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Some Results on Best Proximity Points of Cyclic Meir–Keeler Contraction Mappings

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Abstract. In this paper, we study the existence and uniqueness of best proximity points for cyclic Meir–Keeler contraction mappings in metric spaces with the property W-WUC. Also, the existence of best proximity points for set-valued cyclic Meir-Keeler contraction mappings in metric spaces with the property WUC are obtained

1. Introduction and preliminaries

Let X be a metric space and A, B be nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic mapping, whenever $T(A) \subset B$ and $T(B) \subset A$. If $T : A \cup B \to A \cup B$ is a cyclic mapping, then a point $x \in A \cup B$ is called a best proximity point for T if $d(x, T(x)) = dist(A, B)$, where

$$dist(A, B) = \inf\{d(x, y) : (x, y) \in A \times B\}.$$

Eldred et al. [11] introduced the notion of cyclic contraction mappings and extended the fixed point result of Kirk et al. [16] to a best proximity result in uniformly convex spaces. Later on Suzuki et al. [21] introduced the concept of the UC property for a pair (A, B) of a metric space. Furthermore, Suzuki et al. [21] introduced the notion of cyclic Meir-Keeler contraction as a generalization of cyclic contraction and they obtained a best proximity point theorem for a cyclic Meir-Keeler contraction mapping in a metric space with the property UC. Very recently Fakhar et al. [13] extended the best proximity result in [21] to set-valued mappings. Recently, Espínola and Fernández-León [12] introduced the property WUC and W-WUC as a generalization of the property UC. They also showed that the property WUC is weaker than of the property UC and proved that every pair of nonempty and convex subsets (A, B) of a UKK reflexive Banach space and a strictly convex Banach space has the WUC property. Furthermore, in [12] an existence, uniqueness and convergence theorem for cyclic contraction mappings in metric spaces with the property WUC is proved. Very recently Piatek [19] extended the result in [12] to a cyclic Meir-Keeler contraction mappings under additional conditions. In the recent years many authors studied the existence of a best

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proximity point for single-valued mappings under some suitable contraction conditions, for more details; see [2, 3, 5–9, 12, 14, 19–21, 23] and references therein.
Here, we prove the existence of best proximity points for a Meir-Keeler contraction mapping which is defined on a pair of subsets of a metric space with the property W-WUC. This result improves the result in [19]. Then, by the concept of set-valued cyclic Meir-Keeler contraction mappings we prove the existence of best proximity points for such mappings in metric spaces for pairs of sets verifying the property WUC.
In the sequel, we recall some notions and results which will be used in this paper.

Definition 1.1. *([12]) Let A and B be nonempty subsets of a metric space (X, d). Then pair (A, B) is said to satisfy the WUC property if for any $\{x_n\} \subset A$ such that for every $\varepsilon > 0$ there exists $y \in B$ satisfying that $d(x_n, y) \leq dist(A, B) + \varepsilon$ for $n \geq n_0$, then it is the case that $\{x_n\}$ is convergent.*

Definition 1.2. *([12]) Let A and B be nonempty subsets of a metric space (X, d). Then pair (A, B) is said to satisfy the property W-WUC, if for any $\{x_n\} \subset A$ such that for every $\varepsilon > 0$ there exists $y \in B$ satisfying that $d(x_n, y) \leq dist(A, B) + \varepsilon$ for $n \geq n_0$, then there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$.*

Remark 1.3. *In [12], it was shown that every pair (A, B) of nonempty subsets of a metric space with the UC property such that A is complete, has the WUC property. Also, they prove that every pair of nonempty and convex subsets (A, B) of a UKK reflexive Banach space and a strictly convex Banach space has the WUC property. Also, notice that if A is a nonempty compact subset of a metric space (X, d), then for every nonempty subset B of (X, d) the pair (A, B) has the W-WUC property.*

The following example shows that the W-WUC property is weaker than the WUC property.

Example 1.4. *Let $A = \{1 + \frac{1}{n}, n \in \mathbb{N}\} \cup \{-1 - \frac{1}{n}, n \in \mathbb{N}\} \cup \{\pm 1\}$ and $B = \{3 + \frac{1}{n}, n \in \mathbb{N}\} \cup \{0\}$ be subsets of $(\mathbb{R}, |.|)$. It is clear that $d(A, B) = 1$ and the pair (A, B) has the W-WUC property. Moreover, for the sequence $\{(-1)^n + \frac{(-1)^n}{n}\}$ and for every $\varepsilon > 0$ we have that $|(-1)^n + \frac{(-1)^n}{n} - 0| \leq dist(A, B) + \varepsilon$ for sufficiently large n but this sequence does not converge. Therefore, the the pair (A, B) does not have the WUC property.*

Definition 1.5. *([21]) Let (X, d) be a metric space, A and B be nonempty subsets of X. Then a mapping $T : A \cup B \to A \cup B$ is said to be a cyclic Meir–Keeler contraction if the following conditions hold.*

- *(a1) $T(A) \subset B$ and $T(B) \subset A$.*
- *(a2) For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(x, y) < dist(A, B) + \varepsilon + \delta \text{ implies } d(T(x), T(y)) < dist(A, B) + \varepsilon$$

for all $x \in A$ and $y \in B$.

Theorem 1.6. *([19]) Let A and B be nonempty subsets of a metric space (X, d) such that (A, B) satisfies the WUC property and A is complete. Suppose that $T : A \cup B \to A \cup B$ is a cyclic Meir–Keeler contraction such that*

- *(I) there is $x \in A$ with bonded orbit $\{T^n(x) : n \in \mathbb{N}\}$;*
- *(II) for each $r > dist(A, B)$, there is $\varepsilon > 0$ such that $dist(A, B) + \varepsilon < r < dist(A, B) + \varepsilon + \delta(\varepsilon)$.*

Then

- *(i) T has a unique best proximity point $z \in A$;*
- *(ii) z is a fixed point of T^2;*
- *(iii) for each $x \in A$ the sequence $\{T^{2n}(x)\}$ tends to z.*

Let (X, d) be a metric space, $\mathcal{CB}(X)$ and $\mathcal{K}(X)$ denote the family of all nonempty closed and bounded subsets of X and the family of all nonempty compact subsets of X, respectively. Then, the Pompeiu-Hausdorff metric on $\mathcal{CB}(X)$ is given by

$$H(C, D) = \max\{e(C, D), e(D, C)\},$$

where $e(C, D) = \sup_{a \in C} d(a, D)$ and $d(a, D) = \inf_{b \in D} d(a, b)$. It is well known that if (X, d) is a complete metric space, then $(\mathcal{K}(X), H)$ is a complete metric space.

Definition 1.7. *[13] Let (X, d) be a metric space, let A and B be nonempty subsets of X. A set-valued map $T : A \cup B \multimap A \cup B$ is a set-valued cyclic Meir–Keeler contraction if it satisfies:*

(h1) $T(A) \subset B$ and $T(B) \subset A$.

(h2) For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < dist(A, B) + \varepsilon + \delta \text{ implies } H(T(x), T(y)) < dist(A, B) + \varepsilon$$

for all $x \in A$ and $y \in B$.

Definition 1.8. *([17]) A function φ from $[0, \infty)$ into itself is called an L-function if $\varphi(0) = 0, \varphi(s) > 0$ for $s \in (0, \infty)$, and for every $s \in (0, \infty)$ there exists $\delta > 0$ such that $\varphi(t) \leq s$ for all $t \in [s, s + \delta]$.*

Lemma 1.9. *([9]) Let φ be an L-function. Let $\{s_n\}$ be a nonincreasing sequence of nonnegative real numbers. Suppose $s_{n+1} < \varphi(s_n)$ for all $n \in \mathbb{N}$ with $s_n > 0$. Then, $\lim_n s_n = 0$.*

In the following a characterization of the set-valued cyclic Meir–Keeler contraction mappings is given; see [13].

Proposition 1.10. *[13] Let (X, d) be a metric space, A and B be nonempty subsets of X. Let $T : A \cup B \multimap A \cup B$ be a set-valued mapping. Then, T is set-valued cyclic Meir–Keeler contraction if and only if there exists an (nondecreasing, continuous) L-function φ such that*

$$d(x, y) > dist(A, B) \text{ implies } H(T(x), T(y)) < \varphi(d(x, y) - dist(A, B)) + dist(A, B)$$

and

$$d(x, y) = dist(A, B) \text{ implies } H(T(x), T(y)) = dist(A, B)$$

for all $x \in A$ and $y \in B$.

As a consequence of the above proposition we have the following result.

Lemma 1.11. *[13] Let (X, d) be a metric space, A and B be nonempty subsets of X. Suppose that $T : A \cup B \multimap A \cup B$ is a set-valued cyclic Meir–Keeler contraction and φ is an L-function as in Proposition 1.10. Then,*

(i) $H(T(x), T(y)) \leq d(x, y)$, $\forall x \in A$ and $y \in B$.

(ii) $H(T(x), T(y)) \leq \varphi(d(x, y) - dist(A, B)) + dist(A, B)$, $\forall x \in A$ and $y \in B$.

2. Main Results

In this section, we prove the existence of a best proximity point for Meir–Keeler contraction mappings for pairs of sets verifying the property W-WUC. Then, we study the existence of best proximity points for set-valued cyclic Meir–Keeler contraction mappings.

Definition 2.1. *([12]) If A and B are two nonempty subsets of a metric space (X, d), then we say that A is a Chebyshev set for proximinal points with respect to B if for any $x \in B$ such that $d(x, A) = dist(A, B)$ we have that $P_A(x)$ is singleton, where*

$$P_A(x) = \{y \in A : d(x, y) = d(x, A)\}.$$

In the following we give a best proximity point theorem for a cyclic Meir–Keeler contraction mapping.

Theorem 2.2. *Let A and B be nonempty subsets of a metric space (X, d) and A be closed. Suppose that $T : A \cup B \to A \cup B$ is a cyclic Meir–Keeler contraction mapping and conditions (I), (II) of Theorem 1.6 are satisfied. If the pair (A, B) has the W-WUC property, then T has a best proximity point. Furthermore, if and A is a Chebysev set with respect to B, then*

(c1) T has a unique best proximity point $z \in A$;

(c2) z is a fixed point of T^2;

(c3) for each $x \in A$ the sequence $\{T^{2n}(x)\}$ tends to z.

Proof. Let $x_0 \in A$ be arbitrary and let $x_n = T(x_{n-1})$, $n \in \mathbb{N}$. From the proof of Theorem 4.1 of [19], for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$d(x_{2n}, x_{2m}) < dist(A, B) + \varepsilon$$

for almost all $m \in \mathbb{N}$. Since (A, B) has the W-WUC property, there exists a convergent subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$. Since A closed, there exists $z \in A$ such that $x_{2n_k} \to z$. Now, we will show that z is a best proximity point. Since

$$\begin{aligned} dist(A, B) &\le d(z, T(z)) \\ &\le d(z, x_{2n_k}) + d(x_{2n_k}, T(z)) \\ &\le d(z, x_{2n_k}) + d(x_{2n_k-1}, z) \\ &\le 2d(z, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}). \end{aligned}$$

Then, it suffices to prove that

$$\lim_k d(x_{2n_k}, x_{2n_k-1}) = dist(A, B).$$

Consider the sequence $\{s_n\}$ in $[0, \infty)$ as $s_n = d(x_n, x_{n+1}) - dist(A, B)$ and an L-function φ as in Lemma 1.9. Then, by Proposition 1.10, $s_n > 0$ implies $s_{n+1} < \varphi(s_n)$ and $s_n = 0$ implies $s_{n+1} = 0$. Also, by Lemma 1.11 the sequence $\{s_n\}$ is nonincreasing, therefore by Lemma 1.9 we have $\lim_n s_n = 0$. Therefore, $d(x_{2n_k}, x_{2n_k-1}) = dist(A, B)$ and so z is a best proximity point.

Now, suppose that A is a Chebysev set with respect to B and $z \in A$ is a best proximity point. Then

$$dist(A, B) \le d(Tz, T^2z) \le d(z, Tz) = dist(A, B)$$

and

$$d(z, Tz) \ge d(Tz, A) \ge dist(A, B) = d(z, Tz).$$

Therefore,

$$d(z, Tz) = d(Tz, T^2z) = d(Tz, A) = dist(A, B).$$

Hence, $z, T^2z \in P_A(Tz)$ and so $z = T^2z$. If $z, z' \in A$ are two best proximity points, then by the above $P_A(Tz) = \{z\}$ and $d(Tz, A) = dist(A, B)$. If we show that $z' \in P_A(Tz)$, then $z = z'$. In order to prove this fact, we show that $d(z', Tz) = dist(A, B)$. Suppose on the contrary that $d(z', Tz) > dist(A, B)$, then by Proposition 1.10

$$d(Tz', T^2z) < d(z', Tz) = d(T^2z', T^z) \le d(Tz', z) = d(Tz', T^2z).$$

a contradiction. Thus, $d(z', Tz) = dist(A, B)$ and so $z' = z$.

If $x \in X$, then $d(T^{2n}x, Tz) = d(T^{2n-2}(T^2x), T(T^2z)) \le d(T^{2n-2}x, Tz)$ for all $n \ge 2$ and $d(z, Tz) \le d(T^{2n}x, Tz)$ for all $n \in \mathbb{N}$. Moreover, $d(T^{2n_k}x, Tz) \to dist(A, B) = d(Tz, A)$. Therefore, $d(T^{2n}x, Tz) \to dist(A, B) = d(Tz, A)$. Hence, by the W-WUC property every subsequence of $\{T^{2n}x\}$ has a convergent subsequence. On the other hand A is a Chebysev set with respect to B, then any convergent subsequence of $\{T^{2n}x\}$ is convergent to z. Consequently $T^{2n}x \to z$. □

Piątek [19] showed that every cyclic contraction is a cyclic Meir-keeler contraction and satisfies the conditions (I), (II) of Theorem 2.2.
In the following we give an example which is satisfied in all the conditions of the above theorem.

Example 2.3. *Let* $A = \{(x, 0), x \in [0, 1]\}$ *and* $B = \{(x, 1), x \in [0, 1]\}$ *of* $\mathbb{R}^2$ *equipped with the Euclidean metric and let* $T(x)$ *be defined as follows:*

$$T(x, y) = \begin{cases} \{(\frac{1}{2}x, 1)\} & \text{if } (x, y) \in A, \\ \{(\frac{1}{2}x, 0)\} & \text{if } (x, y) \in B, \end{cases}$$

Then, the pair (A, B) *satisfies the property W-WUC, A is a Chebysev set with respect to B and T is a cyclic contraction map so it is cyclic Meir-keeler contraction. Furthermore, the conditions (I), (II) of Theorem 2.2 are satisfied and* $(0, 0)$ *is a unique best proximity point in A for T.*

Now, we present an example where the set A is not a Chebysev set with respect to B and the best proximity point is not unique.

Example 2.4. *Let A, B be the same as in Example 1.4 and let* $T(x)$ *be defined as follows:*

$$T(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B, \end{cases}$$

Then, the pair (A, B) *has the W-WUC property, T is a Meir-keeler contraction and satisfies the conditions (I), (II) of Theorem 2.2. Also* $x = 1$ *and* $x = -1$ *are best proximity points of T.*

In order to obtain a best proximity result for set-valued cyclic Meir–Keeler contraction mappings, we need the following result.

Proposition 2.5. *Let* (X, d) *be a complete metric space and A and B be nonempty subsets of X such that the pair* (A, B) *satisfies the property WUC. Then, the pair* $(\mathcal{K}(A), \mathcal{K}(B))$ *in* $(\mathcal{CB}(X), H)$ *has the WUC property.*

Proof. In the first step, we show that $dist(A, B) = dist(\mathcal{K}(A), \mathcal{K}(B))$. Since singleton sets are compact, then $dist(\mathcal{K}(A), \mathcal{K}(B)) \leq dist(A, B)$. On the other hand, if $C \in \mathcal{K}(A)$ and $D \in \mathcal{K}(B)$, then for each $x \in C$ there exists $y \in D$ such that $d(x, y) \leq H(C, D)$ and so $dist(A, B) \leq dist(\mathcal{K}(A), \mathcal{K}(B))$. Therefore, $dist(A, B) = dist(\mathcal{K}(A), \mathcal{K}(B))$. Now, let $\{C_m\}$ be a sequence in $\mathcal{K}(A)$ such that for every $\varepsilon > 0$ there exists $Y \subset \mathcal{K}(B)$ satisfying $H(C_m, Y) \leq dist(A, B) + \varepsilon$ for $m \geq m_0$. Since X is a complete metric space, then $(\mathcal{K}(X), H)$ is a complete metric space. We are going to show that $\lim_m H(C_n, C_m) = 0$ for all $m > n \geq m_0$. Since $e(C_n, C_m) = \sup_{x \in C_n} d(x, C_m)$ and C_n is compact, then there exists $x_n \in C_n$ such that

$$e(C_n, C_m) = d(x_n, C_m). \tag{1}$$

On the other hand

$$dist(A, B) \leq d(x_m, Y) \leq e(C_m, Y) \leq H(C_m, Y) \leq dist(A, B) + \varepsilon.$$

Therefore, $d(x_m, Y) \leq dist(A, B) + \varepsilon$. Since Y is compact, there exists $y \in Y$ such that $d(x_m, y) = d(x_m, Y)$ and so $d(x_m, y) \leq dist(A, B) + \varepsilon$. But (A, B) has the property WUC, then $\lim_m d(x_m, x_n) = 0$. Also, by (1), we have $e(C_n, C_m) = d(x_n, C_m) \leq d(x_n, x_m)$. Thus, $\lim_m e(C_n, C_m) = 0$. By the same argument as the above we obtain $\lim_m e(C_m, C_n) = 0$. Hence, $\lim_n H(C_n, C_m) = 0$. □

Now, we are ready to state our main result.

Theorem 2.6. *Let* (X, d) *be a complete metric space, A and B be nonempty subsets of X such that* (A, B) *satisfies the property WUC. Assume that A is closed and* $T : A \cup B \multimap A \cup B$ *is a set-valued cyclic Meir–Keeler contraction such that* $T(D)$ *is compact for any* $D \in \mathcal{K}(A) \cup \mathcal{K}(B)$*. If the following conditions hold:*

(A1) there is $U \in \mathcal{K}(A)$ *with bonded orbit* $\{T^n(U) : n \in \mathbb{N}\}$*,*

(A2) for each $r > dist(A, B)$, there is a $\varepsilon > 0$ such that $dist(A, B) + \varepsilon < r < dist(A, B) + \varepsilon + \delta(\varepsilon)$.

Then, T has a best proximity point x in A, i.e., $d(x, T(x)) = dist(A, B)$. Furthermore, if $y \in T(x)$ and $d(x, y) = dist(A, B)$, then y is a best proximity point in B and x is a fixed point of T^2.

Proof. Let $F : \mathcal{K}(A) \cup \mathcal{K}(B) \to \mathcal{K}(A) \cup \mathcal{K}(B)$ be defined by $F(D) = T(D)$ for all $D \in \mathcal{K}(A) \cup \mathcal{K}(B)$. By our assumption F is well defined. Since T is cyclic, F is cyclic. Also, from Proposition 2.5, the pair $(\mathcal{K}(A), \mathcal{K}(B))$ has the property WUC. We now show that F is a Meir–Keeler contraction mapping. To prove this assertion we repeat an argument given in the proof of Theorem 2.10 in [13]. Since T is a set-valued Meir–Keeler contraction mapping, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \varepsilon + \delta + dist(A, B) \Rightarrow H(T(x), T(y)) < \varepsilon + dist(A, B), x \in A, \ y \in B. \tag{2}$$

Let $C \in \mathcal{K}(A)$ and $D \in \mathcal{K}(B)$ be such that $H(C, D) < \delta + \varepsilon + dist(A, B)$, we show that

$$H(F(C), F(D)) < \varepsilon + dist(A, B).$$

If $H(C, D) < \delta + \varepsilon + dist(A, B)$, then $e(C, D) < \delta + \varepsilon + dist(A, B)$. Therefore, if z is an arbitrary point in C, then $d(z, D) < \delta + \varepsilon + dist(A, B)$. Since D is compact, there exists $w \in D$ such that $d(z, w) = d(z, D)$ and so $d(z, w) < \delta + \varepsilon + dist(A, B)$. Hence, by (7) we have $H(T(z), T(w)) < \varepsilon + dist(A, B)$. Thus, $e(T(z), T(w)) < \varepsilon + dist(A, B)$. It follows that $e(T(z), T(D)) < \varepsilon + dist(A, B)$. Since $z \in C$ is an arbitrary point, then $e(T(C), T(D)) < \varepsilon + dist(A, B)$. Also, $e(D, C) \le H(C, D) < \delta + \varepsilon + dist(A, B)$. Therefore, by the same argument as the above we deduce that $e(T(D), T(C)) < \varepsilon + dist(A, B)$. Hence, $H(T(C), T(D)) < \varepsilon + dist(A, B)$. Thus, $H(C, D) < \varepsilon + \delta + dist(A, B)$ implies

$$H(F(C), F(D)) < \varepsilon + dist(A, B), \ \ C \in \mathcal{K}(A), \quad D \in \mathcal{K}(B).$$

Hence, F is Meir–Keeler contraction. Therefore, by Theorem 1.6 there exists a unique point $E \in \mathcal{K}(A)$ such that $H(E, F(E)) = dist(A, B)$ and $F^2(E) = E$. But T is a set-valued cyclic Meir–Keeler contraction mapping. So by Lemma 1.11, there exists a (nondecreasing, continuous) L-function φ such that

(i) $H(T(x), T(y)) \le d(x, y)$ for all $(x, y) \in A \times B$.

(ii) $H(T(x), T(y)) - dist(A, B) \le \varphi(d(x, y) - dist(A, B))$ for all $(x, y) \in A \times B$.

Let $x_0 \in E$. If for some $y \in T(x_0), d(x_0, y) = dist(A, B)$, then $d(x_0, T(x_0)) = dist(A, B)$. Therefore, x_0 is a best proximity point in A. Suppose that $x_1 \in T(x_0)$ and $d(x_0, x_1) > dist(A, B)$, then by Proposition 1.10

$$H(T(x_1), T(x_0)) - dist(A, B) < \varphi(d(x_1, x_0) - dist(A, B)).$$

Since $d(x_1, T(x_1) \le H(T(x_0), T(x_1))$ and $T(x_1)$ is compact, there exists $x_2 \in T(x_1)$ such that

$$d(x_2, x_1) - dist(A, B) \le H(T(x_1), T(x_0)) < \varphi(d(x_1, x_0) - dist(A, B)).$$

If $d(x_2, x_1) - dist(A, B) = 0$, then

$$dist(A, B) \le d(x_2, T(x_2)) \le H(T(x_1), T(x_2)) \le d(x_1, x_2) = dist(A, B).$$

Hence, x_2 is a best proximity point in A. Otherwise, by the same argument as the above there exists $x_3 \in T(x_2)$ such that

$$d(x_3, x_2) - dist(A, B) \le H(T(x_2), T(x_1)) < \varphi(d(x_2, x_1) - dist(A, B)).$$

By continuing in this way, either T has a best proximity point in A or there is a sequence $\{x_n\}$ in $E \cup T(E)$ such that $x_{n+1} \in T(x_n)$, $x_{2n} \in E$, $x_{2n+1} \in T(E)$ and

$$d(x_{n+1}, x_n) - dist(A, B) \le H(T(x_n), T(x_{n-1})) - dist(A, B) < \varphi(d(x_n, x_{n-1}) - dist(A, B)). \tag{3}$$

Define a sequence $\{s_n\}$ in $(0,\infty)$ by $s_n = d(x_n, x_{n+1}) - dist(A,B)$. Then, by inequality (3), $\{s_n\}$ is nonincreasing sequence and $s_{n+1} < \varphi(s_n)$. Therefore, from Lemma 1.9, we have $\lim_n s_n = 0$. Hence,

$$\lim_n d(x_n, x_{n+1}) = dist(A,B). \tag{4}$$

On the other hand E is compact and $x_{2n} \in E$, then there exists a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ such that

$$\lim_k x_{2n_k} = x \in E. \tag{5}$$

Since

$$dist(A,B) \leq d(x, x_{2n_k-1}) \leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}),$$

then from (4) and (5) we have

$$\lim_k d(x, x_{2n_k-1}) = dist(A,B). \tag{6}$$

Since

$$dist(A,B) \leq d(x_{2n_k}, T(x)) \leq H(T(x_{2n_k-1}), T(x)) < d(x_{2n_k-1}, x) \ \ \forall k \in \mathbb{N},$$

then from (5) and (6), we have $d(x, T(x)) = dist(A,B)$. Therefore, T has a best proximity point in A. Let $y \in T(x)$ such that $d(x,y) = dist(A,B)$, then

$$dist(A,B) \leq d(y, T(y)) \leq H(T(x), T(y)) \leq d(x,y) = dist(A,B)$$

and so $d(y,T(y)) = dist(A,B)$. Hence, y is a best proximity point of T in B. Also, $T(y) \subset T^2(x)$, then $dist(A,B) \leq d(y, T^2(x)) \leq d(y, T(y)) = dist(A,B)$ and so $d(y, T^2(x)) = dist(A,B)$. But $T^2(x)$ is compact, thus there exists $z \in T^2(x)$ such that $d(z,y) = dist(A,B)$. The WUC property of the pair (A,B) implies that $d(x,z) = 0$. Therefore, $x = z \in T^2(x)$, that is x is a fixed point of T^2. □

Now, let us introduce the notion of set-valued cyclic contraction mappings.

Definition 2.7. *Let (X,d) be a metric space, let A and B be nonempty subsets of X. Then $T : A \cup B \multimap A \cup B$ is a set-valued cyclic contraction mapping if it satisfies:*

(i) $T(A) \subset B$ and $T(B) \subset A$.

(ii) There exists $k \in (0,1)$ such that

$$H(T(x), T(y)) \leq kd(x,y) + (1-k)dist(A,B)$$

for all $x \in A$ and $y \in B$.

Proposition 2.8. *[11] Let A and B be nonempty subsets of a metric space X, let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then for $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Corollary 2.9. *Let (X,d) be a complete metric space, A and B be nonempty subsets of X such that (A,B) satisfies the property WUC. Assume that $T : A \cup B \multimap A \cup B$ is a set-valued cyclic contraction mapping such that $T(D)$ is compact for any $D \in \mathcal{K}(A) \cup \mathcal{K}(B)$. Then, T has a best proximity point x in A.*

Proof. Since T is set-valued contraction mapping, then there exists $k \in (0,1)$ such that that

$$H(T(x), T(y)) \leq kd(x,y) + (1-k)dist(A,B), x \in A, \ y \in B. \tag{7}$$

Therefore, T is a set-valued cyclic Meir–Keeler contraction by taking $\delta(\varepsilon) = \frac{1-k}{k}\varepsilon$ for all $\varepsilon > 0$. Now, we show that conditions $(A1)$ and $(A2)$ of Theorem 2.6 are satisfied.

Let $F : \mathcal{K}(A) \cup \mathcal{K}(B) \to \mathcal{K}(A) \cup \mathcal{K}(B)$ be defined by $F(D) = T(D)$ for all $D \in \mathcal{K}(A) \cup \mathcal{K}(B)$. By assumption, F is well defined and it is cyclic. Let $C \in \mathcal{K}(A)$ and $D \in \mathcal{K}(B)$, we show that

$$H(F(C), F(D)) \leq kH(C, D) + (1 - k)dist(A, B).$$

We have $e(C, D) \leq H(C, D)$. Therefore, if z is an arbitrary point in C, then $d(z, D) \leq H(C, D)$. Since D is compact, there exists $w \in D$ such that $d(z, w) = d(z, D)$ and so $d(z, w) \leq H(C, D)$. Hence, by (7) we have

$$H(T(z), T(w)) \leq kd(z, w) + (1 - k)dist(A, B) \leq kH(C, D) + (1 - k)dist(A, B).$$

Thus, $e(T(z), T(w)) \leq kH(C, D) + (1 - k)dist(A, B)$. It follows that

$$e(T(z), T(D)) \leq kH(C, D) + (1 - k)dist(A, B).$$

Since $z \in C$ is an arbitrary point, then $e(T(C), T(D)) \leq kH(C, D) + (1 - k)dist(A, B)$. Therefore, by the above argument, we deduce that

$$e(T(D), T(C)) \leq kH(C, D) + (1 - k)dist(A, B).$$

Hence, $H(T(C), T(D)) \leq kH(C, D) + (1 - k)dist(A, B)$. Therefore,

$$H(F(C), F(D)) \leq kH(C, D) + (1 - k)dist(A, B), \quad C \in \mathcal{K}(A), \quad D \in \mathcal{K}(B).$$

Hence, F is a cyclic contraction mapping with respect to Pompeiu-Hausdorff metric. Thus, by Proposition 2.8 for $\{x_0\} \in \mathcal{K}(A)$ and $F^n(\{x_0\}) = T^n(x_0)$, we have $\lim_{n\to\infty} H(T^n(x_0), T^{n+1}(x_0)) = dist(A, B)$ and sequences $\{T^{2n}(x_0)\}$ and $\{T^{2n+1}(x_0)\}$ are bounded with respect to Pompeiu-Hausdorff metric. Therefore, the set $\{T^n(x_0) : n \in \mathbb{N}\}$ is bounded and so condition (A1) holds. Now, let $r > dist(A, B)$ and putting $\varepsilon = (r - dist(A, B))k + \varepsilon_1$ such that $\varepsilon_1 = \frac{(r-dist(A,B))(1-k)}{2} > 0$. It is clear $dist(A, B) + \varepsilon < r$ and $dist(A, B) + \varepsilon + \delta(\varepsilon) > r$. Thus, condition (A2) holds. Hence, from Theorem 2.6 T has a best proximity point x in A. □

The following example shows that the set-valued contraction condition may be violated while the hypotheses of Theorem 2.6 are fulfilled.

Example 2.10. *Let $A = [0, \frac{1}{3}] \cup \{3, 5, 7, ...\}$ and $B = [\frac{2}{3}, 1) \cup \{2, 4, 6, ...\}$ with the Euclidean distance , and let $T(x)$ be defined as follows:*

$$T(x) = \begin{cases} \{\frac{x+2}{3}, \frac{2}{3}\} & \textit{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{3} & \textit{if } \frac{2}{3} \leq x < 1, \\ 0 & \textit{if } x = 2n, \\ 1 - \frac{1}{n+2} & \textit{if } x = 2n + 1, \end{cases}$$

It is clear that (A, B) satisfies property WUC and T is a set-valued cyclic Meir–Keeler contraction mapping which is satisfied in conditions (A1) and (A2) of Theorem 2.6. But T is not a set-valued contraction mapping.

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