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About Strong Starlikeness Conditions

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Abstract. Some results concerning the strong starlikeness of analytic functions are improved. The techniques of convolutions are used.

1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Let $\mathcal{H}(U)$ denote the class of analytic functions defined on U. Let $\mathcal{A}$ be the subclass of $\mathcal{H}(U)$ which consists of functions of the form: $f(z) = z + a_2z^2 + a_3z^3 + ...$ The subclass of $\mathcal{A}$ consisting of functions for which the domain $f(U)$ is starlike with respect to 0, is denoted by S^*. An analytic characterization of S^* is given by:

$$S^* = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}.$$

In [4] the authors proved that

Theorem 1.1.

$$\text{If } f \in \mathcal{A}, \text{ and } \Re\Big(f'(z) + zf''(z)\Big) > 0,\ \ z \in U, \text{ then } f \in S^*.$$

This result has been improved in [1] and [3] (p.304-307) in the following manner:

Theorem 1.2.

$$\text{If } f \in \mathcal{A}, \text{ and } \Re\Big(f'(z) + \lambda_0 zf''(z)\Big) > 0,\ \ z \in U,\ \ \text{then } f \in S^*,$$

where $\lambda_0 = 0.348\ldots.$

Other kinds of improvements for Theorem 1.1 can be found in [2], [5], [6], [8], [11]. For other results regarding starlikeness we recommend [7], [9], [10]. In [2] Mocanu proved the following theorems:

Theorem 1.3.

$$\text{If } f \in \mathcal{A}, \text{ and } \Re\Big(f'(z) + zf''(z)\Big) > 0,\ \ z \in U \text{ then, } \Big|\arg \frac{zf'(z)}{f(z)}\Big| < \frac{\pi}{3},$$

and

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Theorem 1.4.

$$\text{If } f\in\mathcal{A}, \text{ and } \Re\Big(f'(z)+\frac{1}{2}zf''(z)\Big)>0,\ \ z\in U \text{ then, } \Big|\arg\frac{zf'(z)}{f(z)}\Big|<\frac{4\pi}{9}.$$

In order to deduce these results the differential subordination has been used. We will improve Theorem 1.3, and Theorem 1.4 using a different approach. The results we need in the followings will be presented in the next section.

2. Preliminaries

Lemma 2.1. *[5] If $\theta\in(0,2\pi)$ and $\delta>0$, then the following identities hold:*

$$\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x})}{(\delta^2+x^2)(e^{2\pi x}-1)}dx+i\delta\int_0^\infty \frac{e^{(2\pi-\theta)x}-e^{\theta x}}{(\delta^2+x^2)(e^{2\pi x}-1)}dx=\frac{1}{2\delta}+\sum_{k=1}^\infty\frac{e^{i\theta k}}{k+\delta},$$

$$2\delta\int_0^\infty \frac{x(e^{\theta x}+e^{(2\pi-\theta)x})}{(\delta^2+x^2)^2(e^{2\pi x}-1)}dx+i\int_0^\infty \frac{(\delta^2-x^2)(e^{(2\pi-\theta)x}-e^{\theta x})}{(\delta^2+x^2)^2(e^{2\pi x}-1)}dx=\frac{1}{2\delta^2}+\sum_{k=1}^\infty\frac{e^{i\theta k}}{(k+\delta)^2}.$$

Let $f(z)=\sum_{n=0}^\infty a_nz^n$ and $g(z)=\sum_{n=0}^\infty b_nz^n$ be two analytic functions. The convolution of these functions is defined as follows:

$$\big(f*g\big)(z)=\sum_{n=0}^\infty a_nb_nz^n.$$

Lemma 2.2. *Let $f\in\mathcal{A}$ and $\alpha\in(0,\frac{\pi}{2}]$. The inequality*

$$\Big|\arg\frac{zf'(z)}{f(z)}\Big|<\alpha,\ \ z\in U$$

is equivalent to

$$\frac{f(z)}{z}*\frac{k_T(z)}{z}\neq 0,\ z\in U, T\in[0,\infty),$$

where

$$k_T(z)=z+\sum_{n=1}^\infty\frac{1+n-Te^{\pm i\alpha}}{1-Te^{\pm i\alpha}}z^{n+1}.$$

Proof. Let $p(z)=\frac{zf'(z)}{f(z)}$. Since $p(0)=1$, the condition $\big|\arg\frac{zf'(z)}{f(z)}\big|<\alpha,\ \ z\in U$ is equivalent to the fact that the domain $p(U)$ is inside the angle determined by the following half-lines $\{Te^{i\alpha}\big|\ T\in[0,\infty)\}$ and $\{Te^{-i\alpha}\big|\ T\in[0,\infty)\}$. This means that $p(U)$ does not intersect the sides of the angle or equivalently

$$p(z)\neq Te^{\pm i\alpha},\ (\forall)z\in U, (\forall)T\in[0,\infty).$$

Since the following equivalences hold

$$\begin{aligned}p(z)\neq Te^{\pm i\alpha}\Leftrightarrow Te^{\pm i\alpha}\neq\frac{zf'(z)}{f(z)}&\Leftrightarrow\frac{f(z)}{z}+\frac{1}{1-Te^{\pm i\alpha}}\Big(f'(z)-\frac{f(z)}{z}\Big)\neq 0\\&\Leftrightarrow 1+\sum_{n=1}^\infty a_{n+1}z^n+\frac{1}{1-Te^{\pm i\alpha}}\sum_{n=1}^\infty na_{n+1}z^n\neq 0\\&\Leftrightarrow\frac{f(z)}{z}*\frac{k_T(z)}{z}\neq 0,\end{aligned}$$

the proof is done. □

Lemma 2.3. *If $m > 0$ is a fixed real number and p is an analytic function in U defined by $p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n+m}$, then*

$$\lim_{\substack{z\to 1\\ |z|<1}} \operatorname{Re} p(z) = +\infty. \tag{1}$$

Proof. We have to prove only the particular case

$$\lim_{\substack{z\to 1\\ |z|<1}} \operatorname{Re}\Big(\sum_{n=1}^{\infty} \frac{z^n}{n}\Big) = +\infty. \tag{2}$$

This equality holds because we have

$$\lim_{\substack{z\to 1\\ |z|<1}} \operatorname{Re}\Big(\sum_{n=1}^{\infty} \frac{z^n}{n}\Big) = \lim_{\substack{r\to 1\\ \theta\to 0}} \sum_{n=1}^{\infty} \frac{r^n \cos n\theta}{n} = \lim_{\substack{r\to 1\\ \theta\to 0}} \ln \frac{1}{\sqrt{1 + r^2 - 2r\cos\vartheta}} = +\infty.$$

On the other hand we have

$$\Big|\sum_{n=1}^{\infty} \frac{z^n}{n} - \sum_{n=1}^{\infty} \frac{z^n}{n+m}\Big| = \Big|\sum_{n=1}^{\infty} \frac{z^n}{n(n+m)}\Big| \le \sum_{n=1}^{\infty} \frac{m|z^n|}{n(n+m)} \le \sum_{n=1}^{\infty} \frac{m}{n^2} = m\frac{\pi^2}{6}. \tag{3}$$

Finally (2) and (3) imply (1) and the proof is done. □

3. The Main Result

Theorem 3.1. *Let $\beta \in (0,\infty)$, $\alpha \in (0, \frac{\pi}{2})$ be two real numbers such that*

$$a_\pm(\vartheta) = a_\pm(1,\vartheta) > \delta > 0,\ b_\pm^2(\vartheta) - a_\pm(\vartheta)c_\pm(\vartheta) = b_\pm(1,\vartheta)^2 - a_\pm(1,\vartheta)c_\pm(1,\vartheta) < 0,\ \ \vartheta \in (0,2\pi) \tag{4}$$

where

$$a_\pm(r,\vartheta) = 1 + 2\sum_{n=1}^{\infty} \frac{r^n \cos n\vartheta + r^n \sin n\vartheta \cot(\pm\alpha)}{\beta n(n+1) + n + 1},$$

$$b_\pm(r,\vartheta) = \cos\alpha + 2\sum_{n=1}^{\infty} \frac{r^n \cos n\vartheta \cos\alpha + \frac{n+1+\cos 2\alpha}{2\sin(\pm\alpha)} r^n \sin n\vartheta}{\beta n(n+1) + n + 1}, \tag{5}$$

and

$$c_\pm(r,\vartheta) = 1 + 2\sum_{n=1}^{\infty} \frac{r^n \cos n\vartheta + r^n \sin n\vartheta \cot\alpha}{\beta n + 1}.$$

If $f \in \mathcal{A}$ is a function that satisfies the following condition

$$\Re\Big(f'(z) + \beta z f''(z)\Big) > 0,\ \ z \in U, \tag{6}$$

then

$$\Big|\arg\Big(\frac{zf'(z)}{f(z)}\Big)\Big| < \alpha,\ \ z \in U. \tag{7}$$

Proof. Let f be the function defined in U by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. On the one hand, we have

$$f'(z) + \beta z f''(z) = 1 + \sum_{n=2}^{\infty} (\beta n(n-1) + n) a_n z^{n-1}.$$

The Herglotz representation formula implies the existence of a probability measure μ on $[0, 2\pi]$, such that

$$f'(z) + \beta z f''(z) = 1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-itn} d\mu(t).$$

These imply $a_n = \frac{2}{\beta n(n-1)+n} \int_0^{2\pi} e^{-it(n-1)} d\mu(t), \;\; n \geq 2$, and

$$f(z) = z + 2\sum_{n=2}^{\infty} \frac{z^n}{\beta n(n-1)+n} \int_0^{2\pi} e^{-it(n-1)} d\mu(t).$$

According to Lemma 2.2, condition (7) is equivalent to

$$\frac{f(z)}{z} * \frac{k_T(z)}{z} \neq 0, \; z \in U, T \in [0, \infty), \tag{8}$$

where

$$k_T(z) = z + \sum_{n=1}^{\infty} \frac{1 + n - Te^{\pm i\alpha}}{1 - Te^{\pm i\alpha}} z^{n+1}.$$

Condition (8) in its explicit form is

$$1 + 2\sum_{n=1}^{\infty} \frac{z^n}{\beta n(n+1)+n+1} \frac{1+n-Te^{\pm i\alpha}}{1-Te^{\pm i\alpha}} \int_0^{2\pi} e^{-itn} d\mu(t) \neq 0,$$
$$z \in U, T \in [0, \infty),$$

and this can be rewritten as follows:

$$\int_0^{2\pi} \Big(1 + 2\sum_{n=1}^{\infty} \frac{z^n}{\beta n(n+1)+n+1} \frac{1+n-Te^{\pm i\alpha}}{1-Te^{\pm i\alpha}} e^{-itn}\Big) d\mu(t) \neq 0,$$
$$z \in U, T \in [0, \infty). \tag{9}$$

We introduce the notations

$$A(z, T, t) = \Re\Big(1 + 2\sum_{n=1}^{\infty} \frac{z^n}{\beta n(n+1)+n+1} \frac{1+n-Te^{\pm i\alpha}}{1-Te^{\pm i\alpha}} e^{-itn}\Big),$$

and

$$B(z, T, t) = \Im\Big(1 + 2\sum_{n=1}^{\infty} \frac{z^n}{\beta n(n+1)+n+1} \frac{1+n-Te^{\pm i\alpha}}{1-Te^{\pm i\alpha}} e^{-itn}\Big).$$

We will prove that

$$\int_0^{2\pi} \Big[A(z, T, t)\sin(\pm\alpha) + B(z, T, t)\cos\alpha\Big] d\mu(t) \neq 0, \; z \in U, \; T \in [0, \infty). \tag{10}$$

If (10) holds, then $\int_0^{2\pi} A(z, T, t) d\mu(t) \neq 0, \; z \in U, \; T \in [0, \infty)$, or $\int_0^{2\pi} B(z, T, t) d\mu(t) \neq 0, \; z \in U, \; T \in [0, \infty)$, and this implies (9). In order to prove (10) we will show that

$$\int_0^{2\pi} A(z, T, t) d\mu(t) \sin(\pm\alpha) + \int_0^{2\pi} B(z, T, t) d\mu(t) \cos\alpha > 0, \; z \in U, \; T \in [0, \infty).$$

This inequality is equivalent to

$$\int_0^{2\pi}\Big[A(re^{i\theta},T,t)\sin(\pm\alpha)+B(re^{i\theta},T,t)\cos\alpha\Big]d\mu(t)>0,\ \theta\in[0,2\pi], \tag{11}$$

and $T\in[0,\infty),\ r\in(0,1)$.
In order to prove (11) it is enough to show that

$$\begin{aligned}v(re^{i\theta},T,t)&=A(re^{i\theta},T,t)\sin(\pm\alpha)+B(re^{i\theta},T,t)\cos\alpha>0,\\&\qquad\theta\in[0,2\pi],\ r\in(0,1),\ T\in[0,\infty).\end{aligned} \tag{12}$$

We denote $\vartheta=\theta-t$ and we get

$$\begin{aligned}A(re^{i\theta},T,t)&=1+2\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{r^ne^{in\theta}}{\beta n(n+1)+n+1}\frac{1+n-Te^{\pm i\alpha}}{1-Te^{\pm i\alpha}}e^{-int}=1\\&+\frac{2}{1+T^2-2T\cos\alpha}\Big[\sum_{n=1}^{\infty}\frac{(1+T^2-2T\cos\alpha+n-nT\cos\alpha)r^n\cos n\vartheta}{\beta n(n+1)+n+1}\\&\qquad+\sum_{n=1}^{\infty}\frac{r^n\sin n\vartheta}{\beta n(n+1)+n+1}nT\sin(\mp\alpha)\Big]\end{aligned} \tag{13}$$

and

$$\begin{aligned}B(re^{i\theta},T,t)&=2\,\mathrm{Im}\sum_{n=1}^{\infty}\frac{r^ne^{in\theta}}{\beta n(n+1)+n+1}\frac{1+n-Te^{\pm i\alpha}}{1-Te^{\pm i\alpha}}=\frac{2}{1+T^2-2T\cos\alpha}\\&\sum_{n=1}^{\infty}\frac{(1+T^2+n-(n+2)\cos\alpha)r^n\sin n\vartheta-Tnr^n\cos n\vartheta\sin(\mp\alpha)}{\beta n(n+1)+n+1}\end{aligned} \tag{14}$$

We define the functions $a_\pm,b_\pm,c_\pm$ by the equality

$$\begin{aligned}&\Big[A(re^{i\theta},T,t)\sin(\pm\alpha)+B(re^{i\theta},T,t)\cos\alpha\Big]\frac{1-2T\cos\alpha+T^2}{\sin\alpha}\\&\qquad=a_\pm(r,\vartheta)T^2-2b_\pm(r,\vartheta)T+c_\pm(r,\vartheta).\end{aligned} \tag{15}$$

Comparing (13), (14), and (15) we get (5).
We will prove that

$$\begin{aligned}&a_\pm(r,\vartheta)T^2-2b_\pm(r,\vartheta)T+c_\pm(r,\vartheta)>0,\ (\forall)T\in\mathbb{R},\\&\quad\vartheta\in[0,2\pi],\ \text{and } r\in[0,1)\Leftrightarrow\ re^{i\vartheta}=z\in U.\end{aligned} \tag{16}$$

Equality (1) from Lemma 2.3 implies $\lim_{\substack{z\to1\\|z|<1}}c_\pm(r,\vartheta)=+\infty$. Now taking into account that $a_\pm(r,\vartheta)>\delta>0$ and $a_\pm(r,\vartheta),\ b_\pm(r,\vartheta)$ are bounded functions, we infer that

$$\lim_{\substack{z\to1\\|z|<1}}[a_\pm(r,\vartheta)T^2-2b_\pm(r,\vartheta)T+c_\pm(r,\vartheta)]=+\infty,$$

and the convergence is uniform with respect to T. Thus it follows that for a given positive number $k>0$ there is an other positive number $\epsilon(k)>0$ such that

$$z\in U\text{ and }|z-1|\le\epsilon(k)\Rightarrow\ a_\pm(r,\vartheta)T^2-2b_\pm(r,\vartheta)T+c_\pm(r,\vartheta)>k>0. \tag{17}$$

Consequently the inequality (16) holds on the set

$$U\cap\{z\in\mathbb{C}:|z-1|\le\epsilon(k)\}=D.$$

Further we have to prove that the inequality (16) holds on $U \setminus D$. The mapping

$$w(T,\vartheta,r) = a_\pm(r,\vartheta)T^2 - 2b_\pm(r,\vartheta)T + c_\pm(r,\vartheta)$$

is harmonic on the set $D^* = U \setminus D$. If we check inequality (16) on ∂D^*, then the minimum principle for harmonic functions implies that (16) holds on D^*. According to (17) the inequality (16) hods on the arc of the circle $|z-1| \le \epsilon(k)$, which is inside of U.
Thus we have to check (16) on the arc of ∂U which is autside of the disc $|z-1| \le \epsilon(k)$. Since $a_\pm(\vartheta) > 0$, $\vartheta \in [0, 2\pi]$ it follows that the inequality (16) holds if

$$b_\pm(\vartheta)^2 - a_\pm(\vartheta)c_\pm(\vartheta) = b_\pm(1,\vartheta)^2 - a_\pm(1,\vartheta)c_\pm(1,\vartheta) < 0, \;\; \vartheta \in [\gamma, 2\pi - \gamma], \tag{18}$$

where $\gamma > 0$ is the argument of the point situated in the first quadrant and being the intersection of the unit circle with the circle $|z-1| = \epsilon(k)$. □

Corollary 3.2. *If $f \in \mathcal{A}$ is a function such that*

$$\Re\Big(f'(z) + zf''(z)\Big) > 0, \;\; z \in U,$$

then

$$\left|\arg\Big(\frac{zf'(z)}{f(z)}\Big)\right| < \frac{3\pi}{10}, \;\; z \in U.$$

Proof. According to Theorem 3.1 we have to verify the inequality (4) in case of $\beta = 1$ and $\alpha = \frac{3\pi}{10}$. Let $q_i, p_i : (0, 2\pi) \to \mathbb{R}$ defined by $q_i(\vartheta) = \sum_{n=1}^{\infty} \frac{\cos n\vartheta}{(n+1)^i}$ and $p_i(\vartheta) = \sum_{n=1}^{\infty} \frac{\sin n\vartheta}{(n+1)^i}$, where $i \in \{1, 2\}$. (According to Theorem 3.1 we have to prove inequality (18) only for $\vartheta \in [\gamma, 2\pi - \gamma]$.) Lemma 2.1 implies that $\frac{1}{2} + q_1$ and $\frac{1}{2} + q_2$ are strictly decreasing on $(0, \pi]$ and strictly increasing on $[\pi, 2\pi)$, and p_1, $p_1 + p_2$ and $p_1 + \cos(2\alpha)p_2$ are strictly decreasing on $(0, 2\pi)$.
We consider the case $\vartheta \in (0, \pi]$ in the first step.
In this case he integral representations given in Lemma 2.1 imply that the functions $a(\vartheta) = 1 + 2q_2(\vartheta) + 2p_2(\vartheta)\cot\alpha$, $b(\vartheta) = \cos\alpha + 2q_2(\vartheta)\cos\alpha + \frac{1}{\sin\alpha}(p_1(\vartheta) + p_2(\vartheta)\cos(2\alpha))$ and $c(\vartheta) = 1 + 2q_1(\vartheta) + 2p_1(\vartheta)\cot\alpha$ are positive and the functions q_1, q_2, p_1, p_2 are strictly decreasing. Thus, if we verify the inequalities

$$b_k^2 < a_k c_k, \;\text{ for } k \in \{1, 2, \ldots, 10^2\}, \;\text{ where } \vartheta_k = \frac{k\pi}{10^2}, \tag{19}$$

$a_k = 1 + 2q_2(\vartheta_k) + (p_1 + p_2)(\vartheta_k)\cot\alpha - p_1(\vartheta_{k-1})\cot\alpha$, $b_k = \cos\alpha + 2q_2(\vartheta_{k-1})\cos\alpha + \frac{1}{\sin\alpha}(p_1(\vartheta_{k-1}) + p_2(\vartheta_{k-1})\cos(2\alpha))$ and $c_k = 1 + 2q_1(\vartheta_k) + 2p_1(\vartheta_k)\cot\alpha$, then the monotonicity of $q_i, p_1, p_1 + p_2$, and $p_1 + p_2\cos(2\alpha)$ implies

$$(b(\vartheta))^2 \le {b_k}^2 < a_k c_k \le a(\vartheta)c(\vartheta), \;\; \vartheta \in [\vartheta_{k-1}, \vartheta_k], \; k \in \{1, 2, \ldots, 10^2\},$$

and finally we obtain $(b(\vartheta))^2 - a(\vartheta)c(\vartheta) < 0$ for every $\vartheta \in (0, \pi]$.
The second case is $\vartheta \in [\pi, 2\pi)$.
Lemma 2.1 implies that the functions $\frac{1}{2} + q_1$, $\frac{1}{2} + q_2$ are strictly increasing and positive, and p_1, $p_1 + p_2\cos(2\alpha)$, $p_1 + p_2$ are strictly decreasing and negative on this interval. Let $M, N : [\pi, 2\pi) \to \mathbb{R}$ be defined by $M(\vartheta) = \cos\alpha + 2q_2(\vartheta)\cos\alpha$, and $N(\vartheta) = \frac{1}{\sin\alpha}[p_1(\vartheta) + p_2(\vartheta)\cos 2\alpha]$. Lemma 2.1 implies that M is a strictly increasing and positive function and N is strictly decreasing and negative. We have to prove

$$\begin{aligned} a(\vartheta)c(\vartheta) - b^2(\vartheta) = {}& (1 + 2q_2(\vartheta))(1 + 2q_1(\vartheta)) + 2(1 + 2q_1(\vartheta))\Big(p_1(\vartheta) + p_2(\vartheta)\Big)\cot\alpha \\ & - 2(1 + 2q_1(\vartheta))p_1(\vartheta)\cot\alpha + 2(1 + 2q_2(\vartheta))p_1(\vartheta)\cot\alpha \\ +4\Big(p_1(\vartheta) + p_2(\vartheta)\Big)p_1(\vartheta)\cot^2\alpha & - 4p_1^2(\vartheta)\cot^2\alpha - 2M(\vartheta)N(\vartheta) - M^2(\vartheta) - N^2(\vartheta) > 0, \;\; \vartheta \in [\pi, 2\pi). \end{aligned}$$

In order to prove this inequality we have to verify that

$$\lambda_k > 0, \tag{20}$$

where

$$\begin{aligned}\lambda_k = (1+2q_1(\vartheta_{k-1}))(1+2q_2(\vartheta_{k-1})) + 2(1+2q_1(\vartheta_k))\big(p_1+p_2\big)(\vartheta_k)\cot\alpha\\ -2(1+2q_1(\vartheta_{k-1}))p_1(\vartheta_{k-1})\cot\alpha + 2(1+2q_2(\vartheta_k))p_1(\vartheta_k)\cot\alpha\\ +4\big(p_1(\vartheta_{k-1})+p_2(\vartheta_{k-1})\big)p_1(\vartheta_{k-1})\cot^2\alpha - 4p_1^2(\vartheta_k)\cot^2\alpha\\ -2M(\vartheta_{k-1})N(\vartheta_{k-1}) - M^2(\vartheta_k) - N^2(\vartheta_k)\end{aligned}$$

and

$$\vartheta_k = \frac{k\pi}{10^3},\ \ k \in \{10^3+1, 10^3+2, \dots, 2\cdot 10^3\}.$$

Then the monotonicity of $q_i,\ p_1,\ p_1+p_2,\ M,\ N$ implies

$$0 < \lambda_k \leq a(\vartheta)c(\vartheta) - b^2(\vartheta),\ \ \vartheta \in [\theta_{k-1}, \theta_k],\ k \in \{10^3+1, 10^3+2, \dots, 2\cdot 10^3\}$$

and the inequality $b^2(\vartheta) - a(\vartheta)c(\vartheta) < 0$ follows for every $\vartheta \in [\pi, 2\pi)$. The case $\alpha = -\frac{3\pi}{10}$ can be discussed in the same way. □

Corollary 3.3. *If $f \in \mathcal{A}$ is a function such that*

$$\Re\Big(f'(z) + \frac{1}{2}zf''(z)\Big) > 0,\ \ z \in U,$$

then

$$\Big|\arg\Big(\frac{zf'(z)}{f(z)}\Big)\Big| < \frac{3\pi}{8},\ \ z \in U.$$

Proof. According to Theorem 3.1 we have to verify the inequality (4) in case of $\beta = \frac{1}{2}$ and $\alpha = \frac{3\pi}{8}$. The proof is analogous to the proof of the previous corollary. We use the notation introduced before and we denote by $q_3,\ p_3$ the following functions $q_3(\vartheta) = \sum_{n=1}^{\infty} \frac{\cos n\vartheta}{n+2},\ p_3(\vartheta) = \sum_{n=1}^{\infty} \frac{\sin n\vartheta}{n+2}$. We prove the corollary in two steps. The first case is $\vartheta \in (0, \pi]$.

A short calculation leads to $a(\vartheta) = 1 + 4\big(q_1(\vartheta) - q_3(\vartheta)\big) + 4\big(p_1(\vartheta) - p_3(\vartheta)\big)\cot\alpha,\ b(\theta) = \cos\alpha + 4(q_1(\vartheta) - q_3(\vartheta))\cos\alpha + \frac{2}{\sin\alpha}p_3(\vartheta) + \frac{2\cos 2\alpha}{\sin\alpha}(p_1(\vartheta) - p_3(\vartheta))$ and $c(\vartheta) = 1 + 4q_3(\vartheta) + 4p_3(\vartheta)\cot\alpha$.

Let the functions $P, Q, R : (0, 2\pi) \to \mathbb{R}$ be defined by $P(\vartheta) = 1 + 4\big(q_1(\vartheta) - q_3(\vartheta)\big),\ Q(\vartheta) = \frac{2}{\sin\alpha}p_3(\vartheta) + \frac{2\cos 2\alpha}{\sin\alpha}(p_1(\vartheta) - p_3(\vartheta))$ and $R(\vartheta) = 1 + 4q_3(\vartheta)$. Lemma 2.1 implies that $p_1,\ p_3$ and Q are strictly decreasing on $(0, 2\pi)$, positive on $(0, \pi]$ and negative on $[\pi, 2\pi)$. P and R are decreasing on $(0, \pi]$, increasing on $[\pi, 2\pi)$ and positive on $(0, 2\pi)$. We have

$$\begin{aligned}a(\vartheta)c(\vartheta) - b^2(\vartheta) = P(\vartheta)R(\vartheta) + 4R(\vartheta)p_1(\vartheta)\cot\alpha - 4R(\vartheta)p_3(\vartheta)\cot\alpha + 4P(\vartheta)p_3(\vartheta)\cot\alpha\\ +16p_1(\vartheta)p_3(\vartheta)\cot^2\alpha - 16p_3^2(\vartheta)\cot^2\alpha - P^2(\vartheta)\cos^2\alpha - Q^2(\vartheta) - 2P(\vartheta)Q(\vartheta)\cos\alpha.\end{aligned}$$

If we verify that

$$\mu_k > 0,\ \ k \in \{1, 2, \dots, 10^2\} \tag{21}$$

where μ_k are defined by

$$\begin{aligned}\mu_k = P(\vartheta_k)R(\vartheta_k) + 4R(\vartheta_k)p_1(\vartheta_k)\cot\alpha - 4R(\vartheta_{k-1})p_3(\vartheta_{k-1})\cot\alpha + 4P(\vartheta_k)p_3(\vartheta_k)\cot\alpha\\ +16p_1(\vartheta_k)p_3(\vartheta_k)\cot^2\alpha - 16p_3^2(\vartheta_{k-1})\cot^2\alpha - P^2(\vartheta_{k-1})\cos^2\alpha - Q^2(\vartheta_{k-1}) - 2P(\vartheta_{k-1})Q(\vartheta_{k-1})\cos\alpha,\end{aligned}$$

and $\vartheta_k = \frac{k\pi}{10^2},\ k \in \{1, 2, \dots, 10^2\}$, then the monotonicity of P, Q, R, p_1, p_3 implies

$$a(\vartheta)c(\vartheta) - b^2(\vartheta) \geq \mu_k > 0,\ \ \vartheta \in [\vartheta_{k-1}, \vartheta_k],\ k \in \{1, 2, \dots, 10^2\}.$$

Thus, it follows that $a(\vartheta)c(\vartheta) - b^2(\vartheta) > 0,\ \vartheta \in (0, \pi]$.
The second case is $\vartheta \in [\pi, 2\pi)$.
This time we have to verify

$$\nu_k > 0,\ \ k \in \{10^3 + 1, 10^3 + 2, \dots, 2 \cdot 10^3\} \tag{22}$$

where

$$\begin{aligned}\nu_k &= P(\vartheta_{k-1})R(\vartheta_{k-1}) + 4R(\vartheta_k)p_1(\vartheta_k)\cot\alpha - 4R(\vartheta_{k-1})p_3(\vartheta_{k-1})\cot\alpha + 4P(\vartheta_k)p_3(\vartheta_k)\cot\alpha \\ &+16p_1(\vartheta_{k-1})p_3(\vartheta_{k-1})\cot^2\alpha - 16p_3^2(\vartheta_k)\cot^2\alpha - P^2(\vartheta_k)\cos^2\alpha - Q^2(\vartheta_k) - 2P(\vartheta_{k-1})Q(\vartheta_{k-1})\cos\alpha,\end{aligned}$$

and $\vartheta_k = \frac{k\pi}{10^3},\ k \in \{10^3 + 1, 10^3 + 2, \dots, 2 \cdot 10^3\}$. Just like before we have

$$a(\vartheta)c(\vartheta) - b^2(\vartheta) \geq \nu_k > 0,\ \vartheta \in [\vartheta_{k-1}, \vartheta_k],\ k \in \{10^3 + 1, 10^3 + 2, \dots, 2 \cdot 10^3\},$$

an this implies $a(\vartheta)c(\vartheta) - b^2(\vartheta) > 0,\ \vartheta \in [\pi, 2\pi)$. Consequently, the inequality $a(\vartheta)c(\vartheta) - b^2(\vartheta) > 0$ holds for every $\vartheta \in (0, 2\pi)$. The case $\alpha = -\frac{3\pi}{8}$ can be discussed in the same way. □

Remark 3.4. *Conditions (19), (20), (21) and (22) can be verified easily using a computer program, which applies numerical methods and uses the equalities* $q_1(\vartheta) = -\cos\vartheta \ln(2\sin\frac{\vartheta}{2}) + \frac{\pi-\vartheta}{2}\sin\vartheta - 1$, $p_1(\vartheta) = \frac{\pi-\vartheta}{2}\cos\vartheta + \sin\vartheta\ln(2\sin\frac{\vartheta}{2})$, $q_3(\vartheta) = -\cos 2\vartheta \ln(2\sin\frac{\vartheta}{2}) + \frac{\pi-\vartheta}{2}\sin 2\vartheta - \cos\vartheta - 1/2$, $p_3(\vartheta) = \frac{\pi-\vartheta}{2}\cos 2\vartheta + \sin 2\vartheta \ln(2\sin\frac{\vartheta}{2}) + \sin\vartheta$, *and the estimations* $|q_2(\vartheta) - \sum_{k=1}^{n}\frac{\cos k\vartheta}{(k+1)^2}| < \frac{1}{n+1},\ |p_2(\vartheta) - \sum_{k=1}^{n}\frac{\sin k\vartheta}{(k+1)^2}| < \frac{1}{n+1},\ \vartheta \in (0, 2\pi)$.

4. Final Comments

Corollary 3.2 and Corollary 3.3 are improvements of Theorem 1.3 and Theorem 1.4 respectively. The author of [2] applied some special subordination results in the proof of Theorem 1.3 and Theorem 1.4, which are not applicable for other values of β. Our method presented above is a general one, which is applicable for every $\beta \in (0, 1)$.

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