$$

$$\Rightarrow r = 2n(k + f_1 - f_3).$$

So, r is constant provided that f_1, f_3 are constants.

Proposition 4.6: A Ricci soliton (g, V, λ) in a generalized Sasakian space form where V is a conformal Killing vector field is
a) expanding if $\rho < \frac{8n}{2n+1}f_3 - 4nf_1 - \frac{12n}{2n+1}f_2$
b) shrinking if $\rho > \frac{8n}{2n+1}f_3 - 4nf_1 - \frac{12n}{2n+1}f_2$
c) steady if $\rho = \frac{8n}{2n+1}f_3 - 4nf_1 - \frac{12n}{2n+1}f_2$.

Proof: Since V is a conformal Killing vector field on a generalized Sasakian space form, so we can find

$$S(X, Y) = -(\lambda + \frac{\rho}{2})g(X, Y).$$

Also we have

$$QX = -(\lambda + \frac{\rho}{2})X,$$

and

$$r = -(\lambda + \frac{\rho}{2}).$$

By the equation (16) we have

$$\lambda = \frac{4n}{2n+1}f_3 - 2nf_1 - \frac{6n}{2n+1}f_2 - \frac{\rho}{2}.$$

So the Ricci soliton (g, V, λ) is expanding if $\lambda > 0$, shrinking if $\lambda < 0$ and steady if $\lambda = 0$.
Hence the proposition is established.

Theorem 4.7: A generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ admitting a Ricci soliton (g, V, λ) is Ricci-semi symmetric iff V is conformal Killing vector field.

Proof: A generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ satisfying a Ricci soliton (g, V, λ) satisfies Ricci-semi symmetric if $R(X, Y) \cdot S = 0$.
Then we have

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0.$$

Putting $W = \xi$ we get

$$S(R(X, Y)Z, \xi) + S(Z, R(X, Y)\xi) = 0.$$

Hence we have

$$2n(f_1 - f_3)\eta(R(X, Y)Z) + (f_1 - f_3)\eta(Y)S(X, Z) - (f_1 - f_3)\eta(X)S(Y, Z) = 0. \tag{34}$$

Then we obtain

$$S(Y, Z) = 2n(f_1 - f_3)g(Y, Z),$$

where $f_1 \neq f_3$. Hence the manifold is Einstein manifold.
Using the definition of Ricci soliton we have

$$£_V g = 2[2n(f_3 - f_1) - \lambda]g.$$

Therefore V is conformal Killing vector field.
Now we consider that V is conformal Killing vector field. Then from (1) we have

$$S(X, Y) = -(\lambda + \frac{\rho}{2})g(X, Y).$$

Hence the converse part follows immediately.

Proposition 4.8: A generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ is ξ- concircularly flat iff $r = 2n(2n+1)(f_1 - f_3)$.

Proof: Putting $Z = \xi$ in the equation (18) we have

$$R(X, Y)\xi - \frac{r}{2n(2n+1)}[g(Y, \xi)X - g(X, \xi)Y] = 0.$$

Using (10) in the above equation we have

$$(f_1 - f_3 - \frac{r}{2n(2n+1)})[\eta(Y)X - \eta(X)Y] = 0.$$

This implies that $L(X, Y)\xi = 0$ iff $r = 2n(2n+1)(f_1 - f_3)$.

Proposition 4.9: A generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ is ξ-projectively flat.

Proof: From the definition of projective curvature tensor we have

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2}[g(QY, Z)X - g(QX, Z)Y]. \tag{35}$$

Putting $Z = \xi$ we get

$$P(X, Y)\xi = R(X, Y)\xi - \frac{1}{2}[\eta(QY)X - \eta(QX)Y]. \tag{36}$$

Using the equations (10) and (15) we obtain

$$P(X, Y)\xi = 0.$$

Hence the proposition is proved.

5. η-Ricci Solitons on Generalized Sasakian Space Form:

In this section we deduce some results regarding η-Ricci soliton on generalized Sasakian space form.

From the equation (2) we get

$$2S(X, Y) = -g(\nabla_X \xi, Y) - g(\xi, \nabla_X Y) - 2\lambda g(X, Y) - 2\mu\eta(X)\eta(Y). \tag{37}$$

By using the equation (7) we get

$$S(X, Y) = -\lambda g(X, Y) - \mu\eta(X)\eta(Y) \tag{38}$$

and

$$S(X, \xi) = -(\lambda + \mu)\eta(X). \tag{39}$$

Also from (15) we have

$$\lambda + \mu = 2n(f_3 - f_1). \tag{40}$$

The Ricci operator Q is defined by $g(QX, Y) = S(X, Y)$. Then we get

$$QX = (\mu + 2n(f_1 - f_3))X - \mu\eta(X)\xi. \tag{41}$$

Theorem 5.1: If ξ is a torse-forming η-Ricci soliton on a generalized Sasakian space form then $f = f_3 - 2nf_1 - 3f_2 - \lambda$ and if X is a unit vector field and orthogonal to ξ then the sectional curvature $K(X, \xi) = f_1 - f_3$.

Proof: By the definition of torse-forming vector field we have $g(\nabla_X\xi, \xi) = (f\eta + \gamma)X$.
Using the equation (7) we get $-f\eta = \gamma$. Thus we obtain

$$\nabla_X\xi = f(X - \eta(X)\xi). \tag{42}$$

By the definition of η-Ricci soliton and using the equations (13), (37) and (40) we obtain

$$(f + 2nf_1 + 3f_2 - f_3 + \lambda)g(X, Y) - (f + 2nf_1 + 3f_2 - f_3 + \lambda)\eta(X)\eta(Y) = 0.$$

Therefore we get $f = f_3 - 2nf_1 - 3f_2 - \lambda$.

Now from (10) we have

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y].$$

Taking X as a unit vector field and orthogonal to ξ we get $K(X, \xi) = f_1 - f_3$. .

We know a manifold is ϕ-Ricci symmetric if $\phi^2 \circ \nabla Q = 0$. Now we prove the next proposition.

Proposition 5.2: A ϕ-Ricci symmetric manifold satisfying an η-Ricci soliton on a generalized Sasakian space form with f_1, f_3 constants ($f_1 \neq f_3$) is an Einstein manifold.

Proof: From the equation (41) we have

$$(\nabla_X Q)Y = \nabla_X QY - Q(\nabla_X Y)$$

$$= 2n(Xf_1 - Xf_3)(Y) - \mu(f_3 - f_1)[g(\phi X, Y)\xi + \eta(Y)\phi X].$$

Now applying ϕ^2 both sides we have $\mu = 0$, $\lambda = 2n(f_3 - f_1)$. Hence the result follows.

Corollary 5.3: A generalized Sasakian space form with f_1, f_3 constants ($f_1 = f_3$) is ϕ-Ricci symmetric.

Theorem 5.4: If a (2n+1)-dimensional generalized Sasakian space-form $M^{2n+1}(f_1, f_2, f_3)$ admits an η-Ricci soliton (g, ξ, λ, μ) with $W_2(\xi, Y) \cdot S = 0$ then either $\mu = 0$, $\lambda = 2n(f_3 - f_1)$ or $\mu = 2n(f_3 - f_1)$, $\lambda = 0$.

Proof: First we consider that in a (2n+1)-dimensional generalized Sasakian space-form $M^{2n+1}(f_1, f_2, f_3)$ admitting an η-Ricci soliton (g, ξ, λ, μ) satisfies
$W_2(\xi, Y) \cdot S = 0$. Then we have

$$S(W_2(\xi, Y)Z, U) + S(Z, W_2(\xi, Y)U) = 0.$$

Using the equations (38), (41), (19) we get

$$\lambda(f_3 - f_1)[g(Y,Z)\eta(U) - \eta(Z)g(Y,U)] - \frac{\lambda}{2n}[\eta(Z)S(Y,U) - g(Y,Z)S(U,\xi)]$$

$$+\lambda(f_3 - f_1)[g(Y,U)\eta(Z) - \eta(U)g(Y,Z)] - \frac{\lambda}{2n}[\eta(U)S(Y,Z) - g(Y,U)S(Z,\xi)]$$

$$+\mu\eta(U)(f_3 - f_1)[g(Y,Z) - \eta(Y)\eta(Z)] + \mu\eta(Z)(f_3 - f_1)[g(Y,U) - \eta(Y)\eta(U)]$$

$$-\frac{\mu}{2n}\eta(U)[\eta(Z)S(Y,\xi) - g(Y,Z)S(\xi,\xi)] - \frac{\mu}{2n}\eta(Z)[\eta(U)S(Y,\xi) - g(Y,U)S(\xi,\xi)] = 0.$$

Putting $U = \xi$ we have

$$[\mu(f_3 - f_1)] - \frac{\mu}{2n}(\lambda + \mu) - \frac{\lambda}{2n}(\lambda + \mu) + \frac{\lambda^2}{2n}]g(\phi Y, \phi Z) = 0.$$

Since $g \neq 0$ and from the equation (40) we have either $\mu = 0$, $\lambda = 2n(f_3 - f_1)$ or $\mu = 2n(f_3 - f_1)$, $\lambda = 0$.

Theorem 5.5: In a (2n+1)-dimensional generalized Sasakian space-form $M^{2n+1}(f_1, f_2, f_3)$ admitting an η-Ricci soliton (g, ξ, λ, μ) satisfying $S(\xi, X) \cdot W_2 = 0$ yields either $\mu = 0$, $\lambda = 2n(f_3 - f_1)$ or $\mu = 2n(f_3 - f_1)$, $\lambda = 0$.

Proof: First we consider that in a (2n+1)-dimensional generalized Sasakian space-form $M^{2n+1}(f_1, f_2, f_3)$ admitting an η-Ricci soliton (g, ξ, λ, μ) satisfies $S(\xi, X) \cdot W_2 = 0$. So we have

$$S(X, W_2(Y,Z)V)\xi - S(\xi, W_2(Y,Z)V)X + S(X,Y)W_2(\xi,Z)V - S(\xi,Y)W_2(X,Z)V$$

$$+S(X,Z)W_2(Y,\xi)V - S(\xi,Z)W_2(Y,X)V + S(X,V)W_2(Y,Z)\xi - S(\xi,V)W_2(Y,Z)X = 0.$$

Taking inner product with ξ in the above equation we obtain

$$S(X, W_2(Y,Z)V) - S(\xi, W_2(Y,Z)V)\eta(X) + S(X,Y)\eta(W_2(\xi,Z)V)$$

$$-S(\xi,Y)\eta(W_2(X,Z)V) + S(X,Z)\eta(W_2(Y,\xi)V) - S(\xi,Z)\eta(W_2(Y,X)V)$$

$$+S(X,V)\eta(W_2(Y,Z)\xi) - S(\xi,V)\eta(W_2(Y,Z)X) = 0. \tag{43}$$

Putting $V = \xi$ and using the equations (38), (39), (41), (19) we get

$$-\lambda g(X, W_2(Y,Z)\xi) + (\lambda + \mu)\eta(W_2(Y,Z)X) = 0. \tag{44}$$

Using the equations (41) and (19), the equation (44) becomes

$$[-\frac{\lambda^2}{2n} - \lambda(f_1 - f_3) + (\lambda + \mu)(f_1 - f_3) + \frac{(\lambda + \mu)}{2n}][g(X,Y)\eta(Z) - g(X,Z)\eta(Y)] = 0.$$

Thus from the equation (40) we find either either $\mu = 0$, $\lambda = 2n(f_3 - f_1)$ or $\mu = 2n(f_3 - f_1)$, $\lambda = 0$.

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