



General Fixed Point Theorems on Metric Spaces and 2-metric Spaces

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Abstract. In this paper, we state a new form of fixed point theorems on metric spaces and 2-metric spaces which unifies and generalizes many results in the literature. Examples are given to illustrate the results.

1. Introduction and Preliminaries

There were many kinds of fixed point theorems in the literature. To unify these results, some authors used certain implicit relations. In 2008, Akram *et al.* [3] introduced the class of A -contractions in a metric space which includes the contractions studied in [6], [20]. Also, several results in [1], [7], [16] and some others were extended to the A -contractions. In 2010, Saha and Dey [23] proved some fixed point theorems for A -contractions in a 2-metric space. Also in 2010, Akinbo *et al.* [2] generalized the main results in [3] for four self-maps on a metric space. In 2012, Gupta and Kaur [14] proved some common fixed point theorems for A -contractions on 2-metric spaces. In 2013, Dey and Saha [10] established a common fixed point theorem for four self-maps of a complete 2-metric space using the weak commutativity condition and A -contraction type condition and then extended the theorem for a class of self-maps. The key of these results is the class \mathcal{A} of three-variable functions $\alpha : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ satisfying

1. α is continuous on the set \mathbb{R}_+^3 .
2. There exists $k \in [0, 1)$ such that if $y \leq \alpha(x, x, y)$ or $y \leq \alpha(x, y, x)$ or $y \leq \alpha(y, x, x)$ then $y \leq kx$ for all $x, y \in \mathbb{R}_+$.

In this paper, we generalize the class \mathcal{A} to a class \mathcal{B} of nine-variable functions. By using the class \mathcal{B} , we state some general fixed point theorems on metric spaces and 2-metric spaces which unify and generalize many results in the literature. Examples are given to illustrate the results.

Now, we recall notions and lemmas which are useful in what follows. As a generalization of a metric space, the concept of a 2-metric space was introduced by Gähler in [13].

Definition 1.1 ([13]). Let X be non-empty and $\sigma : X \times X \times X \rightarrow \mathbb{R}$ be a map satisfying the following.

1. For every $a, b \in X$ with $a \neq b$ there exists $c \in X$ such that $\sigma(a, b, c) \neq 0$.

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2. If at least two of $a, b, c \in X$ are the same, then $\sigma(a, b, c) = 0$.
3. The symmetry: $\sigma(a, b, c) = \sigma(a, c, b) = \sigma(b, c, a) = \sigma(b, a, c) = \sigma(c, a, b) = \sigma(c, b, a)$ for all $a, b, c \in X$.
4. The rectangle inequality: $\sigma(a, b, c) \leq \sigma(a, b, d) + \sigma(b, c, d) + \sigma(c, a, d)$ for all $a, b, c, d \in X$.

Then σ is called a 2-metric on X and (X, σ) is called a 2-metric space which will be sometimes denoted by X if there is no confusion. Every member $x \in X$ is called a point of X .

Definition 1.2 ([15]). Let (X, σ) be a 2-metric space.

1. A sequence $\{x_n\}$ is called convergent to x in (X, σ) , written as $\lim_{n \rightarrow \infty} x_n = x$, if for every $a \in X$, $\lim_{n \rightarrow \infty} \sigma(x_n, x, a) = 0$.
2. A sequence $\{x_n\}$ is called Cauchy in (X, σ) if for every $a \in X$, $\lim_{n, m \rightarrow \infty} \sigma(x_n, x_m, a) = 0$, that is, for each $\varepsilon > 0$ and $a \in X$, there exists n_0 such that $\sigma(x_n, x_m, a) < \varepsilon$ for all $n, m \geq n_0$.
3. (X, σ) is called complete if every Cauchy sequence is a convergent sequence.

Definition 1.3 ([13]). Let (X, σ) be a 2-metric space and $a, b \in X, r \geq 0$. The set

$$B(a, b, r) = \{x \in X : \sigma(a, b, x) < r\}$$

is called a 2-ball centered at a and b with radius r . The topology generated by the collection of all 2-balls as a subbasis is called a 2-metric topology on X .

Lemma 1.4 ([17], Lemma 4). $\lim_{n \rightarrow \infty} x_n = x$ in a 2-metric space (X, σ) if and only if $\lim_{n \rightarrow \infty} x_n = x$ in the 2-metric topological space X .

For fixed point theorems on 2-metric spaces, readers may refer to [4], [11], [12], [18] and references therein.

Definition 1.5 ([3], [23]). A map $T : X \rightarrow X$ on a metric space (X, d) is called an A -contraction if

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty))$$

for all $x, y \in X$ and some $\alpha \in \mathcal{A}$.

A map $T : X \rightarrow X$ on a 2-metric space (X, σ) is called an A -contraction if

$$\sigma(Tx, Ty, u) \leq \alpha(\sigma(x, y, u), \sigma(x, Tx, u), \sigma(y, Ty, u))$$

for all $x, y, u \in X$ and some $\alpha \in \mathcal{A}$.

Lemma 1.6 ([13]). If $\lim_{n \rightarrow \infty} x_n = x$ in a 2-metric space (X, σ) , then $\lim_{n \rightarrow \infty} \sigma(x_n, y, z) = \sigma(x, y, z)$ for all $x, y, z \in X$.

Lemma 1.7 ([17], Lemma 5). Let $f : X \rightarrow Y$ be a continuous map from a 2-metric space X into a 2-metric space Y . If $\lim_{n \rightarrow \infty} x_n = x$ in X , then $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ in Y .

Lemma 1.8 ([24]). Let $\{y_n\}$ be a sequence in a complete 2-metric space (X, σ) . If there exists $h \in (0, 1)$ such that $\sigma(y_n, y_{n+1}, a) \leq h\sigma(y_{n-1}, y_n, a)$ for all $n \geq 1$ and $a \in X$, then $\{y_n\}$ converges to a point in X .

The main results of the paper are presented in two following sections. Section 2 is devoted to some general fixed point theorems on metric spaces and 2-metric spaces. In section 3, we give some examples and compare our results with known ones to show that our fixed point theorems are generalizations of some results in [3], [21], [23] and are better than some known results.

2. General fixed Point Theorems on Metric Spaces and 2-Metric Spaces

In this section, we investigate some general fixed point theorems on metric spaces and 2-metric spaces. First, we extend the class \mathcal{A} in [3] to the class \mathcal{B} of nine-variable functions $\beta : \mathbb{R}_+^9 \rightarrow \mathbb{R}_+$ satisfying

1. β is continuous on the set \mathbb{R}_+^9 .
2. For all $x, y, z \in \mathbb{R}_+$,
 - (a) If $x \leq \beta(0, 0, x, x, 0, 0, 0, 0, x)$ or $x \leq \beta(x, 0, 0, x, x, 0, 0, x, x)$, then $x = 0$.
 - (b) There exists $k \in [0, 1)$ such that $y \leq kx$ provided $z \leq x + y$ and

$$y \leq \beta(x, x, y, z, 0, z, y, y, 0) \text{ or } y \leq \beta(x, y, x, z, 0, z, y, y, 0) \text{ or } y \leq \beta(y, x, x, z, 0, z, y, y, 0).$$

By using the class \mathcal{B} , we set up a general fixed point theorem on metric spaces as follows.

Theorem 2.1. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a map such that

$$d(Tx, Ty) \leq \beta(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(T^2x, x), d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)) \quad (1)$$

for all $x, y \in X$ and some $\beta \in \mathcal{B}$. Then T has a unique fixed point x^* and $\lim_{n \rightarrow \infty} T^n x = x^*$ for all $x \in X$.

Proof. For each $x \in X$, put $x_0 = x$ and $x_n = Tx_{n-1}$ for all $n \geq 1$.

Step 1. We prove that $\lim_{n \rightarrow \infty} x_n = x^*$ for some x^* in (X, d) . For all $n \geq 1$, from (1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \beta(d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}), \\ &\quad d(T^2x_{n-1}, x_{n-1}), d(T^2x_{n-1}, Tx_{n-1}), d(T^2x_{n-1}, x_n), d(T^2x_{n-1}, Tx_n)) \\ &= \beta(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n), \\ &\quad d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_n), d(x_{n+1}, x_n), d(x_{n+1}, x_{n+1})) \\ &= \beta(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0, \\ &\quad d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1}), 0). \end{aligned}$$

Since $\beta \in \mathcal{B}$ and $d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$, there exists $k \in [0, 1)$ such that $d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$ for all $n \geq 1$. It implies $d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$. So, for $n > m$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + \dots + d(x_{n-1}, x_n) \\ &\leq (k^m + \dots + k^{n-1})d(x_0, x_1) \\ &\leq \frac{k^m}{1-k}d(x_0, x_1). \end{aligned} \quad (2)$$

Taking the limit as $n, m \rightarrow \infty$ in (2), we get $\lim_{n, m \rightarrow \infty} d(x_m, x_n) = 0$. This proves that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d) . Then there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = x^*. \quad (3)$$

Step 2. We prove that x^* is a fixed point of T . By using (1) again, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, Tx_n) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \beta(d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, Tx_n), \\ &\quad d(T^2x_n, x_n), d(T^2x_n, Tx_n), d(T^2x_n, x^*), d(T^2x_n, Tx^*)) \\ &= d(x^*, x_{n+1}) + \beta(d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, x_{n+1}), \\ &\quad d(x_{n+2}, x_n), d(x_{n+2}, x_{n+1}), d(x_{n+2}, x^*), d(x_{n+2}, Tx^*)). \end{aligned} \quad (4)$$

Note that β is continuous, taking the limit as $n \rightarrow \infty$ in (4) and using (3), we get

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x^*) + \beta(d(x^*, x^*), d(x^*, x^*), d(x^*, Tx^*), d(x^*, Tx^*), d(x^*, x^*), d(x^*, x^*), d(x^*, x^*), d(x^*, x^*), d(x^*, Tx^*)) \\ &= \beta(0, 0, d(x^*, Tx^*), d(x^*, Tx^*), 0, 0, 0, 0, d(x^*, Tx^*)). \end{aligned}$$

Since $\beta \in \mathcal{B}$, $d(x^*, Tx^*) = 0$. Then $Tx^* = x^*$, that is, x^* is a fixed point of T .

Step 3. We prove that the fixed point of T is unique. Let $x^*, y^* \in X$ such that $Tx^* = x^*, Ty^* = y^*$. By using (1) again, we have

$$\begin{aligned} &d(x^*, y^*) \\ &= d(Tx^*, Ty^*) \\ &\leq \beta(d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), d(x^*, Ty^*), d(y^*, Tx^*), d(T^2x^*, x^*), d(T^2x^*, Tx^*), d(T^2x^*, y^*), d(T^2x^*, Ty^*)) \\ &= \beta(d(x^*, y^*), d(x^*, x^*), d(y^*, y^*), d(x^*, y^*), d(y^*, x^*), d(x^*, x^*), d(x^*, x^*), d(x^*, y^*), d(x^*, y^*)) \\ &= \beta(d(x^*, y^*), 0, 0, d(x^*, y^*), d(x^*, y^*), 0, 0, d(x^*, y^*), d(x^*, y^*)). \end{aligned}$$

Since $\beta \in \mathcal{B}$, $d(x^*, y^*) = 0$. Then $x^* = y^*$. This proves that the fixed point of T is unique.

Step 4. For each $x \in X$, we prove that $\lim_{n \rightarrow \infty} T^n x = x^*$. By using (3), $\lim_{n \rightarrow \infty} T^n x = \lim_{n \rightarrow \infty} x_n = x^*$. \square

Next, we prove a technical lemma on 2-metric spaces.

Lemma 2.2. Let (X, σ) be a 2-metric space and $T : X \rightarrow X$ be a map such that

$$\begin{aligned} \sigma(Tx, Ty, u) &\leq \beta(\sigma(x, y, u), \sigma(x, Tx, u), \sigma(y, Ty, u), \sigma(x, Ty, u), \sigma(y, Tx, u), \\ &\sigma(T^2x, x, u), \sigma(T^2x, Tx, u), \sigma(T^2x, y, u), \sigma(T^2x, Ty, u)) \end{aligned} \tag{5}$$

for all $x, y, u \in X$ and some $\beta \in \mathcal{B}$. Then $\sigma(x, Tx, T^2x) = 0$. Also,

$$\sigma(x, T^2x, u) \leq \sigma(x, Tx, u) + \sigma(Tx, T^2x, u)$$

for all $x, y, u \in X$.

Proof. For each $x \in X$, by choosing $u = x$ and $y = Tx$ in (5), we have

$$\begin{aligned} \sigma(Tx, T^2x, x) &\leq \beta(\sigma(x, Tx, x), \sigma(x, Tx, x), \sigma(Tx, T^2x, x), \sigma(x, T^2x, x), \sigma(Tx, Tx, x), \\ &\sigma(T^2x, x, x), \sigma(T^2x, Tx, x), \sigma(T^2x, Tx, x), \sigma(T^2x, T^2x, x)) \\ &= \beta(0, 0, \sigma(T^2x, Tx, x), 0, 0, 0, \sigma(T^2x, Tx, x), \sigma(T^2x, Tx, x), 0). \end{aligned}$$

Since $\beta \in \mathcal{B}$, there exists $k \in [0, 1)$ such that $\sigma(Tx, T^2x, x) \leq k \cdot 0 = 0$. This proves $\sigma(Tx, T^2x, x) = 0$.

Now, for each $u \in X$, we have

$$\sigma(x, T^2x, u) \leq \sigma(x, Tx, u) + \sigma(Tx, T^2x, u) + \sigma(x, Tx, T^2x).$$

It implies that $\sigma(x, T^2x, u) \leq \sigma(x, Tx, u) + \sigma(Tx, T^2x, u)$. \square

The following is an analogue of Theorem 2.1 for self-maps on 2-metric spaces.

Theorem 2.3. Let (X, σ) be a complete 2-metric space and $T : X \rightarrow X$ be a map such that

$$\begin{aligned} \sigma(Tx, Ty, u) &\leq \beta(\sigma(x, y, u), \sigma(x, Tx, u), \sigma(y, Ty, u), \sigma(x, Ty, u), \sigma(y, Tx, u), \\ &\sigma(T^2x, x, u), \sigma(T^2x, Tx, u), \sigma(T^2x, y, u), \sigma(T^2x, Ty, u)) \end{aligned} \tag{6}$$

for all $x, y, u \in X$ and some $\beta \in \mathcal{B}$. Then T has a unique fixed point x^* and $\lim_{n \rightarrow \infty} T^n x = x^*$ for all $x \in X$.

Proof. For each $x \in X$, put $x_0 = x$ and $x_n = Tx_{n-1}$ for all $n \geq 1$.

Step 1. We prove that $\lim_{n \rightarrow \infty} x_n = x^*$ for some x^* in (X, σ) . For each $u \in X$, from (6), we have

$$\begin{aligned} \sigma(x_n, x_{n+1}, u) &= \sigma(Tx_{n-1}, Tx_n, u) \\ &\leq \beta\left(\sigma(x_{n-1}, x_n, u), \sigma(x_{n-1}, Tx_{n-1}, u), \sigma(x_n, Tx_n, u), \sigma(x_{n-1}, Tx_n, u), \sigma(x_n, Tx_{n-1}, u), \right. \\ &\quad \left. \sigma(T^2x_{n-1}, x_{n-1}, u), \sigma(T^2x_{n-1}, Tx_{n-1}, u), \sigma(T^2x_{n-1}, x_n, u), \sigma(T^2x_{n-1}, Tx_n, u)\right) \\ &= \beta\left(\sigma(x_{n-1}, x_n, u), \sigma(x_{n-1}, x_n, u), \sigma(x_n, x_{n+1}, u), \sigma(x_{n-1}, x_{n+1}, u), \sigma(x_n, x_n, u), \right. \\ &\quad \left. \sigma(x_{n+1}, x_{n-1}, u), \sigma(x_{n+1}, x_n, u), \sigma(x_{n+1}, x_n, u), \sigma(x_{n+1}, x_{n+1}, u)\right) \\ &= \beta\left(\sigma(x_{n-1}, x_n, u), \sigma(x_{n-1}, x_n, u), \sigma(x_n, x_{n+1}, u), \sigma(x_{n-1}, x_{n+1}, u), 0, \right. \\ &\quad \left. \sigma(x_{n-1}, x_{n+1}, u), \sigma(x_n, x_{n+1}, u), \sigma(x_n, x_{n+1}, u), 0\right). \end{aligned}$$

Note that, by Lemma 2.2,

$$\begin{aligned} \sigma(x_{n-1}, x_{n+1}, u) &= \sigma(x_{n-1}, TTx_{n-1}, u) \\ &\leq \sigma(x_{n-1}, Tx_{n-1}, u) + \sigma(Tx_{n-1}, TTx_{n-1}, u) \\ &= \sigma(x_{n-1}, x_n, u) + \sigma(x_n, x_{n+1}, u). \end{aligned}$$

Then, since $\beta \in \mathcal{B}$, there exists $k \in [0, 1)$ such that

$$\sigma(x_n, x_{n+1}, u) \leq k\sigma(x_{n-1}, x_n, u). \tag{7}$$

From (7), by using Lemma 1.8, there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = x^*. \tag{8}$$

Step 2. We prove that x^* is a fixed point of T . By using (6) again, we have

$$\begin{aligned} \sigma(x^*, Tx^*, u) &\leq \sigma(x^*, Tx^*, x_{n+1}) + \sigma(Tx^*, u, x_{n+1}) + \sigma(u, x^*, x_{n+1}) \\ &= \sigma(Tx^*, x_{n+1}, x^*) + \sigma(Tx_n, Tx^*, u) + \sigma(u, x^*, x_{n+1}) \\ &\leq \sigma(Tx^*, x_{n+1}, x^*) + \beta\left(\sigma(x_n, x^*, u), \sigma(x_n, Tx_n, u), \sigma(x^*, Tx^*, u), \sigma(x_n, Tx^*, u), \right. \\ &\quad \left. \sigma(x^*, Tx_n, u), \sigma(T^2x_n, x_n, u), \sigma(T^2x_n, Tx_n, u), \sigma(T^2x_n, x^*, u), \sigma(T^2x_n, Tx^*, u)\right) \\ &\quad + \sigma(u, x^*, x_{n+1}) \\ &= \sigma(Tx^*, x_{n+1}, x^*) + \beta\left(\sigma(x_n, x^*, u), \sigma(x_n, x_{n+1}, u), \sigma(x^*, Tx^*, u), \sigma(x_n, Tx^*, u), \right. \\ &\quad \left. \sigma(x^*, x_{n+1}, u), \sigma(x_{n+2}, x_n, u), \sigma(x_{n+2}, x_{n+1}, u), \sigma(x_{n+2}, x^*, u), \sigma(x_{n+2}, Tx^*, u)\right) \\ &\quad + \sigma(u, x^*, x_{n+1}). \end{aligned} \tag{9}$$

By the proof of Lemma 1.8, $\{x_n\}$ is a Cauchy sequence in the complete 2-metric space (X, σ) . Then we have

$$\lim_{n \rightarrow \infty} \sigma(x_n, x_{n+1}, u) = \lim_{n \rightarrow \infty} \sigma(x_n, x_{n+2}, u) = 0. \tag{10}$$

Note that β is continuous, using Lemma 1.6 and (10) and taking the limit as $n \rightarrow \infty$ in (9), we have

$$\begin{aligned} \sigma(x^*, Tx^*, u) &= \sigma(Tx^*, x^*, x^*) + \beta\left(\sigma(x^*, x^*, u), \sigma(x^*, x^*, u), \sigma(x^*, Tx^*, u), \sigma(x^*, Tx^*, u), \right. \\ &\quad \left. \sigma(x^*, x^*, u), \sigma(x^*, x^*, u), \sigma(x^*, x^*, u), \sigma(x^*, x^*, u), \sigma(x^*, Tx^*, u)\right) + \sigma(u, x^*, x^*) \\ &= \beta\left(0, 0, \sigma(x^*, Tx^*, u), \sigma(x^*, Tx^*, u), 0, 0, 0, 0, \sigma(x^*, Tx^*, u)\right). \end{aligned}$$

Since $\beta \in \mathcal{B}$, $\sigma(x^*, Tx^*, u) = 0$. Then $\sigma(x^*, Tx^*, u) = 0$ for all $u \in X$. So we have $Tx^* = x^*$, that is, x^* is a fixed point of T .

Step 3. We prove that the fixed point of T is unique. Let $x^*, y^* \in X$ such that $Tx^* = x^*, Ty^* = y^*$. By using (6) again, we have

$$\begin{aligned} \sigma(x^*, y^*, u) &= \sigma(Tx^*, Ty^*, u) \\ &\leq \beta\left(\sigma(x^*, y^*, u), \sigma(x^*, Tx^*, u), \sigma(y^*, Ty^*, u), \sigma(x^*, Ty^*, u), \sigma(y^*, Tx^*, u), \right. \\ &\quad \left. \sigma(T^2x^*, x^*, u), \sigma(T^2x^*, Tx^*, u), \sigma(T^2x^*, y^*, u), \sigma(T^2x^*, Ty^*, u)\right) \\ &= \beta\left(\sigma(x^*, y^*, u), \sigma(x^*, x^*, u), \sigma(y^*, y^*, u), \sigma(x^*, y^*, u), \sigma(y^*, x^*, u), \right. \\ &\quad \left. \sigma(x^*, x^*, u), \sigma(x^*, x^*, u), \sigma(x^*, y^*, u), \sigma(x^*, y^*, u)\right) \\ &= \beta\left(\sigma(x^*, y^*, u), 0, 0, \sigma(x^*, y^*, u), \sigma(x^*, y^*, u), 0, 0, \sigma(x^*, y^*, u), \sigma(x^*, y^*, u)\right). \end{aligned}$$

Since $\beta \in \mathcal{B}$, $\sigma(x^*, y^*, u) = 0$. This proves that $x^* = y^*$. Then the fixed point of T is unique.

Step 4. For each $x \in X$, we prove that $\lim_{n \rightarrow \infty} T^n x_n = x^*$. For each $u \in X$, by using (8), we have

$$\lim_{n \rightarrow \infty} \sigma(T^n x, x^*, u) = \lim_{n \rightarrow \infty} \sigma(x_n, x^*, u) = 0.$$

This proves that $\lim_{n \rightarrow \infty} T^n x = x^*$. \square

3. Examples and Comparison with Known Results

In this section, we give some examples and compare our results with known ones to show that our fixed point theorems are generalizations of some results in [3], [21], [23] and are better than some known results. The following lemma is easy to prove.

Lemma 3.1. 1. If $\alpha \in \mathcal{A}$ and

$$\beta(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = \alpha(x_1, x_2, x_3)$$

for all $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \in \mathbb{R}_+$, then $\beta \in \mathcal{B}$.

2. If $\beta \in \mathcal{B}$ and $\alpha(x, y, z) = \beta(x, y, z, 0, 0, 0, 0, 0, 0)$ for all $x, y, z \in \mathbb{R}_+$, then $\alpha \in \mathcal{A}$.

By using Lemma 3.1.(1), Theorem 2.1 and Theorem 2.3, we get following corollaries.

Corollary 3.2 ([3], Theorem 5). Let T be an A -contraction on a complete metric space X . Then T has a unique fixed point in X such that the sequence $\{T^n x\}$ converges to the fixed point for any $x \in X$.

Corollary 3.3 ([23], Theorem 2.1). Let T be an A -contraction on a complete 2-metric space X . Then T has a unique fixed point in X .

In 2010, Popa et al. [19] proved some common fixed point theorems in 2-metric spaces for two pairs of weakly compatible mappings satisfying an implicit relation where every 2-metric was assumed to be continuous, see [19, page 106]. This assumption was used in the proof of [19, Lemma 4.3]. To show the difference between the above results and the main results in [19], we give the following example.

Example 3.4. Let $X = \left\{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\right\}$ and

$$\sigma(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \text{ are distinct and } \left\{\frac{1}{n}, \frac{1}{n+1}\right\} \subset \{x, y, z\} \text{ for some } n \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then (X, σ) is a complete 2-metric space and σ is not continuous, see [18, Example 0.1].

Define $T, S : X \rightarrow X$ by $Tx = Sx = 0$ for all $x \in X$. Then Theorem 2.3 is applicable to T, S . Since σ is not continuous, the main results of [19] are not applicable to T, S .

The following example is easy to prove.

Example 3.5. Let

$$\beta(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = k \max \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$$

for all $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \in \mathbb{R}_+$ and some $k \in [0, \frac{1}{2})$. Then $\beta \in \mathcal{B}$. Moreover, if $x_4 = x_5 = x_6 = 0$, then $\beta \in \mathcal{B}$ for some $k \in [0, 1)$.

Recall that there were many fixed point theorems on metric spaces. For a self-map T on a metric space (X, d) , contraction conditions in the literature usually contained at most five values $d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)$, see [5], [9], [21], [22] for example. By using Theorem 2.1 and Example 3.5, we get Corollary 3.6 which is exactly a new form of fixed point theorems in metric spaces.

Corollary 3.6. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a map such that

$$d(Tx, Ty) \leq k \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty) \right\} \tag{11}$$

for all $x, y \in X$ and some $k \in [0, 1)$. Then T has a unique fixed point x^* and $\lim_{n \rightarrow \infty} T^n x = x^*$ for all $x \in X$.

Proof. Let

$$\beta(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = k \max \left\{ x_1, x_2, x_3, \frac{x_4 + x_5}{2}, \frac{x_6 + x_9}{2}, x_7, x_8, x_9 \right\}$$

for all $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \in \mathbb{R}^+$. Then β is continuous.

If $x \leq \beta(0, 0, x, x, 0, 0, 0, 0, x) = k \max \left\{ 0, 0, x, \frac{x}{2}, \frac{x}{2}, 0, 0, x \right\} = kx$, then $x = 0$.

If $y \leq \beta(x, x, y, z, 0, z, y, y, 0) = k \max \left\{ x, x, y, \frac{z+0}{2}, \frac{z+0}{2}, y, y, 0 \right\}$ and $z \leq x + y$, then $y \leq k \max \{x, y\}$. It implies that $y \leq kx$.

The remaining is proved similarly. Then $\beta \in \mathcal{B}$ and note that the condition (11) becomes the condition (1). Then, by using Theorem 2.1, we get the conclusion. \square

As similar as the proof of Corollary 3.6, we get the following.

Corollary 3.7. Let (X, σ) be a complete 2-metric space and $T : X \rightarrow X$ be a map such that

$$\sigma(Tx, Ty, u) \leq k \max \left\{ \sigma(x, y, u), \sigma(x, Tx, u), \sigma(y, Ty, u), \frac{\sigma(x, Ty, u) + \sigma(y, Tx, u)}{2}, \frac{\sigma(T^2x, x, u) + \sigma(T^2x, Ty, u)}{2}, \sigma(T^2x, Tx, u), \sigma(T^2x, y, u), \sigma(T^2x, Ty, u) \right\} \tag{12}$$

for all $x, y, u \in X$ and some $k \in [0, 1)$. Then T has a unique fixed point x^* and $\lim_{n \rightarrow \infty} T^n x = x^*$ for all $x \in X$.

Note that Corollary 3.6 is a generalization of the following well-known result, where the contraction condition (13) is the contraction condition (19'') in [21].

Corollary 3.8. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a map such that

$$d(Tx, Ty) \leq k \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \tag{13}$$

for all $x, y \in X$ and some $k \in [0, 1)$. Then T has a unique fixed point x^* and $\lim_{n \rightarrow \infty} T^n x = x^*$ for all $x \in X$.

The following example shows the validity of Corollary 3.6 in comparison with Corollary 3.8, also with Ćirić fixed point theorem, see [8, Theorem 1].

Example 3.9. Let $X = \{-2, -1, 0, 1, 2\}$ and let d be defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } (x, y) \in \{(-2, 1), (-2, 2), (1, -2), (2, -2)\} \\ 1 & \text{otherwise.} \end{cases}$$

Then (X, d) is a complete metric space. Let $T : X \rightarrow X$ be defined by

$$T(-2) = T(-1) = T0 = -2, T1 = -1, T2 = 0.$$

We have

$$\begin{aligned} d(Tx, Ty) &= d(-2, -2) = 0 \text{ if } x, y \in \{-2, -1, 0\} \\ d(T(-2), T1) &= d(T(-1), T1) = d(T0, T1) = d(-2, -1) = 1 \\ d(T(-2), 1) &= d(T(-1), 1) = d(T0, 1) = d(-2, 1) = 2 \\ d(T(-2), T2) &= d(T(-1), T2) = d(T0, T2) = d(-2, 0) = 1 \\ d(T(-2), 2) &= d(T(-1), 2) = d(T0, 2) = d(-2, 2) = 2 \\ d(T1, T2) &= d(-1, 0) = 1 \\ d(1, 2) &= d(1, T1) = d(2, T2) = d(1, T2) = d(2, T1) = 1 \\ d(T^2 1, 1) &= d(T(-1), 1) = d(-2, 1) = 2. \end{aligned}$$

The above calculations show that condition (13) and condition (B) in [8, Definition 1] do not hold for $x = 1, y = 2$ but (11) holds for all $x, y \in X$ and some $q \in [\frac{1}{2}, 1)$. Then Corollary 3.8 and [8, Theorem 1] are not applicable to T and (X, d) while Corollary 3.6 is applicable to T and (X, d) .

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References

- [1] B. Ahmad and F. U. Rehman, Some fixed point theorems in complete metric spaces, *Math. Japonica* 36 (1991), no. 2, 239 – 243.
- [2] G. Akinbo, M. O. Olatinwo, and A. O. Bosedede, A note on A -contractions and common fixed points, *Acta Univ. Apulensis* (2010), no. 23, 91 – 98.
- [3] M. Akram, A. A. Zafar, and A. A. Siddiqui, A general class of contractions: A -contractions, *Novi Sad J. Math.* 38 (2008), no. 1, 25 – 33.
- [4] A. Aliouche and C. Simpson, Fixed points and lines in 2-metric spaces, *Adv. Math.* 229 (2012), no. 1, 668 – 690.
- [5] V. Berinde, Iterative approximation of fixed points, *Lecture Notes in Mathematics*, Springer, 2006.
- [6] R. Bianchini, Su un problema di S. Reich riguardante la teori dei punti fissi, *Boll. Un. Math. Ital.* 5 (1972), 103 – 108.
- [7] Z. Chuanyi, The generalized set-valued contraction and completeness of the metric space, *Math. Japonica* 35 (1990), no. 1, 111 – 118.
- [8] L. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* 45 (1974), 267 – 273.
- [9] P. Collaço and J. C. E. Silva, A complete comparison of 25 contraction conditions, *Nonlinear Anal.* 30 (1997), no. 1, 471 – 476.
- [10] D. Dey and M. Saha, Some common fixed point theorems in a complete 2-metric space, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* 52 (2013), no. 1, 79 – 87.
- [11] N. V. Dung and V. T. L. Hang, Fixed point theorems for weakly C -contractive mappings in ordered 2-metric spaces, *Fixed Point Theory Appl.* 2013:161 (2013), 1 – 14.
- [12] N. V. Dung, N. T. Hieu, N. T. T. Ly, and V. D. Thinh, Remarks on the fixed point problem of 2-metric spaces, *Fixed Point Theory Appl.* 2013:167 (2013), 1 – 7.

- [13] V. S. Gähler, 2-metrische Räume und ihre topologische struktur, *Math. Nachr.* 26 (1963/64), 115 – 118.
- [14] V. Gupta and R. Kaur, Some common fixed point theorems for a class of A -contractions on 2-metric space, *Int. J. Pure Appl. Math.* 78 (2012), no. 6, 909 – 916.
- [15] K. Iseki, Fixed point theorems in 2-metric spaces, *Math. Seminar Notes, Kobe Univ.* 3 (1975), 133 – 136.
- [16] M. S. Khan, On fixed point theorems, *Math. Japonica* 23 (1978), no. 2, 201 – 204.
- [17] B. K. Lahiri, P. Das, and L. K. Dey, Cantor’s theorem in 2-metric spaces and its applications to fixed point problems, *Taiwanese J. Math.* 15 (2011), no. 1, 337 – 352.
- [18] S. V. R. Naidu and J. R. Prasad, Fixed point theorems in 2-metric spaces, *Indian J. Pure Appl. Math.* 17 (1986), no. 8, 974 – 993.
- [19] V. Popa, M. Imdad, and J. Ali, Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces, *Bull. Malays. Math. Sci. Soc.* (2) 33 (2010), no. 1, 105 – 120.
- [20] S. Reich and T. Kannan, Fixed point theorem, *Boll. Un. Math. Ital.* 4 (1971), 1 – 11.
- [21] B. E. Rhoades, A comparison of various definition of contractive mappings, *Trans. Amer. Math. Soc.* 226 (1977), 257 – 290.
- [22] B. E. Rhoades, A biased discussion of fixed point theory, *Carpathian J. Math.* 23 (2007), no. 1 - 2, 11 – 26.
- [23] M. Saha and D. Dey, Fixed point theorems for a class of A -contractions on a 2-metric space, *Novi Sad J. Math.* 40 (2010), no. 1, 3 – 8.
- [24] S. L. Singh, Some contractive type principles on 2-metric spaces and applications, *Math. Sem. Notes Kobe Univ.* 7 (1979), no. 1, 1 – 11.