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Nonlinear Differential Equations Arising from Boole Numbers and their Applications

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Abstract. In this paper, we study nonlinear differential equations satisfied by the generating function of Boole numbers. In addition, we derive some explicit and new interesting identities involving Boole numbers and higher-order Boole numbers arising from our nonlinear differential equations.

1. Introduction

The Boole polynomials, $Bl_n(x \mid \lambda)$, $(n \geq 0)$, are given by the generating function

$$\frac{1}{1+(1+t)^{\lambda}}(1+t)^x = \sum_{n=0}^{\infty} Bl_n(x \mid \lambda)\frac{t^n}{n!}, \quad (\text{see [7–10]}), \tag{1}$$

where we assume that $\lambda \neq 0$.

When $x = 0$, $Bl_n(\lambda) = Bl_n(0 \mid \lambda)$, $(n \geq 0)$, are called the Boole numbers. The higher-order Boole polynomials (also called Peters polynomials) are defined by the generating function

$$\left(\frac{1}{1+(1+t)^{\lambda}}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Bl_n^{(r)}(x \mid \lambda)\frac{t^n}{n!}, \quad (r \in \mathbb{N}), \quad (\text{see [16]}). \tag{2}$$

The first few Boole and higher-order Boole polynomials are as follows:

$$Bl_0(x \mid \lambda) = \frac{1}{2}, \quad Bl_1(x \mid \lambda) = \frac{1}{4}(2x-\lambda), \quad Bl_2(x \mid \lambda) = \frac{1}{4}(2x(x-\lambda-1)+\lambda),$$

and

$$Bl_0^{(r)}(x \mid \lambda) = 2^{-r}, \quad Bl_1^{(r)}(x \mid \lambda) = 2^{-(r+1)}(2x-\lambda),$$
$$Bl_2^{(r)}(x \mid \lambda) = 2^{-(r+2)}\left(4x(x-1)+(2-4x)\lambda r + r(r-1)\lambda^2\right), \cdots .$$

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Boole numbers and polynomials have been studied by several authors (see [7–9, 15]). For Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials, one is referred to [1–5, 11–14, 17–19]).

The purpose of this paper is to give some explicit and new identities for the Boole numbers and the higher-order Boole numbers arising from nonlinear differential equations.

The following Theorems A and B are the main results of this paper which are stated as Theorems 2.2 and 2.3, respectively.

Theorem A. *The family of nonlinear differential equations*

$$F^{(N)} = \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) F^i, \quad (N \in \mathbb{N}), \tag{3}$$

have a solution $F = F(t,\lambda) = \frac{1}{(1+t)^\lambda + 1}$,
where $a_0(N;\lambda) = (N+\lambda-1)_{N-1}$, $a_N(N;\lambda) = (-1)^N \lambda^{N-1} N!$, *and with* $a_j(N;\lambda)$ $(1 \le j \le n-1)$ *as in* (26)

Theorem B. *For* $N \in \mathbb{N}$ *and* $k \in \mathbb{N} \cup \{0\}$, *we have*

$$Bl_{k+N}(\lambda) = (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) \sum_{k=0}^{k} \binom{k}{l} (-1)^l (N+l-1)_l \, Bl_{k-l}^{(i)}(\lambda). \tag{4}$$

2. Nonlinear Differential Equations Arising from the Generating Function of Boole Numbers

Let

$$F = F(t;\lambda) = \frac{1}{(1+t)^\lambda + 1}. \tag{5}$$

Then, by (5), we get

$$\begin{aligned} F^{(1)} &= \frac{d}{dt} F(t) \\ &= \left(\frac{1}{(1+t)^\lambda + 1} \right)^2 \frac{(-1)\lambda}{(1+t)} (1+t)^\lambda \\ &= \frac{(-1)\lambda}{1+t} \frac{1}{\left((1+t)^\lambda + 1\right)^2} \left((1+t)^\lambda - 1 + 1\right) \\ &= \frac{(-1)\lambda}{1+t} \left(F - F^2\right), \end{aligned} \tag{6}$$

and

$$\begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= \frac{(-1)^2 \lambda}{(1+t)^2} \left(F - F^2\right) - \frac{\lambda}{1+t} \left(F^{(1)} - 2FF^{(1)}\right) \\ &= \frac{(-1)^2 \lambda}{(1+t)^2} \left(F - F^2\right) + \frac{(-1)^2 \lambda^2}{(1+t)^2} (1-2F) \left(F - F^2\right) \\ &= \frac{(-1)^2 \lambda}{(1+t)^2} \left\{(1+\lambda) F - (1+3\lambda) F^2 + 2\lambda F^3\right\}. \end{aligned} \tag{7}$$

Continuing this process, we set

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t) = \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) F^i, \tag{8}$$

where $N = 0, 1, 2, \ldots$.

From (8), we have

$$\begin{aligned} &F^{(N+1)} \\ &= \frac{d}{dt} F^{(N)} \\ &= \frac{(-1)^{N+1} \lambda N}{(1+t)^{N+1}} \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) F^i + \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N;\lambda)\, i F^{i-1} F^{(1)} \\ &= \frac{(-1)^{N+1} \lambda N}{(1+t)^{N+1}} \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) F^i + \frac{(-1)^{N+1} \lambda^2}{(1+t)^{N+1}} \sum_{i=1}^{N+1} i a_{i-1}(N;\lambda) F^{i-1} \left(F - F^2\right) \\ &= \frac{(-1)^{N+1} \lambda}{(1+t)^{N+1}} \left\{ \sum_{i=1}^{N+1} (N + i\lambda) a_{i-1}(N;\lambda) F^i - \sum_{i=2}^{N+2} (i-1) \lambda a_{i-2}(N;\lambda) F^i \right\} \\ &= \frac{(-1)^{N+1} \lambda}{(1+t)^{N+1}} \Big\{ (N+\lambda) a_0(N;\lambda) F - (N+1) \lambda a_N(N;\lambda) F^{N+2} \\ &\quad + \sum_{i=2}^{N+1} \left((N+i\lambda) a_{i-1}(N;\lambda) - (i-1) \lambda a_{i-2}(N;\lambda) F^i \right) \Big\}. \end{aligned} \tag{9}$$

On the other hand, replacing N by $N+1$ in (8), we get

$$F^{(N+1)} = \frac{(-1)^{N+1} \lambda}{(1+t)^{N+1}} \sum_{i=1}^{N+2} a_{i-1}(N+1;\lambda) F^i. \tag{10}$$

From (9) and (10), we can derive the following relations:

$$a_0(N+1;\lambda) = (N+\lambda) a_0(N;\lambda), \tag{11}$$

$$a_{N+1}(N+1;\lambda) = -(N+1) \lambda a_N(N;\lambda) \tag{12}$$

and

$$a_{i-1}(N+1;\lambda) = -(i-1) \lambda a_{i-2}(N;\lambda) + (N+i\lambda) a_{i-1}(N;\lambda), \tag{13}$$

where $2 \leq i \leq N+1$.

By (5) and (8), it is easy to show that

$$F = F^{(0)} = \lambda a_0(0;\lambda) F. \tag{14}$$

By comparing the coefficients on both sides of (14), we have

$$a_0(0;\lambda) = \frac{1}{\lambda}. \tag{15}$$

From (6) and (8), we note that

$$\begin{aligned} \frac{(-1)\lambda}{1+t} \left(F - F^2\right) &= F^{(1)} \\ &= \frac{(-1)\lambda}{1+t} \left(a_0(1;\lambda) F + a_1(1;\lambda) F^2\right). \end{aligned} \tag{16}$$

Thus, by (16), we get

$$a_0(1;\lambda) = 1, \text{ and } a_1(1;\lambda) = -1.$$

$$\begin{aligned} a_0(N+1;\lambda) &= (N+\lambda)\, a_0(N;\lambda) \\ &= (N+\lambda)(N+\lambda-1)\, a_0(N-1;\lambda) \\ &\vdots \\ &= (N+\lambda)(N+\lambda-1)\cdots(1+\lambda)\, a_0(1;\lambda) \\ &= (N+\lambda)(N+\lambda-1)\cdots(1+\lambda)\cdot 1 \\ &= (N+\lambda)_N, \end{aligned} \tag{17}$$

and

$$\begin{aligned} a_{N+1}(N+1;\lambda) &= -(N+1)\,\lambda a_N(N;\lambda) \\ &= (-1)^2\lambda^2 (N+1)\, N a_{N-1}(N-1;\lambda) \\ &\vdots \\ &= (-1)^N \lambda^N (N+1)\, N\cdots 2 a_1(1;\lambda) \\ &= (-1)^{N+1}\lambda^N (N+1)!, \end{aligned} \tag{18}$$

where

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1), \quad (n \geq 0).$$

From (13), we can derive the following equations:

$$\begin{aligned} & a_1(N+1;\lambda) \\ &= -\lambda a_0(N;\lambda) + (N+2\lambda)\, a_1(N;\lambda) \\ &= -\lambda a_0(N;\lambda) + (N+2\lambda)\{-\lambda a_0(N-1;\lambda) + ((N-1)+2\lambda)\, a_1(N-1;\lambda)\} \\ &= -\lambda\left(a_0(N;\lambda) + (N+2\lambda)\, a_0(N-1;\lambda)\right) + (N+2\lambda)(N+2\lambda-1)\, a_1(N-1;\lambda) \\ &= -\lambda\left(a_0(N;\lambda) + (N+2\lambda)\, a_0(N-1;\lambda)\right) \\ &\quad + (N+2\lambda)(N+2\lambda-1)\{-\lambda a_0(N-2;\lambda) + (N+2\lambda-2)\, a_1(N-2;\lambda)\} \\ &= -\lambda\{a_0(N;\lambda) + (N+2\lambda)\, a_0(N-1;\lambda) + (N+2\lambda)(N+2\lambda-1)\, a_0(N-2;\lambda)\} \\ &\quad + (N+2\lambda)(N+2\lambda-1)(N+2\lambda-2)\, a_1(N-2;\lambda) \\ &\vdots \\ &= -\lambda\sum_{i=0}^{N-1}(N+2\lambda)_i\, a_0(N-i;\lambda) + (N+2\lambda)_N\, a_1(1;\lambda) \\ &= -\lambda\sum_{i=0}^{N}(N+2\lambda)_i\, a_0(N-i;\lambda), \end{aligned} \tag{19}$$

Similarly to $i=1$ case, for $i=2$ and $i=3$, we obtain

$$a_2(N+1;\lambda) = -2\lambda\sum_{i=0}^{N-1}(N+3\lambda)_i\, a_1(N-i;\lambda), \tag{20}$$

and

$$a_3(N+1;\lambda) = -3\lambda \sum_{i=0}^{N-2} (N+4\lambda)_i \, a_2(N-i;\lambda). \tag{21}$$

Proceeding in this way, we get

$$a_k(N+1;\lambda) = -k\lambda \sum_{i_1=0}^{N-k+1} (N+(k+1)\lambda)_{i_1} a_{k-1}(N-i_1;\lambda), \tag{22}$$

where $1 \le k \le N$.

Therefore, we obtain the following theorem.

Theorem 2.1. *We have the following recurrence relations:*

(i) $a_0(0;\lambda) = \frac{1}{\lambda}$, $a_0(1;\lambda) = 1$, $a_1(1;\lambda) = -1$,

(ii) $a_0(N+1;\lambda) = (N+\lambda)_N$, $a_{N+1}(N+1;\lambda) = (-1)^{N+1}\lambda^N (N+1)!$,

(iii) $a_k(N+1;\lambda) = -k\lambda \sum_{i_1=0}^{N-k+1} (N+(k+1)\lambda)_{i_1} a_{k-1}(N-i_1;\lambda)$,

for $1 \le k \le N$.

Now, we observe that

$$\begin{aligned} a_1(N+1;\lambda) &= -\lambda \sum_{i_1=0}^{N} (N+2\lambda)_{i_1} a_0(N-i_1;\lambda) \\ &= -\lambda \sum_{i_1=0}^{N} (N+2\lambda)_{i_1} (N+\lambda-i_1-1)_{N-i_1-1}, \end{aligned} \tag{23}$$

Continuing this process, we have

$$\begin{aligned} & a_j(N+1;\lambda) \\ &= (-1)^j j! \lambda^j \\ &\times \sum_{i_j=0}^{N-j+1} \sum_{i_{j-1}=0}^{N-j+1-i_j} \cdots \sum_{i_1=0}^{N-j+1-i_j-\cdots-i_2} (N+(j+1)\lambda)_{i_j} \left(N+j\lambda-i_j-1\right)_{i_{j-1}} \\ &\times \cdots \times \left(N+2\lambda-i_j-\cdots-i_2-(j-1)\right)_{i_1} \\ &\times \left(N+\lambda-i_j-\cdots-i_1-j\right)_{N-i_j-\cdots-i_1-j}, \end{aligned} \tag{24}$$

where $1 \le j \le N$.

From (24), we note that the matrix $(a_i(j;\lambda))_{0 \le i,j \le N}$ is given by

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & & N \\ \hline 0 & \frac{1}{\lambda} & 1 & (1+\lambda) & (2+\lambda)_2 & \cdots & (N+\lambda-1)_{N-1} \\ 1 & & -1 & & & & \\ 2 & & & (-1)^2\lambda 2! & & & \\ 3 & & & & (-1)^3\lambda^2 3! & & \\ & & 0 & & & \ddots & \\ N & & & & & & (-1)^N \lambda^{N-1} N! \end{array} \tag{25}$$

Therefore, by Theorem 1, (8), and (24), we obtain the following theorem.

Theorem 2.2. *The family of nonlinear differential equations*

$$F^{(N)} = \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) F^i, \quad (N \in \mathbb{N}),$$

have a solution $F = F(t,\lambda) = \frac{1}{(1+t)^\lambda + 1}$,
where $a_0(N;\lambda) = (N+\lambda-1)_{N-1}$, $a_N(N;\lambda) = (-1)^N \lambda^{N-1} N!$,

$$\begin{aligned} & a_j(N;\lambda) \\ &= (-1)^j j! \lambda^j \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (N+(j+1)\lambda-1)_{i_j} \\ &\quad \times \left(N+j\lambda-\lambda_j-2\right)_{i_{j-1}} \cdots \left(N+2\lambda-i_j-\cdots-i_2-j\right)_{i_1} \\ &\quad \times \left(N+\lambda-i_j-\cdots-i_1-j-1\right)_{N-i_j-\cdots-i_1-j-1}, \quad (1 \le j \le N-1). \end{aligned} \tag{26}$$

Recall that the Boole numbers, $Bl_k(\lambda)$, $(k \ge 0)$, are given by the generating function

$$\frac{1}{(1+t)^\lambda + 1} = \sum_{k=0}^{\infty} Bl_k(\lambda) \frac{t^k}{k!}. \tag{27}$$

From (2), Theorem 2.2 and (27), we have

$$\begin{aligned} & \sum_{k=0}^{\infty} Bl_{k+N}(\lambda) \frac{t^k}{k!} \\ &= F^{(N)} \\ &= \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) \left(\frac{1}{(1+t)^\lambda + 1} \right)^i \\ &= (-1)^N \lambda (1+t)^{-N} \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) \left(\frac{1}{(1+t)^\lambda + 1} \right)^i \\ &= (-1)^N \lambda \left(\sum_{l=0}^{\infty} (-1)^l (N+l-1)_l \frac{t^l}{l!} \right) \left(\sum_{i=1}^{N+1} a_{i-1}(N;\lambda) \sum_{m=0}^{\infty} Bl_m^{(i)}(\lambda) \frac{t^m}{m!} \right) \\ &= (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) \left(\sum_{l=0}^{\infty} (-1)^l (N+l-1)_l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} Bl_m^{(i)}(\lambda) \frac{t^m}{m!} \right) \\ &= (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) \left(\sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k}{l} (-1)^l (N+l-1)_l Bl_{k-l}^{(i)}(\lambda) \right) \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} \left\{ (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) \sum_{l=0}^{k} \binom{k}{l} (-1)^l (N+l-1)_l Bl_{k-l}^{(i)}(\lambda) \right\} \frac{t^k}{k!}, \end{aligned} \tag{28}$$

where $N \in \mathbb{N}$.

By comparing the coefficients on both sides of (28), we obtain the following theorem.

Theorem 2.3. *For* $N \in \mathbb{N}$ *and* $k \in \mathbb{N} \cup \{0\}$, *we have*

$$Bl_{k+N}(\lambda) = (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N;\lambda) \sum_{k=0}^{k} \binom{k}{l} (-1)^l (N+l-1)_l Bl_{k-l}^{(i)}(\lambda).$$

As is well known, Euler numbers are given by the generating function

$$\left(\frac{2}{e^t+1}\right)=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}. \tag{29}$$

By (2) and (29), we easily get

$$\begin{aligned}\sum_{n=0}^{\infty}2^iBl_n^{(i)}(\lambda)\frac{t^n}{n!}&=\left(\frac{2}{(1+t)^{\lambda}+1}\right)^i\\&=\left(\frac{2}{e^{\lambda\log(1+t)}+1}\right)^i\\&=\sum_{k=0}^{\infty}E_k^{(i)}\frac{1}{k!}\lambda^k\left(\log(1+t)\right)^k\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}E_k^{(i)}\lambda^kS_1(n,k)\right)\frac{t^n}{n!},\quad(i\in\mathbb{N}).\end{aligned} \tag{30}$$

From (30), we have

$$2^iBl_n^{(i)}(\lambda)=\sum_{k=0}^{n}E_k^{(i)}\lambda^kS_1(n,k),\quad(n\geq 0,i\in\mathbb{N}). \tag{31}$$

Therefore, by Theorem 2.3 and (31), we obtain the following theorem.

Theorem 2.4. *For $k\in\mathbb{N}\cup\{0\}$ and $N\in\mathbb{N}$, we have*

$$\begin{aligned}&\frac{1}{2}\sum_{n=0}^{k+N}E_n\lambda^nS_1(k+N,n)\\&=(-1)^N\lambda\sum_{i=1}^{N+1}a_{i-1}(N;\lambda)\sum_{l=0}^{k}\binom{k}{l}(-1)^l(N+l-1)_l\sum_{n=0}^{k-l}2^{-i}E_n^{(i)}\lambda^nS_1(k-l,n).\end{aligned}$$

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