Filomat 31:12 (2017), 3715–3726
https://doi.org/10.2298/FIL1712715A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Regular Ideals in Reduced Rings

A.R. Aliabad[a], R. Mohamadian[a], S. Nazari[a]

[a]*Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran*

Abstract. Let R be a commutative ring with identity and X be a Tychonoff space. An ideal I of R is Von Neumann regular (briefly, regular) if for every $a \in I$, there exists $b \in R$ such that $a = a^2b$. In the present paper, we obtain the general form of a regular ideal in $C(X)$ which is O^A, for some closed subset A of βX, for which $A^c \cap X \subseteq (P(X))^\circ$, where $P(X)$ is the set of all P-points of X. We show that the ideals and subrings such as $C_K(X)$, $C_\psi(X)$, $C_\infty(X)$, $Soc_mC(X)$ and $M^{\beta X\setminus X}$ are regular if and only if they are equal to the socle of $C(X)$. We carry further the study of the maximal regular ideal, for instance, it is shown that for a vast class of topological spaces (we call them OPD-spaces) the maximal regular ideal is $O^{X\setminus I(X)}$, where $I(X)$ is the set of isolated points of X. Also, for this class, the socle of $C(X)$ is the maximal regular ideal if and only if $I(X)$ contains no infinite closed set. We also show that $C(X)$ contains an ideal which is both essential and regular if and only if $(P(X))^\circ$ is dense in X. Finally it is shown that, for semiprimitive rings pure ideals are of the form O^A which A is a closed subset of $Max(R)$, also a P-point of $X = Max(R)$ is introduced and it is shown that the maximal regular ideal of an arbitrary ring R is $O^{X\setminus P(X)}$, which $P(X)$ is the set of P-points of $X = Max(R)$.

1. Introduction and Preliminary Results

Throughout this paper, all rings R, are commutative with identity and all topological spaces are Tychonoff. We denote by $C(X)$ (resp., $C^\star(X)$) the ring of all real-valued (resp., bounded) continuous functions on X. In this paper, we denote by βX the Stone-Cech compactification of X. For every $f \in C(X)$, $Z(f)$ is the set of zeros of f and $Coz(f) = X\setminus Z(f)$. For each ideal I in $C(X)$, $\eta(I) = \cap_{f\in I}Z(f)$ and $\theta(I) = \cap_{f\in I}cl_{\beta X}Z(f)$. For every $p \in \beta X$, we set $O^p = \{f \in C(X) : p \in int_{\beta X}cl_{\beta X}Z(f)\}$ and $M^p = \{f \in C(X) : p \in cl_{\beta X}Z(f)\}$. More generally M^A and O^A, for $A \subseteq \beta X$ are defined similarly. If $A \subseteq X$, we know that $M^A = M_A = \{f \in C(X) : A \subseteq Z(f)\}$ and $O^A = O_A = \{f \in C(X) : A \subseteq Z^\circ(f)\}$. An ideal I in $C(X)$ is called fixed if $\eta(I) \neq \emptyset$, otherwise I is free. The reader is referred to [23] and [37], for undefined terms and notations about the ring of continuous functions and topology.

We denote the annihilator of a subset S of a ring R by $Ann(S)$. By P_S, M_S and $< S >$ (or SR) we mean the intersection of minimal prime ideals containing S, maximal ideals containing S and the ideal generated by S, respectively. $Spec(R), Min(R), Max(R), Rad(R)$ and $Jac(R)$ denote the set of all prime ideals, all minimal prime ideals, all maximal ideals of R and their intersections, respectively. If I is an ideal of R, then the pure

2010 *Mathematics Subject Classification*. Primary 16E50; Secondary 13A15, 13A18, 16E50, 54C40, 54G10
Keywords. Von Neumann regular rings, socle, essential ideals, P-ideal, P-space, maximal regular ideal, SVNL-ring, Gelfand ring, PM-ring, essentially P-space, pure ideal
Received: 31 January 2016; Revised: 26 November 2016; Accepted: 04 December 2016
Communicated by Ljubiša Kočinac
Email addresses: aliabady_r@scu.ac.ir (A.R. Aliabad), mohamadian_r@scu.ac.ir (R. Mohamadian), s-nazari@phdstu.scu.ac.ir (S. Nazari)

part of I is $m(I) = \{a \in R : I + Ann(a) = R\}$=$\{a \in R : a = ai$ for some $i \in I\}$. An ideal I is called pure if $m(I) = I$. The socle of a ring is the sum of all minimal ideals of R. We denote the largest regular ideal of a ring (the maximal regular ideal) by $M(R)$. For more information about the maximal regular ideal, see [16] and [4]. A non-zero ideal in a commutative ring is said to be essential if it intersects every non-zero ideal non-trivially. A ring R is said to be reduced if it contains no non-zero idempotent or $Rad(R) = (0)$. If $Jac(R) = (0)$, then we call R a semiprimitive ring. By a semiprime ideal we mean an ideal I that $x^2 \in I$ implies $x \in I$. An ideal I is said to be a P-ideal if every proper prime ideal in I is maximal in I, or equivalently, every prime ideal which does not contain I is maximal in R, for instance see [8] and [34] .

In [8], it is shown that for a reduced ring these two concepts, regular and P-ideal, are equivalent. Motivated by this, we consider regular ideals in reduced rings (specially in $C(X)$) with attention to this new view. Although some papers which contains some facts about P-ideals (for instance [3], [8], [14], [34] and [27]) helped us in this work, but it have to be mentioned that many of their results can be found in books and papers about regular rings, because they have not noticed this equivalency. We avoid considering this matter in this paper.

In Section 2, we study the regularity of some well-known ideals and subrings of $C(X)$. First we obtain the general form of the regular ideals of $C(X)$. Then we observe that being an SVNL-ring has a close relation with regularity in Gelfand rings. We also see that regularity of some of the ideals of $C(X)$ is equivalent to this fact that, they are the socle of $C(X)$. The technique that we use in this section may help one to study the regularity of other ideals and subrings of $C(X)$.

As we said, in Section 2, we will see that the regularity of some ideals of $C(X)$ make them to be $C_F(X)$, hence naturally this question arises: in which class of spaces the socle of $C(X)$ is the maximal regular ideal? It is shown in [4, Corollary 5.5], that the maximal regular ideal is $O^{X\setminus P(X)}$. In Section 3, by using this fact, we answer to this question. We also present a prime representation of the maximal regular ideal in reduced rings. Other questions such as when the maximal regular ideal is $O^{X\setminus I(X)}$and when it is $M^{X\setminus I(X)}$, is answered in this section. Section 4 is about studying the relation between essential ideals and regular ideals. We observe when the regular ideals are not essential at all. What will happen when we have an ideal which is both regular and essential in $C(X)$. Finally, some facts about essential ideals is considered in Gelfand rings. In the last section naturally, because of the close relation between Gelfand rings and $C(X)$, we simulate some of our results into Gelfand rings. We define some concepts of $C(X)$ in an arbitrary ring and achieve some results; for instance, we obtain a general form of regular ideals and pure ideals in Gelfand rings. Also we obtain a representation for the maximal regular ideal.

For our purpose we need the following lemmas and propositions which are established and can be found in [8, Theorem 3.3 and Theorem 3.5], [6], [4, Theorem 5.3] and [5, Theorem 2.4], respectively.

Lemma 1.1. *Let R be a ring and $Q \in Spec(R)$. If $A = \{P \in Min(R) : P \subseteq Q\}$ and $O(Q) = \{a \in R : Ann(a) \nsubseteq Q\}$, then:*

(a) $m(Q) \subseteq O(Q) \subseteq \cap_{P\in A} P$;

(b) *if Q is a pure ideal, then $Q \in Min(R)$.*

Furthermore, if R is reduced, then:

(c) *if $Q \in Max(R)$, then $m(Q) = O(Q) = \cap_{p\in A} P$;*

(d) *if $Q \in Max(R)$, then Q is a pure ideal if and only if $Q \in Min(R)$.*

Lemma 1.2. *For a reduced ring an ideal I is a P-ideal if and only if it is a regular ideal.*

Lemma 1.3. *If $f \in C(X)$, then $< f >$ is a regular ideal if and only if $Coz(f)$ is a clopen (closed and open) P-space.*

We know that there are many results about regular ideals in an arbitrary ring, for instance see [24]. Clearly studying of regular ideals, with considering their new equivalent (P-ideal) in reduced rings, implies some new facts about them in reduced rings. In the following we state some of these facts.

[24, Lemma 1.3], says that, if I is an ideal of R, then R is regular if and only if I and R/I are both regular, this yields the following proposition.

Proposition 1.4. *If one maximal ideal of R is regular, then R is regular.*

Proof. Let M be the regular maximal ideal of R, then R/M is a field, so it is regular, hence R is regular. □

Recall that an ideal I of R is called sz°-ideal (resp., z°-ideal) if for every finite subset $F \subseteq I$ (resp., $a \in I$), $P_F \subseteq I$ (resp., $P_a \subseteq I$). Also an ideal I of R is called sz-ideal (resp., z-ideal) if for every finite subset $F \subseteq I$ (resp., $a \in I$), $M_F \subseteq I$ (resp., $M_a \subseteq I$). For more information about these ideals for instance see [9]. In some papers such as [31], sz°-ideal (resp., z°-ideal) are called ξ-ideal (resp., d-ideal). We need the following lemma to say that a regular ideal in a semiprimitive ring is all the above ideals. The following lemma is proved for a regular element in [8, Lemma 1.7], but one can easily see that it is also true for a finite set of regular elements. We know that if F is a finite set of regular elements, then there exists a regular element a such that $< F >=< a >$. Hence by using [9, Proposition 1.4], we have the following lemma.

Lemma 1.5. *Let F be a finite subset of regular elements of a reduced ring R, then $P_F = Ann(Ann(F)) =< F >=< a >= P_a$.*

Proposition 1.6. *For a reduced ring R every regular ideal is an sz°-ideal (hence z°-ideal), furthermore if R is a semiprimitive ring then a regular ideal is an sz-ideal (and so z-ideal).*

Proof. By combining Lemma 1.5 and this fact that $M_F \subseteq P_F$, for a ring R that $Rad(R) = Jac(R)$ and finite set F (see [9, Proposition 2.9]), these results are clear. □

In the next proposition we give new equivalent for regular rings when R is an arbitrary ring. Easily this proposition follows from [24, Corollary 1.2] and [8, Proposition 3.7(a)].

Proposition 1.7. *For a ring R, the following are equivalent:*
(a) *R is a regular ring;*
(b) *For every $a \in R$, aR is a semiprime ideal;*
(c) *For every $a \in R$, aR is a pure ideal.*

Remark 1.8. It is easy to see that a minimal ideal in any ring is a P-ideal, and since the sum of P-ideals is a P-ideal, the socle of any ring is a P-ideal. In special case when R is reduced, then minimal ideals of R and the socle of R are regular ideals. Also if R is a semiprimitive ring (such as $C(X)$), then they are sz-ideal (z-ideal). In [27], it is shown that the socle of $C(X)$ is a z-ideal. This remark shows that the assertion is true for all semiprimitive rings and in particular $C(X)$.

We need the following lemma in this paper.

Lemma 1.9. *If a regular ideal I contains a prime ideal, then it is minimal prime.*

Proof. Since I is pure, $I = m(I) \subseteq \cap_{P\subseteq I, P\in Min(R)} P \subseteq I$. Hence I is a minimal prime ideal. □

2. Regularity of Some Ideals of $C(X)$

In this section, we study the regularity of some well-known ideals of $C(X)$. First we introduce some notations that we will use in this section. $C_K(X)$ and $C_\psi(X)$ denote, respectively, the ideals of $C(X)$, consisting of functions with compact support ($Supp(f) = \overline{Coz(f)}$) and pseudo-compact support. Also we know from [23] that $C_K(X) = \cap\{I : I \text{ is a free ideal of } C(X)\} = O^{\beta X\setminus X}$. $C_\infty(X)$ is the subring, consisting of functions vanishing at infinity. $C_F(X)$, consists of functions with finite cozero-set which is equal to the socle of $C(X)$, see [27]. It is also shown in [36, Theorem 3.6] and [14, Lemma 2.4] that, the socle of $C(X)$ is $O^{\beta X\setminus I(X)}$ which is the intersection of essential ideals. For a topological space X, a point $p \in \beta X$ such that for every $f \in C(X)$, $p \in cl_{\beta X}Z(f)$ implies that $p \in int_{\beta X}cl_{\beta X}Z(f)$ (or $O^p = M^p$) is called a P-point. X is called a P-space if each of its points is a P-point. We denote by $P(X)$, the set of all P-points of X, also the set of P-points of βX, with respect to X, is denoted by $P_\beta(X)$. An essentially P-space, is a topological space X that at most one point of X fails to be a P-point. The set of isolated points of X is denoted by $I(X)$. The space X is said to be pseudo-discrete if the interior of all compact sets are finite. In [4], it is shown that the maximal regular ideal of $C(X)$ is of the form $O^{X\setminus P(X)}$ and also $O^{\beta X\setminus P_\beta(X)}$.

We know that for any two subsets A and B of βX, if $O^A \subseteq O^B$, then $cl_{\beta X}B \subseteq cl_{\beta X}A$. In the following we characterize the general form of the regular ideals in $C(X)$. First we need the following proposition.

Remark 2.1. We know that $O^p = \bigcap\{P : P \in Min(R), P \subseteq M^p\} = m(M^p)$, also by [19] we have, $\theta(O^A) = \theta(M^A) = cl_{\beta X}A$. In addition, [26, Corollary 4.5] says that for an ideal I in $C(X)$ we have, $m(I) = \bigcap_{I\subseteq M} m(M)$. By using all these facts we have the following proposition.

Proposition 2.2. *The following facts hold:*

(a) *For each ideal I in $C(X)$, $m(I) = O^{\theta(I)}$ and if I is pure, then $I = O^{\theta(I)}$;*

(b) *If $A \subseteq \beta X$ and O^A is pure, then $O^A = O^{cl_{\beta X}A}$;*

(c) *If $A \subseteq X$ and O^A is pure, then $O^A = O^{cl_X A}$;*

(d) *If $A \subseteq \beta X$, then $m(M^A) = m(O^A) = O^{cl_{\beta X}A}$.*

Proof. (a) By Remark 2.1, when I is pure, then $I = m(I) = \bigcap_{I\subseteq M^p} m(M^p) = \bigcap_{p\in\theta(I)} O^p = O^{\theta(I)}$.

(b) If O^A is pure, then by (a), $O^A = O^{\theta(O^A)} = O^{cl_{\beta X}A}$.

(c) We know that $A \subseteq cl_X A \subseteq cl_{\beta X}A$, hence $O^A = O^{cl_{\beta X}A} \subseteq O^{cl_X A} \subseteq O^A$, thus $O^{cl_X A} = O^A$.

(d) $m(M^A) = O^{\theta(M^A)} = O^{cl_{\beta X}A} = O^{\theta(O^A)} = m(O^A)$. □

Compare part (a) with [2, Theorem 2.2], part (b) with [1, Theorem 3.2] and part (d) with [2, Theorem 2.3 and Theorem 2.6]. A useful corollary of the above proposition is as follows:

Corollary 2.3. *Other forms of the maximal regular ideal are $O^{X\setminus(P(X))^\circ}$ and $O^{\beta X\setminus int_{\beta X}(P_\beta(X))}$.*

Proof. Since regular ideals are pure, then by Proposition 2.2, $O^{X\setminus P(X)} = O^{cl_X(X\setminus P(X))} = O^{X\setminus(P(X))^\circ}$. The second part has a similar proof. □

Proposition 2.4. *Let I be an ideal of $C(X)$. Then:*

(a) *I is a regular ideal if and only if it is of the form O^A in which A is a closed subset of βX such that $A^c \cap X \subseteq (P(X))^\circ$;*

(b) *I is a regular ideal if and only if it is of the form O^A in which A is a closed subset of βX such that $A^c \subseteq int_{\beta X}P_\beta(X)$.*

Proof. (a). Let I is a regular ideal, then by Proposition 2.2 and Corollary 2.3, $I = O^{\theta(I)} \subseteq O^{X\setminus(P(X))^\circ}$. Thus, $X\setminus(P(X))^\circ \subseteq cl_{\beta X}(X\setminus(P(X))^\circ) \subseteq cl_{\beta X}(\theta(I)) = \theta(I)$. Hence, if we put $A = \theta(I)$, then we are done. Conversely, if $I = O^A$, for which $A^c \cap X \subseteq (P(X))^\circ$, then $X\setminus(P(X))^\circ \subseteq X\setminus(A^c \cap X) = X \cap A \subseteq A$, so $O^A \subseteq O^{X\setminus(P(X))^\circ}$ and hence O^A is regular.

(b). This is similar to the part (a). □

We recall that an SVNL-ring is a ring that if $< S >= R$, for some non-empty subset $S \subseteq R$, then at least one element of S is regular, or equivalently, all maximal ideals of R except maybe one of them are pure. By a Gelfand ring (or PM-ring) we mean a ring for which every prime ideal is contained in a unique maximal ideal. As we mentioned before, when a maximal ideal of a ring is regular, then the whole ring is regular. Naturally this question arises: what will happen if we have a prime regular ideal? We reply to this question in Gelfand rings, and in special case $C(X)$. In [5], it is stated that every SVNL-ring is a Gelfand ring, we will see in the next theorem when the converse is true. For other facts about SVNL-rings and Gelfand rings see [5] and [17]. We frequently use the following lemma in this paper, which is established in [18, Theorem 1.2]. We note that one can easily see that when R is reduced then O_M which is defined in [18], coincides $m(M)$.

Lemma 2.5. *A reduced ring R is a Gelfand ring if and only if for each $M \in Max(R)$, $m(M)$ is contained in the unique maximal ideal M.*

Theorem 2.6. *Let R be a reduced ring. The following are equivalent:*

(a) *R is an SVNL-ring that is not a regular ring;*

(b) *R is a Gelfand ring and $M(R) = O(M)$ $(= m(M))$ for some non-pure maximal ideal M in R;*

(c) *R is a non-regular Gelfand ring and contains a regular ideal which is contained in a unique maximal ideal M.*

Proof. From [4, Theorem 2.2] and Lemma 1.1, it is evident that (a) implies (b). Trivially by using Lemma 2.5, (b) implies (c). To prove $(c) \Rightarrow (a)$, let I be the regular ideal contained in a unique maximal ideal M, then for every $M \neq N \in Max(R)$, we have $I \nsubseteq N$. Hence N is maximal and minimal prime and therefore pure, thus R is an SVNL-ring. □

Corollary 2.7. *From the above theorem we can say that if a reduced Gelfand ring contains a regular ideal which contains a prime ideal (briefly by Lemma 1.9, if a reduced Gelfand ring contains a prime regular ideal), then it is an SVNL-ring. Furthermore, in particular, for a topological space X, if O_x, for some $x \in X$ is regular, then by this theorem, X is an essentially P-space. But if $x \in \beta X \setminus X$ and O^x is regular, then since O^x is a free regular ideal, then by [34], X is a P-space. Moreover if for two distinct points $x, y \in X$, O_x and O_y are regular, then their sum $C(X) = O_x + O_y$ is regular and therefore X is a P-space. But if $C(X)$ contains a non-maximal prime regular ideal, then the following corollary holds. For an example that satisfies the conditions of the next corollary see Example 3.7.*

Corollary 2.8. *For a topological space X, $C(X)$ contains a non-maximal prime regular ideal if and only if X is an essentially P-space and non-P-space and the non-P-point of X is an F-point.*

Proof. Necessity. Trivially, X is an essentially P-space and non-P-space. Let x be the non-P-point of X. If P is the mentioned prime regular ideal, then by Lemma 1.9, P is minimal prime and $O_x \subseteq P$. But the maximal regular ideal is O_x. Hence, $P = O_x$ and thus x is an F-point.

Sufficiency. If all points of X are P-points except for one point x, then M_x is the only non-pure maximal ideal, so by [4, Theorem 2.2], $M(C(X)) = mM_x = O_x$. This means that $M(C(X))$ is the required non-maximal prime regular ideal. □

In the sequel we study the regularity of some well-known ideals and subrings of $C(X)$. The first one is an improvement and generalization, in an easier way of [14, Proposition 2.5] and [3, Theorem 3.2, Corollary 3.3 and Corollary 3.4].

Proposition 2.9. *If an ideal $I \subseteq C_\psi(X)$ is a regular ideal, then $I \subseteq C_F(X)$.*

Proof. Suppose that $f \in I$, then $< f >$ is regular and $f \in C_\psi(X)$. Therefore, by Lemma 1.3, $Coz(f)$ is a pseudo-compact P-space. Hence, by [23, 4K], $Coz(f)$ is finite and so $f \in C_F(X)$. □

Corollary 2.10. *In the above proposition, if $C_F(X) \subseteq I \subseteq C_\psi(X)$, then I is regular if and only if $I = C_F(X)$. For instance $C_K(X)$ and $C_\psi(X)$ have this condition. Also by [30, Theorem 2.2], the intersection of all free maximal ideals (or $M^{\beta X \setminus X}$) and the intersection of all maximal essential ideals (or $Soc_m(C(X)) = M^{\beta X \setminus I(X)}$, see [22]), have this condition, in fact we have $C_F(X) \subseteq C_K(X) \subseteq M^{\beta X \setminus X} \subseteq C_\psi(X)$ and $C_F(X) \subseteq Soc_m(C(X)) \subseteq M^{\beta X \setminus X} \subseteq C_\psi(X)$. Therefore, each one of these ideals is regular if and only if it is equal to $C_F(X)$.*

A subset B of βX is called a round subset if $O^B = M^B$, also by a μ-compact space we mean a topological space X for which $M^{\beta X \setminus X} = O^{\beta X \setminus X}$, see [30]. Since $Soc_m(C(X))$ is regular if and only if $Soc_m(C(X)) = C_F(X)$, we can add the regularity of $Soc_m(C(X))$ as another equivalent to [12, Theorem 2.4]. In addition, by Corollary 2.10, [11, Theorem 4.5] and by this fact that $C_F(X) \subseteq O^{\beta X \setminus X} \subseteq M^{\beta X \setminus X}$, the following proposition is clear:

Proposition 2.11. *For a topological space X the following are equivalent:*

(a) *$M^{\beta X \setminus X}$ is a regular ideal;*

(b) *$M^{\beta X \setminus X} = C_F(X)$;*

(c) *X is psudo-discrete and μ-compact (or $\beta X \setminus X$ is a round subset).*

Clearly, $C_F(X) \subseteq Soc_m(C(X)) \subseteq M^{X \setminus I(X)}$ and for example if $Soc_m(C(X)) \neq C_F(X)$, then $M^{X \setminus I(X)}$ is not regular in general (whereas $O^{X \setminus I(X)}$ is always regular, because $O^{X \setminus I(X)} \subseteq O^{X \setminus P(X)}$). By this fact and according to Corollary 2.10, a natural question arises that: Can we conclude that $M^{X \setminus I(X)} = C_F(X)$ when $M^{X \setminus I(X)}$ is regular? By the following proposition when $I(X)$ is infinite the answer is no, see example 3.7. Also, in the following proposition we try to find an equivalent topological property for the regularity of $M^{X \setminus I(X)}$.

Proposition 2.12. *Let X be a topological space that $I(X) = Coz(f)$, for some $f \in C(X)$, then $M^{X\setminus I(X)}$ is a regular ideal if and only if $I(X)$ is closed.*

Proof. Since $I(X) = Coz(f)$, for some $f \in M^{X\setminus I(X)}$ and this ideal is regular, it follows that $I(X)$ is closed. Conversely, if $I(X)$ is closed, then all its subsets are closed and since $f \in M^{X\setminus I(X)}$ we have that $Coz(f) \subseteq I(X)$, so $Coz(f)$ is also closed. Therefore, $M^{X\setminus I(X)}$ is regular. □

Proposition 2.13. *The following statements are equivalent:*
(*a*) $C_\infty(X)$ *is a regular ring;*
(*b*) $C_\infty(X)$ *is a regular ideal;*
(*c*) $C_\infty(X) = C_F(X)$;
(*d*) *X is psudo-discrete and* $I(X)$ *is finite.*

Proof. By [7, Theorem 4.1], $C_\infty(X) = C_K(X)$, so $(a) \Rightarrow (b)$. If (b) holds, then by [15, Proposition 1.8], we must have $C_\infty(X) = C_K(X)$. On the other hand by Corollary 2.10, regularity of $C_K(X)$ implies that $C_K(X) = C_F(X)$. $(c) \Rightarrow (a)$ is trivial and by [11, Theorem 4.5], $(c) \Leftrightarrow (d)$ is trivial. □

Remark 2.14. First we note that we can add these equivalent to [7, Theorem 4.1], but we need a little correction of that theorem. According to this proposition and since $I(X)$ is finite, [14, Proposition 2.8 and Corollary 2.9], are trivial. As another result of this proposition and by using the chains in Corollary 2.10 and this fact from [25, Theorem 3.2] that $M^{\beta X\setminus X} \subseteq C_\infty(X)$, one can easily see that if one of the above conditions holds, then we have:

$$C_\infty(X) = C_K(X) = Soc_m(C(X)) = M^{\beta X\setminus X} = C_F(X).$$

3. On the Maximal Regular Ideal of a Reduced Ring via $C(X)$

By [16], the maximal regular ideal of a ring R is $M(R) = \{x \in R : xR \text{ is regular}\}$. In [4, Theorem 1.3], it is shown that, the maximal regular ideal is the intersection of the pure parts of those maximal ideals M of R, that are not pure. Now according to Lemma 1.1, which says that, in a reduced ring an ideal M is maximal and minimal prime if and only if M is pure, the following proposition can be extracted easily, which is a prime representation of the maximal regular ideal in a reduced ring.

Proposition 3.1. *In reduced rings the maximal regular ideal has the following representation in the form of the intersection of prime ideals*

$$M(R) = \bigcap_{P\in Min(R)\setminus Max(R)} P = \bigcap_{P\in Spec(R)\setminus Max(R)} P.$$

In [8], it is shown that the maximal P-ideal is of the above form. Hence we state the above facts to make clear the relation between existing facts about the maximal regular ideal and the maximal P-ideal.

Clearly, $C_F(X)$ is a regular ideal and consequently the maximal regular ideal of $C(X)$ must contain $C_F(X)$. Naturally, it is important to know when the maximal regular ideal has its smallest form, on the other words, when the maximal regular ideal is $C_F(X)$. The next concept helps us to answer this question. Also it helps us to show when the maximal regular ideal is $O^{X\setminus I(X)}$ and when is $M^{X\setminus I(X)}$. For finding definitions and other facts about the following topological concepts, the reader is referred to [20].

Definition 3.2. A topological space X is said to be an *OPD*-space if each open subset of P- points of X is discrete.

Clearly, a topological space X is an *OPD*-space if and only if $(P(X))^\circ = I(X)$.

Remark 3.3. A vast class of topological spaces are *OPD*-space. For instance we state some of them:

(1) Let X be a K-space, by [33, Proposition 4.1] and by the fact that every open subset of a K-space is a K-space, we have that $(P(X))^\circ$ is a K-space and also a P-space, hence it is discrete and open, thus we have $(P(X))^\circ = I(X)$. Therefore K-spaces and their subclasses such as compact spaces, first countable spaces, locally compact spaces, metric spaces and sequential spaces are *OPD*-space, see [20, 3.3].

(2) The spaces for which $P(X)$ is discrete are trivially *OPD*-space. For instance when $P(X)$ is countable, then by [23, 4K.1], $P(X)$ is discrete.

(3) Another example of *OPD*-spaces are those spaces for which each point is a G_δ-set, in this case, because of [23, 4L.1], we have $P(X) = I(X)$ and therefore $(P(X))^\circ = I(X)$.

We mentioned before that $O^{X\setminus I(X)}$ is a regular ideal. But it is shown in the next proposition when it is the maximal regular ideal.

Proposition 3.4. *$O^{X\setminus I(X)}$ is the maximal regular ideal if and only if X is an OPD-space.*

Proof. $O^{X\setminus I(X)}$ is the maximal regular ideal if and only if $O^{X\setminus I(X)} = O^{X\setminus (P(X))^\circ}$; if and only if $X\setminus I(X) = X\setminus(P(X))^\circ$; if and only if $I(X) = (P(X))^\circ$; if and only if X is an *OPD*-space. □

Although if $M(C(X)) = C_F(X)$, then X has no infinite clopen discrete subspace, but for *OPD*-spaces the converse is also true.

Theorem 3.5. *Let X be an OPD-space, then the maximal regular ideal is the socle of $C(X)$ if and only if $I(X)$ contains no infinite closed set.*

Proof. Necessity. Let $M(C(X)) = C_F(X)$, then suppose on the contrary that $I(X)$ contains a closed infinite set A, then $A = Coz(f)$ and $Coz(f)$ is a clopen infinite P-space. Hence, by Lemma 1.3, $f \in M(C(X))\setminus C_F(X)$ which is a contradiction.

Sufficiency. We show that $M(C(X)) = C_F(X)$. If $f \in M(C(X))$, then $Coz(f)$ is an open P-space, so by hypothesis $Coz(f) \subseteq I(X)$. Hence, again by hypothesis and closedness of $Coz(f)$, it is finite and thus $f \in C_F(X)$. □

There exists another class of topological spaces which are not necessarily an *OPD*-space but for them the maximal regular ideal is $C_F(X)$. As we show below for pseudo-compact spaces and their subclasses such as sequentially compact spaces and countably compact spaces the maximal regular ideal is $C_F(X)$.

Proposition 3.6. *If X is a pseudo-compact space, then $C_F(X)$ is the maximal regular ideal.*

Proof. Clearly $C_F(X)$ is regular. Now, suppose that I is a regular ideal and $f \in I$. Hence, $Coz(f)$ is a clopen P-space. But every clopen subset of a pseudo-compact space is pseudo-compact, so $Coz(f)$ is a pseudo-compact P-space and hence it is finite, thus $f \in C_F(X)$. This implies that $C_F(X)$ is the maximal regular ideal. □

Finally we give an example that can be used for many parts of this paper.

Example 3.7. The topological space Σ in [23, 4M], is an example of an F-space and an essentially P-space (σ is an F-point). The maximal regular ideal of Σ is $O^{X\setminus P(X)} = O_\sigma$, which is a prime regular ideal. In addition, the space Σ is an example of a topological space X (that is even a normal space), for which $C_F(X)$ is not the maximal regular ideal, because $C_F(X)$ cannot be a prime ideal, see [21, Proposition1.2]. Also this space is an example of a topological space for which $M^{X\setminus I(X)} = M^\sigma$ is not a regular ideal.

4. Essential Ideals and Regular Ideals

In this section, we study essential ideals and look for their relation with regular ideals. First we give some well-known facts and lemmas about essential ideals. Note that, by $Coz(I)$ we mean $\cup_{f\in I}Coz(f)$. Trivially we have that $(\eta(I))^c = Coz(I)$.

Lemma 4.1. *For an ideal I in* $C(X)$, *the following statements are equivalent* ([10, Theorem 3.1]*:*
(a) *An ideal I in* $C(X)$ *is an essential ideal;*
(b) $(\bigcap Z[I])^\circ = (\eta(I))^\circ = \emptyset$*;*
(c) $Coz(I)$ *is a dense subset of X.*

Lemma 4.2. *For a reduced ring R, I is essential if and only if* $Ann(I) = (0)$.

Lemma 4.3. *For a semiprimitive ring R we have* $Ann(I) = \cap_{I\not\subseteq M}M$.

Remark 4.4. Since for any ring R, $Jac(R) \cap M(R) = (0)$, [16, Theorem 5], we conclude that if $Jac(R) \neq (0)$, then regular ideals are not essential at all. Hence, in studying the relation between regular ideals and essential ideals in a ring R, without loss of generality, we can assume that $Jac(R) = (0)$. There are many rings R for which $Jac(R) = (0)$ and have no regular essential ideal. For instance, we know that every finitely generated regular ideal is principal. But if $I =< a >$ is a principal regular ideal, then for some b, $0 \neq 1 - ab \in Ann(I)$, so $Ann(I) \neq (0)$. Therefore principal regular ideals of a reduced ring, such as minimal ideals, are non-essential. Furthermore, we can say that the rings for which every ideal is finitely generated (i.e., Noetherian rings), have no regular essential ideal.

Now first we characterize those spaces for which there is an ideal in $C(X)$ which is both regular and essential. The following lemma is similar to [22, Lemma 2.9].

Lemma 4.5. *Let A be a closed subset of X. Then the following are equivalent:*
(a) *The ideal* O^A *is essential;*
(b) *The ideal* M^A *is essential;*
(c) $A^\circ = \emptyset$.

Proof. Trivially (a) ⇒ (b). Since $\eta(M^A) = \eta(O^A) = \overline{A} = A$ and by Lemma 4.1, (b) ⇒ (c) and (c) ⇒ (a). □

Theorem 4.6. *Let X be a topological space, then* $C(X)$ *contains an ideal which is both regular and essential if and only if* $(P(X))^\circ$ *is dense in X.*

Proof. If $C(X)$ contains an essential and regular ideal, then $M(C(X))$ is also essential. Therefore, to prove this proposition we need to show that $M(C(X))$ is essential if and only if $\overline{(P(X))^\circ} = X$. By the previous lemma and by Corollary 2.3, $M(C(X)) = O^{X\setminus(P(X))^\circ}$ is essential if and only if $(X\setminus(P(X))^\circ)^\circ = \emptyset$, by using easy topological techniques the latter is equivalent to the density of $(P(X))^\circ$. □

Remark 4.7. In *OPD*-spaces, since we have $(P(X))^\circ = I(X)$, by [11, Corollary 2.3] and [22, Proposition 2.10], the following are equivalent:
(a) There is an ideal which is both essential and regular;
(b) The maximal regular ideal (or $O^{X\setminus I(X)}$) is essential;
(c) $I(X)$ is dense in X;
(d) The socle of $C(X)$ is an essential ideal;
(e) $Soc_m(C(X))$ is an essential ideal;
(f) Every intersection of essential ideals is essential.

We now generalize some results about essential ideals in $C(X)$ to Gelfand rings. Clearly, the results in $C(X)$ are an especial case of these results. First, we show that in a semiprimitive Gelfand ring essentiality of each $M_0 \in Max(R)$ yields essentiality of $m(M_0)$. Then motivated by [10, Corollary 3.3], we show that in an arbitrary semiprimitive Gelfand ring every ideal containing a prime ideal is an essential ideal or minimal prime and maximal ideal. We next show that having no essential ideal in a reduced ring (not necessarily a Gelfand ring) implies the regularity of the ring.

Lemma 4.8. *Let R be a semiprimitive Gelfand ring. If $M_0 \in Max(R)$, then M_0 is an essential ideal if and only if $m(M_0)$ is an essential ideal.*

Proof. Suppose that M_0 is essential. Since R is semiprimitive and by using Lemma 2.5 and Lemma 4.3, we have $Ann(m(M_0)) = \bigcap_{m(M_0)\nsubseteq M} M = \bigcap_{M\neq M_0} M$, thus $Ann(m(M_0)) \cap M_0 = \bigcap_{M\in Max(R)} M = (0)$, hence $Ann(m(M_0)) = (0)$, so $m(M_0)$ is an essential ideal. The converse is trivial. □

Lemma 4.9. *If R is a reduced ring and I is an ideal containing a prime ideal of R, then I is essential or minimal prime.*

Proof. Let I be a non-essential ideal, then by Lemma 4.2 $Ann(I) \neq (0)$. Let I contains a minimal prime ideal, say P. We show that $I = P$. Let $0 \neq x \in Ann(I)$, since R is reduced $x \notin I$, it follows that $xI = (0) \subseteq P$ and $x \notin P$. Therefore, $I \subseteq P$ and this implies that $I = P$. □

These lemmas imply the following facts. We note that Proposition 4.10 is an improvement of [32, Corollary 5].

Proposition 4.10. *Let R be a semiprimitive Gelfand ring, then each ideal containing a prime ideal, is an essential ideal or a minimal prime and maximal ideal, hence a pure maximal ideal.*

Proof. Let I be a non-essential ideal in R, we show that I is a maximal and minimal prime ideal. Suppose that M is the unique maximal ideal containing I. Then by Lemma 4.8, M is also non-essential, hence by Lemma 4.9 and by the fact that M is a non-essential ideal, it is a minimal prime ideal. This implies that I is a minimal prime and maximal ideal. □

Remark 4.11. As a corollary of Lemma 4.9, if a reduced ring R contains no essential ideal or equivalently all the maximal ideals of R are non-essential, then all maximal ideals are also minimal prime; this implies the regularity of the ring. An easy corollary of Lemma 4.8 is that, in $C(X)$ as an especial case, M_x is essential if and only if $m(M_x) = O_x$ is essential. The following remark is also the topological form of Lemma 4.8 in $C(X)$.

Remark 4.12. Let X be a topological space, then by [10, Remark 3.2] whenever $x \in X$ is a non-isolated point, then O_x is an essential ideal. The converse is also true; because, O_x is an essential ideal if and only if $(\cap Z[O_x])^\circ = (\{x\})^\circ = \emptyset$; if and only if x is a non- isolated point; if and only if M_x is an essential ideal. Hence, we can say that all prime ideals of $C(X)$ are essential if and only if $I(X) = \emptyset$.

5. Regularity in Gelfand Rings

In this section the maximal spectrum of R ($Max(R)$) is topologized by Zariski topology; i.e., by assuming as a base for closed sets:

$$h(a) = \{M \in Max(R) : a \in M\}.$$

Hence closed sets are of the form $h(I) = \bigcap_{a\in I} h(a) = \{M \in Max(R) : I \subseteq M\}$, for some ideal I in R. We define $h^c(a) = Max(R)\backslash h(a)$. We recall that $Max(R)$ is a T_1 and compact topological space. We mentioned before that for a reduced ring $O(M) = \{a \in R : Ann(a) \nsubseteq M\} = m(M) = \bigcap_{P\subseteq M} P$. An easy calculation shows that when $Jac(R) = (0)$, then $h^c(I) \subseteq h(J)$ if and only if $IJ = (0)$ and $h(I) \subseteq h^c(J)$ if and only if $< I, J >= R$. Let $A \subseteq Max(R)$, we define the following ideals which are defined in [18] and [36] in a similar way:

$$O^A = \{a \in R : A \subseteq (h(a))^\circ\} \text{ and } M^A = \{a \in R : A \subseteq h(a)\}.$$

One can easily observe that when $Jac(R) = (0)$, then $M \in (h(a))^\circ$ if and only if $a \in O(M)$. Hence, we have the following facts:

$$O^A = \{a \in R : A \subseteq (h(a))^\circ\} = \bigcap_{M\in A}\{a \in R : M \in (h(a))^\circ\} = \bigcap_{M\in A} O(M),$$
$$M^A = \{a \in R : A \subseteq h(a)\} = \bigcap_{M\in A}\{a \in R : M \in h(a)\} = \bigcap_{M\in A} M.$$

Now, we are ready to transfer some of the results of this paper, which are about regular ideals and maximal regular ideals, into Gelfand rings. We first need the following lemmas:

Lemma 5.1. *A reduced ring R is a Gelfand ring if and only if for every ideal I we have $h(m(I)) = h(I)$.*

Proof. Necessity. We show that for a Gelfand ring, $h(m(I)) = h(I)$. According to [26, Theorem 5.2], it is enough to show that for every $x, y \in R$ having $< x > + < y >= R$, there exists $u, v \in R$ for which $< x > + < u >=< y > + < v >= R$ and $uv = 0$. Suppose that $< x > + < y >= R$, then there exist $s, t \in R$ such that $xt + ys = 1$. By [17, Theorem 4.1], there are $a, b \in R$ such that $(1 - axt)(1 - bys) = 0$. Now, if we set $u = 1 - axt$ and $v = 1 - bys$, then $uv = 0$ and $< x > + < u >=< y > + < v >= R$.

Sufficiency. By Lemma 2.5, it is clear. □

Lemma 5.2. *For a Gelfand ring R, the set $B = \{h(a) : a \in R\}$ form a neighborhood base for the topology of $Max(R)$.*

Proof. Let $h^c(I)$ be an open set in $Max(R)$ and $M_0 \in h^c(I)$, thus by the previous lemma $M_0 \in h^c(m(I))$. Therefore, there exists $a \in m(I)$ such that $a \notin M_0$. Since $a \in m(I)$, it follows that $a = ai$. If we set $b = 1 - i$, then $ab = 0$, and so $M_0 \in h^c(a) \subseteq h(b)$. On the other hand, clearly $< I, b >= R$, hence $h(b) \subseteq h^c(I)$. Therefore, $M_0 \in h^c(a) \subseteq h(b) \subseteq h^c(I)$, this complete the proof. □

Lemma 5.3. *A closed set in $Max(R)$ is an intersection of neighborhoods of the form $h(a)$ if and only if $\{\overline{h^c(a)}\}_{a\in R}$ form a neighborhood base for the topology of $Max(R)$.*

Proof. Necessity. Let $M \in h^c(I)$, thus $M \notin h(I)$, hence there exists $a \in R$ such that $M \notin h(a)$ and $h(I) \subseteq ((h(a))^\circ$. Therefore,

$$M \in h^c(a) \subseteq Max(R) \backslash (h(a))^\circ \subseteq h(I) \Rightarrow M \in h^c(a) \subseteq \overline{h^c(a)} \subseteq h(I).$$

Sufficiency. Let $h(I)$ be a closed set in $Max(R)$ and $M \notin h(I)$, thus $M \in h^c(I)$. Hence. there exists $x_M \in R$ such that $\overline{h^c(x_M)} \subseteq h^c(I)$ and $\overline{h^c(x_M)}$ is a neighborhood of M. i.e. there exists $a_M \in R$ such that:

$$M \in h^c(a_M) \subseteq \overline{h^c(x_M)} \text{ so } M \notin h(a_M) \text{ and } h(I) \subseteq Max(R) \backslash \overline{h^c(x_M)} \subseteq h(a_M).$$

Finally, it is clear that $h(I) = \bigcap_{M \notin h(I)} h(a_M)$. □

Proposition 5.4. *For an arbitrary Gelfand ring R each closed set is an intersection of neighborhoods of the form $h(a)$.*

Proof. If U is an open set in $Max(R)$ and $M \in U$, then by Lemma 5.2 there exist $a, b \in R$ such that $M \in h^c(b) \subseteq h(a) \subseteq U$, and so $M \in h^c(b) \subseteq \overline{h^c(b)} \subseteq h(a) \subseteq U$. Hence, again by Lemma 5.3, we are done. □

The following lemma has a similar statement for $C(X)$.

Lemma 5.5. *For a Gelfand ring $O^B \subseteq O^A$ implies that $\overline{A} \subseteq \overline{B}$.*

Proof. Let $M \notin \overline{B}$, then by Proposition 5.4 there exists $a \in R$ such that $\overline{B} \subseteq (h(a))^\circ$ and $M \notin h(a)$. Therefore, $a \in O^B \subseteq O^A$, and hence $A \subseteq (h(a))^\circ \subseteq h(a)$. Consequently, $\overline{A} \subseteq h(a)$, $M \notin h(a)$ and therefore $M \notin \overline{A}$. □

Proposition 5.6. *Pure ideals of any semiprimitive ring are of the form O^A which A is a closed set in $Max(R)$.*

Proof. For this purpose we show that for a semiprimitive ring $m(I) = O^{h(I)}$. Suppose that $a \in O^{h(I)}$, so we have that $h(I) \subseteq int(h(a))$. Hence, there exists an ideal J of R such that $h(I) \subseteq h^c(J) \subseteq h(a)$. Therefore, $< I, J >= R$ and $aJ = 0$. Thus, there are $j \in J$ and $i \in I$ such that $i + j = 1$. Hence, $a = ai$ and consequently $a \in m(I)$. Conversely, let $a \in m(I)$, hence $a = ai$, for some $i \in I$. Now if we set $x = 1 - i$, then $h(I) \subseteq h^c(x) \subseteq h(a)$ and hence $a \in O^{h(I)}$. □

Example 5.7. This example shows that the semiprimitivity of R is necessary. Let R be a local ring, so it is not semiprimitive. If the unique maximal ideal of R is M and if $J \subsetneq M$, then pick $a \in M \backslash J$. Now $h(a) = \{M\}$ is a neighborhood of $h(J)$ but $a \notin m(J)$. Thus, $a \in O^{h(J)} \backslash m(J)$, consequently $m(J) \neq O^{h(J)}$.

By a P-point in $Max(R)$ we mean $M \in Max(R)$ such that $O(M) = M$. As before we denote by $P(X)$, the set of all P-points of $Max(R)$.

If we set $X = Max(R)$, then the following theorem is exactly like [4, Corollary 5.5].

Theorem 5.8. *The maximal regular ideal of an arbitrary ring is of the form* $O^{X \backslash P(X)}$.

Proof. By [AHAS06, Theorem 1.3], $M(R) = \bigcap_{M \neq O(M)} O(M) = O^{X \backslash P(X)}$. □

We end the paper by finding a general form of regular ideals in semiprimitive Gelfand rings.

Theorem 5.9. *Let R be a semiprimitive Gelfand ring, then I is regular in R if and only if $I = O^B$, for some B such that $X \backslash B \subseteq P(X)$, where $X = Max(R)$.*

Proof. Since I is regular, it is pure. Then by Proposition 5.6, there exists a closed subset B of X such that $I = O^B$. On the other hand $O^{X \backslash P(X)}$ is the greatest maximal ideal and so $O^B \subseteq O^{X \backslash P(X)}$. Therefore, by Lemma 5.5 we have

$$X \backslash P(X) \subseteq \overline{X \backslash P(X)} \subseteq \overline{B} = B.$$

The converse is trivial. □

References

[1] E. Abu Osba, Purity of the ideal of continuous functions with pseudo-compact support, Internat. J. Math. Sci. 29 (2002) 381–388.
[2] E.A. Abu-Osba, H. Al-Ezeh, The pure part of the ideals in C(X), Math. J. Okayama Univ. 45 (2003) 73–82.
[3] E.A. Abu-Osba, H. Al-Ezeh, Some properties of the ideal of continuous functions with pseudo-compact support, IJMMS 27 (2001) 169–176.
[4] E.A. Abu-Osba, M. Henriksen, O. Alkam, F.A. Smith, The maximal regular ideal of some commutative rings, Comment. Math. Univ. Carolin. 47 (2006) 1–10.
[5] E.A. Abu-Osba, M. Henriksen, O. Alkam, Combining local and von Neumann regular rings, Comm. Algebra 32 (2004) 2639–2653.
[6] H. Al-ezeh, Pure ideals in reduced gelfand rings with unity, Arch. Math. 35 (1989) 266–269.
[7] A.R. Aliabad, F. Azarpanah, M. Namdari, Rings of cotinuous functions vanishing at infinity, Comment. Math. Univ. Carolin. 45 (2004) 519–533.
[8] A.R. Aliabad, J. Hashemi, R. Mohamadian, *P*- ideals and *PMP*-ideals in commutative rings, J. Math. Extension, Vol 10, No. 4, (2016) 19-33.
[9] A.R. Aliabad, R. Mohamadian, On $sz°$-ideals in polynomial rings, Comm. Algebra 39 (2011) 701–717.
[10] F. Azarpanah, Essential ideals in $C(X)$, Period. Math. Hungar. 31 (1995) 105–112.
[11] F. Azarpanah, Intersection of essential ideals in $C(X)$, Proc. Amer. Math. Soc. 125 (1997) 2149–2154.
[12] F. Azarpanah, M. Ghirati, A.Taherifar, When is $C_F(X) = M^{\beta X \backslash I(X)}$?, Topol. Appl. 194 (2015) 22–25.
[13] F. Azarpanah, O.A.S. Karamzadeh, A.R. Aliabad, On ideals consisting entirely of zero- divisors, Comm. Algebra 28 (2000) 1061–1073.
[14] F. Azarpanah, A.R. Olfati, On ideals of ideals in $C(X)$, Bull. Iran. Math. Soc. 41 (2015) 23–41.
[15] F. Azarpanah, T. Sondararajan, When the family of functions vanishing at infinity is an ideal of $C(X)$, Rocky Mountain J. Math. 31 (2001) 1–8.
[16] B. Brown, N. McCoy, The maximal regular ideal of a ring, Proc. Amer. Math. Soc. 1 (1950) 165–171.
[17] M. Contessa, On *PM*-rings, Comm. Algebra 10 (1982) 93-108.
[18] G. De Marco, A. Orsatti, Commutative rings in which every prime ideal is contained in a unique maximal ideal, Proc. Amer. Math. Soc. 30 (1971) 549–566.
[19] W. Dietrich, On the ideal structure of C(X), Trans. Amer. Math. Soc. 152 (1970) 61–77.
[20] R. Engelking, General Topology, PWN-Polish Scientific Publishers, Warsaw, 1997.
[21] A.A. Estaji, O.A.S. Karamzadeh, On C(X) modulo its socle, Comm. Algebra 31 (2003) 1561–1571.
[22] M. Ghirati, A. Taherifar, Intersection of essential (resp., free) maximal ideals of $C(X)$, Topol. Appl. 167 (2014) 62–68.
[23] L. Gillman, M. Jerison, Rings of Continuous Functions, Van Nostrand Reinhold, New York, 1960.
[24] K.R. Goodearl, Von Neumann Regular Rings, Pitman, London, 1979.
[25] D.G. Johnson, M. Mandelker, Functions with pseudo-compact support, General Topol. Appl. 3 (1973) 331–338.
[26] T.R. Jenkins, J.D. McKnight, Jr., Coherence classes of ideals in rings of continuous functions, Indag. Math. 24 (1962) 299–306.

[27] O.A.S. Karamzadeh, M. Rostami, On the intrinsic topology and some related ideals of $C(X)$, Proc. Amer. Math. Soc. 93 (1985) 179–184.
[28] I. Kaplansky, Commutative Rings, Allyn and Bacon, Boston, 1970.
[29] T.Y. Lam, Lectures on Moduls and Rings, Springer-Verlag, 1999.
[30] M. Mandelker, Supports of continuous functions, Trans. Amer. Math. Soc. 156 (1971) 73–83.
[31] G. Mason, Prime ideals and quotient rings of reduced rings. Math. Japan 34 (1989) 941–956.
[32] A.A. Mehrvaz, K. Samei, On noncommutative Gelfand rings, Sci. I. R. Iran 10 (1999) 193–196.
[33] A.K. Misra, A topological view of P-spaces, Gen. Top. Appl. 2 (1972) 349–362.
[34] D. Rudd, P-ideals and F-ideals rings of continuous functions, Fund. Math. 88 (1975) 53–59.
[35] A. Taherifar, Intersections of essential minimal prime ideals, Comment. Math. Univ. Carol. 55 (2014) 121–130.
[36] A. Taherifar, Some new classes of topological spaces and annihilator ideals, Topol. Appl. 165 (2014) 89–97.
[37] S. Willard, General Topology, Addison Wesly, Reading Mass., 1970.